Research Article

Extension of Antoine's Ring Construction

Tai Keun Kwak

Department of Mathematics, Daejin University, Pocheon 11159, Republic of Korea

Correspondence should be addressed to Tai Keun Kwak; tkkwak@daejin.ac.kr

Received 11 August 2019; Accepted 9 November 2019; Published 22 November 2019

Academic Editor: Aloys Krieg

Copyright © 2019 Tai Keun Kwak. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This article concerns a kind of extension of Antoine's ring construction which is unit-IFP and Armendariz. Such extensions are also shown to be unit-IFP and Armendariz, by which we may extend the classes of unit-IFP and Armendariz rings.

1. Introduction

Antoine constructed a kind of ring coproduct and proved that such a ring is Armendariz in [1]. Kim et al. showed that such an Armendariz ring is also unit-IFP [2]. We consider an extended situation of Antoine's construction and show that such construction also provides examples of unit-IFP rings and Armendariz rings.

Throughout this note, every ring is an associative ring with identity unless otherwise stated. Let R be a ring. The group of units in R is denoted by $U(R)$. The polynomial ring with an indeterminate x over R is denoted by $R[x]$, and let $C_{f(x)}$ denote the set of all coefficients of given a polynomial $f(x)$. Use $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ to denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R , respectively. It is well known that $N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$.

2. Unit-IFP Property

In this section, we observe the unit-IFP property of a kind of ring which extends the situation of Antoine's construction in [1, Theorem 4.7].

Due to Bell [3], a ring R is said to have the *Insertion-of-Factors-Property* (simply, be an IFP ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$, while Narbonne [4] and Shin [5] used the terms *semicommutative* and *SI* for the IFP, respectively. Following Kim et al. [2], a ring R is said to satisfy *Insertion-of-Factors-Property on units* (simply, be a unit-IFP ring) if

$ab = 0$ for $a, b \in R$ implies $aU(R)b = 0$. They proved that Köthe's conjecture (i.e., the upper nilradical contains every nil left ideal) holds for unit-IFP rings in [2, Theorem 1.3(1)]. IFP rings are clearly unit-IFP but not conversely by [2, Example 1.1].

A ring is usually called *Abelian* if every idempotent is central. Following the literature, a ring R is said to be *directly finite* (or *Dedekind finite*) if $ab = 1$ for $a, b \in R$ implies $ba = 1$. Abelian rings are clearly directly finite. Unit-IFP rings are Abelian by [2, Lemma 1.2].

In the following which is generalizations of [6, Theorem 1.2 and Theorem 1.3], the free algebra generated by a set A over a field K is denoted by $K\langle A \rangle$.

Theorem 1. *Let K be a field and S be a noncommutative set of cardinality ≥ 2 . Let $n \geq 2$, $T \subseteq S$, and $S_0 = S \setminus T$. Set I be the ideal of $K\langle S \rangle$ generated by $t_1 t_2, \dots, t_n$ with $t_1, t_2, \dots, t_n \in T$, and J be the ideal of $K\langle T \rangle$ generated by T^n . Let $R = K\langle S \rangle / I$, $R_1 = K\langle S_0 \rangle$ and $R_2 = K\langle T \rangle / J$. Then, we have the following:*

- (1) R is a unit-IFP ring with $U(R) = \{k + g + \gamma\delta \mid k \in K \setminus \{0\}, g \in R'_2, \gamma \in R_2 T^p R_2, \delta \in R, \epsilon \in R_2 T^q R_2 \text{ for some } p, q \geq 1 \text{ with } p + q \geq n\}$, where R'_2 is the subalgebra of R_2 of all polynomials of zero constant term.
- (2) $N(R) = \{g + \gamma\delta \mid g \in R'_2, \gamma \in R_2 T^p R_2, \delta \in R, \epsilon \in R_2 T^q R_2 \text{ for some } p, q \geq 1 \text{ with } p + q \geq n\}$ and $N(R)^n = 0$.
- (3) R is a prime ring.

Proof. First, note that R_1 is a domain (hence Abelian) and R_2 is Abelian by Lemma 1 to follow.

- (1) We apply the method in the proof of [6, Theorem 1.2] to here. Note that $T^n \subset I$. Identify $k \in K$ and $s \in S$ with their images in R for simplicity, respectively. Then $T^n = 0$. Note that R is isomorphic to $R_1 *_{\kappa} R_2$ that is the ring coproduct of R_1 and R_2 over K .

By [7, Corollary 2.16(i)] (refer to [1, Lemma 4.4(a)] also), $U(R)$ is generated by the units of R_1 and R_2 , together with elements of the form $1 - \gamma\delta\epsilon$, where $\delta \in R$ and $\gamma, \epsilon \in R_i$ for some i , such that $\epsilon\gamma = 0$. Note that R_1 is a domain with

$$U(R_1) = U(K\langle S_0 \rangle) = \frac{K}{\{0\}}, \quad (1)$$

$$U(R_2) = \{f \in R_2 \mid \text{the constant term of } f \text{ is nonzero}\},$$

because every polynomial of zero constant term in R_2 is nilpotent. So, if both γ and ϵ are nonzero, then they are contained in R_2 such that $\gamma \in R_2 T^p R_2$ and $\epsilon \in R_2 T^q R_2$ with $p + q \geq n$. Therefore,

$$U(R) = \left\{ k + g + \gamma\delta\epsilon \mid k \in \frac{K}{\{0\}}, g \in R'_2, \delta \in R, \right. \\ \left. \gamma \in R_2 T^p R_2, \epsilon \in R_2 T^q R_2 \right\}, \quad (2)$$

where R'_2 is the subalgebra of R_2 of all polynomials of zero constant term, and $p, q \geq 1$, $p + q \geq n$. Here, note $(g + \gamma\delta\epsilon)^{2n} = 0$.

Next suppose that $\alpha\beta = 0$ for $\alpha, \beta \in R \setminus \{0\}$. Then, by [7, Corollary 2.16(ii)] (refer to [1, Lemma 4.4(b)] also), there exist $u \in U(R)$ and sets C, D in some R_i with $CD = 0$ such that $\alpha \in RCu$ and $\beta \in u^{-1}DR$. When C, D are nonzero, $CD = 0$ implies $C, D \subseteq R_2$. Furthermore, $CD = 0$ implies that

$$C \subseteq R_2 T^l R_2 \text{ and } D \subseteq R_2 T^m R_2 \text{ for some } l, m \geq 1 \text{ with } l + m \geq n. \quad (3)$$

Then α and β are sums of nonzero elements of the forms

$$r_1 t_1 t_2 \cdots t_l u \text{ and } u^{-1} t'_1 t'_2 \cdots t'_m r_2, \text{ respectively,} \quad (4)$$

where $t_i, t'_j \in T$ and $r_1, r_2 \in R$.

We will show $\alpha v \beta = 0$ for all $v \in U(R)$. Here, $uvu^{-1} \in U(R)$, say $w = uvu^{-1}$. By the argument above, we have $w = k + g + \gamma\delta\epsilon$ with $k \in K \setminus \{0\}$, $g \in R'_2$, $\delta \in R$, $\gamma \in R_2 T^p R_2$, $\epsilon \in R_2 T^q R_2$ for some $p, q \geq 1$ with $p + q \geq n$. Then we obtain

$$\begin{aligned} & [r_1 t_1 t_2 \cdots t_l u] v [u^{-1} t'_1 t'_2 \cdots t'_m r_2] \\ &= [r_1 t_1 t_2 \cdots t_l] w [t'_1 t'_2 \cdots t'_m r_2] \\ &= [r_1 t_1 t_2 \cdots t_l] [k + g + \gamma\delta\epsilon] [t'_1 t'_2 \cdots t'_m r_2] \\ &= [r_1 t_1 t_2 \cdots t_l] [k + g] [t'_1 t'_2 \cdots t'_m r_2] \\ &\quad + [r_1 t_1 t_2 \cdots t_l] [\gamma\delta\epsilon] [t'_1 t'_2 \cdots t'_m r_2] \\ &= [r_1 t_1 t_2 \cdots t_l] [\gamma\delta\epsilon] [t'_1 t'_2 \cdots t'_m r_2], \end{aligned} \quad (5)$$

noting that $[r_1 t_1 t_2 \cdots t_l] [k + g] [t'_1 t'_2 \cdots t'_m r_2] = 0$ because $l + m \geq n$ and $T^n = 0$. But $l + m \geq n$ and $p + q \geq n$, so $l + p \geq n$ or $q + m \geq n$. Note that every term of $r_1 t_1 t_2 \cdots t_l \gamma$ contains an element in T^{l+p} and every term of $\epsilon t'_1 t'_2 \cdots t'_m r_2$ contains an element in T^{q+m} . Thus, $[r_1 t_1 t_2 \cdots t_l] [\gamma\delta\epsilon] [t'_1 t'_2 \cdots t'_m r_2] = 0$, proving that $\alpha U(R) \beta = 0$. Therefore, R is a unit-IFP ring. Notice that R is not IFP, since $t_1 t_2 \cdots t_n = 0$ but $(t_1) s_0 (t_2 \cdots t_n) \neq 0$ for some $s_0 \in S_0$.

- (2) We apply the proof of [6, Theorem 1.3]. Let $c \in N(R)$ and $u \in U(R)$. Since R is unit-IFP by (1), $uc \in N(R)$ and $u + c \in U(R)$ by the argument in the proof of [6, Theorem 1.3]. So, by (1), there exist $p, q \geq 1$ with $p + q \geq n$ such that $u + c = k + g + \gamma\delta\epsilon$ for some $k \in K \setminus \{0\}$, $g \in R'_2$, $\delta \in R$, $\gamma \in R_2 T^p R_2$, $\epsilon \in R_2 T^q R_2$. Then, c is of the form $g + \gamma\delta\epsilon$, and hence, $N(R) \subseteq \{g + \gamma\delta\epsilon \mid k \in K \setminus \{0\}, g \in R'_2, \delta \in R, \gamma \in R_2 T^p R_2, \epsilon \in R_2 T^q R_2\}$.

For the converse inclusion, consider the expansion of $(g + \gamma\delta\epsilon)^n$ and let e be a term of it. Note that $g^n = 0$ because $T^n = 0$.

If $\gamma\delta\epsilon$ does not occur in e , then $e = g^n = 0$.

If two or more $\gamma\delta\epsilon$'s occur in e , then e is of the form $r_1 (\gamma\delta\epsilon) g^h (\gamma\delta\epsilon) r_2$ with $h \geq 0$ and $r_i \in R$; hence, e is zero because every term of $\gamma\delta\epsilon$ contains a monomial in T^{p+q} .

Assume that exactly one $\gamma\delta\epsilon$ occurs in e , i.e., $e = g^h (\gamma\delta\epsilon) g^k$ with $h, k \geq 0$ and $h + k = n - 1$. Then every term of e is of the form $r'_1 s_1 b^{h+p} f' s'_2 r'_2$ with $s_1 \in T^{h+p}$, $s_2 \in T^{q+k}$, $h \geq 0$, and $f', r'_i \in R$. But $h + p + q + k \geq 2n - 1$ and so $h + p \geq n$ or $q + k \geq n$; hence, e is zero.

This implies $(g + \gamma\delta\epsilon)^n = 0$, and we get $N(R) \subseteq \{g + \gamma\delta\epsilon \mid k \in K \setminus \{0\}, g \in R'_2, \delta \in R, \gamma \in R_2 T^p R_2, \epsilon \in R_2 T^q R_2\}$. Therefore,

$$N(R) = \{g + \gamma\delta\epsilon \mid g \in R'_2, \delta \in R, \gamma \in R_2 T^p R_2, \epsilon \in R_2 T^q R_2 \text{ for some } p, q \geq 1 \text{ with } p + q \geq n\}. \quad (6)$$

We next show $N(R)^n = 0$. Consider an element in $N(R)^n$ of the form

$$F = (g_1 + \gamma_1 \delta_1 \epsilon_1) (g_2 + \gamma_2 \delta_2 \epsilon_2) \cdots (g_n + \gamma_n \delta_n \epsilon_n), \quad (7)$$

with $g_i + \gamma_i \delta_i \epsilon_i \in N(R)$ for all i . Then $g_i \in R'_2$, $\delta_i \in R$, $\gamma_i \in R_2 T^{p_i} R_2$, $\epsilon_i \in R_2 T^{q_i} R_2$ for some $p_i, q_i \geq 1$ with $p_i + q_i \geq n$. Suppose that $g_i + \gamma_i \delta_i \epsilon_i \neq 0$ for all i . Since $T^n = 0$, $g_i \cdots g_n = 0$ for any choice $(i_1, \dots, i_n)$. Let E be a product of n times of g_i 's and $\gamma_i \delta_i \epsilon_i$'s in the expansion of F .

Suppose that any $\gamma_i \delta_i \epsilon_i$ does not occur in E , then $E = g_1 \cdots g_n = 0$.

Suppose that exactly one $\gamma_i \delta_i \epsilon_i$ occurs in E , i.e., $E = g_{i_1} \cdots g_{i_i} (\gamma_i \delta_i \epsilon_i) g_{i_{i+1}} \cdots g_{i_{n-1}}$. Then every term of E contains a product of the form

$$r_1 b_1 \cdots b_{s+p_i} f' b'_1 \cdots b'_{q_i+(n-1-s)} r_2 \text{ with } f', \\ r_1, r_2 \in R, b_j, b'_l \in T. \quad (8)$$

But $s + p_i + q_i + (n - 1 - s) = p_i + q_i + n - 1 \geq 2n - 1$, and so $s + p_i \geq n$ or $q_i + (n - 1 - s) \geq n$; hence, E is zero.

Suppose that two or more $\gamma_i \delta_i \epsilon_i$'s occur in E . Then E is of the form

$$g_{i_1} \cdots g_{i_{s_1}} (\gamma_{j_1} \delta_{j_1} \epsilon_{j_1}) g_{i_{s_1+1}} \cdots g_{i_{s_2}} (\gamma_{j_2} \delta_{j_2} \epsilon_{j_2}) g_{i_{s_2+1}} \cdots g_{i_{s_3}} \cdots g_{i_{s_{(t-1)+1}}} \cdots g_{i_{s_t}} (\gamma_{j_t} \delta_{j_t} \epsilon_{j_t}) g_{i_{s_t+1}} \cdots g_{i_{n-t}}, \quad (9)$$

with $2 \leq t \leq n$. So every term of E contains a product of the form

$$\begin{aligned} & \left(b(1)_1 \cdots b(1)_{s_1+p_{j_1}} \right) f'_1 \left(b(2)_1 \cdots b(2)_{q_{j_1}+(s_2-s_1)+p_{j_2}} \right) \\ & \cdot f'_2 \left(b(3)_1 \cdots b(3)_{q_{j_2}+(s_3-s_2)+p_{j_3}} \right) \\ & \cdot f'_3 \left(b(t)_1 \cdots b(t)_{q_{j_t}+(s_t-s_{t-1})+p_{j_t}} \right) \\ & \cdot f'_t \left(b(t+1)_1 \cdots b(t+1)_{q_{j_t}+n-t-s_t} \right), \end{aligned} \quad (10)$$

where $f'_j \in R$ and $b(h)_k \in T$. Assume here that $m_l = q_{j_l} + (s_{l+1} - s_l) + p_{j_{l+1}} \leq n-1$ for all $l = 1, 2, \dots, t-1$. But $(s_1 + p_{j_1}) + (m_1 + \cdots + m_{t-1}) + (q_{j_t} + n - t - s_t) \geq tn + (n-t)$ and $\sum_{l=1}^{t-1} m_l \leq (t-1)n - (t-1)$. This yields $(s_1 + p_{j_1}) + (q_{j_t} + n - t - s_t) \geq [tn + n - t] - [(t-1)n - (t-1)] = 2n - 1$; hence, we have that $s_1 + p_{j_1} \geq n$ or $q_{j_t} + n + 1 - t - s_t \geq n$. So $b^{s_1+p_{j_1}} = 0$ or $b^{q_{j_t}+n-t-s_t} = 0$. This yields $E = 0$.

These results imply $F = 0$, and $N(R)^n = 0$ follows.

- (3) Suppose that $\alpha R \beta = 0$ for $\alpha, \beta \in R$. Assume on the contrary that both α and β are nonzero. Note that $\alpha\beta = 0$. Then, by the argument above, α is a sum of nonzero elements of the form $r_1 t_1 t_2 \cdots t_l u$ and β is a sum of nonzero elements of the form $u^{-1} t'_1 t'_2 \cdots t'_m r_2$, where $l, m \geq 1$ with $l + m \geq n$, $t_i, t'_j \in T$, and $r_1, r_2 \in R$. Let $a \in S_0$. Then, $\alpha a \beta$ is the sum of nonzero elements of the form $[r_1 t_1 t_2 \cdots t_l u] a [u^{-1} t'_1 t'_2 \cdots t'_m r_2]$.

Next we use the method in [8, Example 14]. There exists a finite subset S_1 of S such that $\alpha a \beta$ is contained in the subring of R generated by S_1 , say R_3 . Let $S_1 = \{a_1, a_2, \dots, a_{m_1}, b_{m_1+1}, b_{m_1+2}, \dots, b_{m_1+m_2}\}$, where $a = a_1$, $a_i \in S_0$, and $b_j \in T$.

Take $u \geq 1$ such that $10^u > m_1 + m_2$. Observe that nonzero monomials in R_3 can be embedded into the set of natural numbers through the corresponding $a_1 \rightarrow 10^u + 1, a_2 \rightarrow 10^u + 2, \dots, a_{m_1} \rightarrow 10^u + m_1, b_{m_1+1} \rightarrow 10^u + m_1 + 1, b_{m_1+2} \rightarrow 10^u + m_1 + 2, \dots, b_{m_1+m_2} \rightarrow 10^u + m_1 + m_2$, σ say. Then they are totally ordered via σ (for example, $a_1 < b_{m_1+1}$ and $a_1 a_2 < b_{m_1+1} a_2$ because $10^u + 1 < 10^u + m_1 + 1$ and $10^u + 1, 10^u + 2 < 10^u + m_1 + 1, 10^u + 2$, where $\alpha, \beta = \alpha \cdot 10^{f+1} + \beta$ when $10^{f+1} > \beta \geq 10^f$). We identify f with $\sigma(f)$ for a nonzero monomial f in R_3 .

We can write

$$\begin{aligned} \alpha &= c_1 d_1 + \cdots + c_{v_1} d_{v_1}, \\ \alpha\beta &= c'_1 d'_1 + \cdots + c'_{w_1} d'_{w_1}, \end{aligned} \quad (11)$$

with $d_1 < \cdots < d_{v_1}$, $d'_1 < \cdots < d'_{w_1}$, where d_i 's and d'_j 's, are nonzero monomials of coefficient 1, and $c_i, c'_j \in K \setminus \{0\}$. Then every $(c_i d_i)(c'_j d'_j) = (c_i c'_j)(d_i d'_j)$ is of the form $[r_1 t_1 t_2 \cdots t_l u] a [u^{-1} t'_1 t'_2 \cdots t'_m r_2]$, hence nonzero. Furthermore,

$d_{v_1} d'_{w_1}$ is the largest in the set $\{d_i d'_j\}$; hence, this must be zero because $\alpha a \beta = 0$, contrary to $d_{v_1} d'_{w_1} \neq 0$. So $\alpha = 0$ or $\beta = 0$. Therefore, R is prime.

In the following, we further study the structure of zero divisors in the unit-IFP ring R in Theorem 1. $\square$

Proposition 1. Let R be the ring in Theorem 1. Let $\alpha\beta = 0$ for $0 \neq \alpha, \beta \in R$. Then $\alpha \in RT^{l'}$ and $\beta \in T^{m'} R$ for some $l', m' \geq 1$ with $l' + m' \geq n$.

Proof. Let R be the unit-IFP ring in Theorem 1. Suppose that $\alpha\beta = 0$ for $0 \neq \alpha, \beta \in R$. Then, α is a sum of nonzero elements of the form $r_1 t_1 t_2 \cdots t_l u$ and β is a sum of nonzero elements of the form $u^{-1} t'_1 t'_2 \cdots t'_m r_2$, where $l, m \geq 1$ with $l + m \geq n$, $t_i, t'_j \in T$, and $r_1, r_2 \in R$. Write $u = k + g + \gamma\delta\epsilon$, where $k \in K \setminus \{0\}$, $g \in R'_2$, $\delta \in R$, $\gamma \in R_2 T^p R_2$, $\epsilon \in R_2 T^q R_2$ with $p + q \geq n$. Then, $u^{-1} = k' + g' + \gamma'\delta'\epsilon'$ with $k' \in K \setminus \{0\}$, $g' \in R'_2$, $\delta' \in R$, $\gamma' \in R_2 T^p R_2$, $\epsilon' \in R_2 T^q R_2$. Note that $l + p \geq n$ or $q + m \geq n$.

Suppose $l + p \geq n$. Then,

$$\begin{aligned} r_1 t_1 t_2 \cdots t_l u &= (r_1 t_1 t_2 \cdots t_l) (k + g + \gamma\delta\epsilon) = k(r_1 t_1 t_2 \cdots t_l) \\ &+ (r_1 t_1 t_2 \cdots t_l) g + (r_1 t_1 t_2 \cdots t_l) \gamma\delta\epsilon \\ &= k(r_1 t_1 t_2 \cdots t_l) + (r_1 t_1 t_2 \cdots t_l) g \in RT^l, \\ u^{-1} t'_1 t'_2 \cdots t'_m r_2 &= (k' + g' + \gamma'\delta'\epsilon') (t'_1 t'_2 \cdots t'_m r_2) \\ &= k(t'_1 t'_2 \cdots t'_m r_2) + g'(t'_1 t'_2 \cdots t'_m r_2) \\ &+ \gamma'\delta'\epsilon' (t'_1 t'_2 \cdots t'_m r_2) \in T^{m_1} R, \end{aligned} \quad (12)$$

where $m_1 = \min\{m, p\}$. Here, we have $l + m_1 \geq n$ because $l + m, l + p \geq n$.

Suppose $q + m \geq n$. Then,

$$\begin{aligned} r_1 t_1 t_2 \cdots t_l u &= (r_1 t_1 t_2 \cdots t_l) (k + g + \gamma\delta\epsilon) \\ &= k(r_1 t_1 t_2 \cdots t_l) + (r_1 t_1 t_2 \cdots t_l) g \\ &+ (r_1 t_1 t_2 \cdots t_l) \gamma\delta\epsilon \in RT^{l_1}, \\ u^{-1} t'_1 t'_2 \cdots t'_m r_2 &= (k' + g' + \gamma'\delta'\epsilon') (t'_1 t'_2 \cdots t'_m r_2) \\ &= k(t'_1 t'_2 \cdots t'_m r_2) + g'(t'_1 t'_2 \cdots t'_m r_2) \in T^{m_1} R, \end{aligned} \quad (13)$$

where $l_1 = \min\{l, q\}$. Here, we have $l_1 + m \geq n$ because $l + m, q + m \geq n$.

Therefore, $\alpha \in RT^{l'}$ and $\beta \in T^{m'} R$ for some $l', m' \geq 1$ with $l' + m' \geq n$. $\square$

3. Armendariz Property

In this section we study Armendariz property of the ring R in Theorem 1. Following Rege and Chhawchharia [9], a ring R (possibly without identity) is said to be Armendariz provided that if $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$ then $ab = 0$ for all $a \in C_{f(x)}, b \in C_{g(x)}$. A ring R is usually called *reduced* if $N(R) = 0$. Reduced rings are Armendariz by [10, Lemma 1].

Armendariz rings are Abelian by the proof of [11, Theorem 6]. Some conditions under which Armendariz rings are unit-IFP are provided in [2].

Lemma 1. *The ring R_2 in Theorem 1 is Armendariz. Furthermore, R_2 is a local ring with $J(R_2) = N_*(R_2) = N^*(R_2) = N(R_2)$.*

Proof. Suppose that $f(x)g(x) = 0$ for $f(x), g(x) \in R_2[x] \setminus \{0\}$. We again use the method in the proof of Theorem 1(3). There exists a finite subset T_1 of T such that $C_{f(x)}$ and $C_{g(x)}$ are contained in the subring of R_2 generated by T_1 , R_4 say. Let $T_1 = \{b_1, b_2, \dots, b_m\}$. Observe that nonzero monomials in R_4 can be embedded into the set of natural numbers through a similar corresponding to one in the proof of Theorem 1(3). Express this corresponding by σ_1 . We identify f with $\sigma_1(f)$ for a nonzero monomial f in R_4 .

We can write $f(x) = \sum_{i=1}^M h_i k_i$ and $g(x) = \sum_{j=1}^N t_j l_j$ for some $k_i, l_j \in K[x]$ and some monomials h_i, t_j in R_2 , where every monomial in $\{h_i, t_j\}$ can be assumed to be of coefficient 1. In this situation, we can assume that

$$\begin{aligned} h_1 < h_2 < \dots < h_M, \\ t_1 < t_2 < \dots < t_N. \end{aligned} \quad (14)$$

Then $h_1 t_1$ is smaller than other $h_i t_j$'s. But $f(x)g(x) = 0$, and so $h_1 t_1 k_1 l_1 = 0$. It then follows that $h_1 t_1 k_1 l_1 \in T^n K[x]$. Since $h_1 t_1$ is smallest in $\{h_i t_j\}$, other $h_i t_j$'s are also contained in T^n . This implies $h_i t_j = 0$ for all i and j . From this result, we can say that $\alpha\beta = 0$ for all $\alpha \in C_{f(x)}$ and $\beta \in C_{g(x)}$. Therefore, R_2 is Armendariz.

$(R_2/R_2') \cong K$ and $R_2'^n = 0$; hence, $R_2' = N(R_2) = N_*(R_2) = N^*(R_2) = J(R_2)$. Especially, R_2 is a local ring. $\square$

Theorem 2. *The ring R in Theorem 1 is Armendariz.*

Proof. We apply the proof of [1, Theorem 4.7] to this extended situation. First, we check a property of the ring R_2 . Suppose that $\alpha\beta = 0$ and $\gamma\delta = 0$ for $\alpha, \beta, \gamma, \delta \in R_2$. Then, $\alpha\beta, \gamma\delta \in R_2'$ because every element in $R_2 \setminus R_2'$ is invertible in R_2 . So we have that $\alpha \in R_2 T^p R_2, \beta \in R_2 T^q R_2$ and $\gamma \in R_2 T^{p'} R_2, \delta \in R_2 T^{q'} R_2$ for some $p, q, p', q' \geq 1$ with $p + q \geq n$ and $p' + q' \geq n$. It then follows that $p + q' \geq n$ or $q + p' \geq n$; hence,

$$“\beta\alpha = 0 \text{ and } \delta\gamma = 0” \text{ and } “\alpha\delta = 0 \text{ or } \gamma\beta = 0”. \quad (15)$$

Suppose that $f(x)g(x) = 0$ for $f(x), g(x) \in R[x] \setminus \{0\}$. Note that R_2 is Armendariz (hence Abelian) by Lemma 1; hence, R_2 is directly finite. R_1 is clearly a domain, so directly finite. Then, by [1, Proposition 4.5], there exist $p(x) \in K[x]$, subsets $U, V \subseteq R_i[x]$, and $\alpha(x), \beta(x) \in R[x]$ such that $UV = 0$, $p(x)f(x) \in R[x]U\alpha(x)$, and $p(x)g(x) \in \beta(x)VR[x]$. Furthermore, by [1, Proposition 4.6], we may assume that $C_{f(x)} \subseteq C_{p(x)f(x)}$ and $C_{g(x)} \subseteq C_{p(x)g(x)}$.

Since R_1 is a domain, U and V are contained in $R_2[x]$. But R_2 is Armendariz. So, since $UV = 0$, we have $uv = 0$ for any $u \in C_{t_1(x)}$ and $v \in C_{t_2(x)}$, where $t_1(x) \in U$ and $t_2(x) \in V$.

Write $C_U = \{C_{t_1(x)} | t_1(x) \in U\}$ and $C_V = \{C_{t_2(x)} | t_2(x) \in V\}$. Then, we obtain $uv = 0$ for any $u \in C_U$ and $v \in C_V$.

Observe that any coefficient of $p(x)f(x)$ is a sum of elements of the form $ru\alpha_*$ where $r \in R$, $u \in C_U$, and $\alpha_* \in C_{\alpha(x)}$ and that the coefficients in $p(x)g(x)$ are sums of elements of the form β_*vs where $s \in R$, $v \in C_V$, and $\beta_* \in C_{\beta(x)}$. Hence the product of the coefficients of $f(x)$ and $g(x)$ will be the sums of elements of the form $ru\alpha_*\beta_*vs$.

We next observe the structure of $\alpha_*\beta_*$. Note that $\alpha(x), \beta(x)$ are products of units in $R_1[x]$, units in $R_2[x]$, and elements of the form $h(x) - \gamma(x)\delta(x)\epsilon(x)$ where $h(x) \in K[x] \setminus \{0\}$, $\gamma(x), \epsilon(x) \in R_2[x]$ such that $\epsilon(x)\gamma(x) = 0$ and $\delta(x) \in R[x]$. Since R_2 is Armendariz, $\epsilon(x)\gamma(x) = 0$ implies that $\epsilon\gamma = 0$ for all $\epsilon \in C_{\gamma(x)}, \gamma \in C_{\epsilon(x)}$. Hence, $\alpha_*\beta_*$ will be in the subring generated by the coefficients of polynomials in $U(R_1[x]), U(R_2[x])$, and elements of the form $\gamma\delta\epsilon$ where $\delta \in R$ and $\gamma, \epsilon \in R_2$ such that $\epsilon\gamma = 0$.

Consider the element $\gamma'\delta'\epsilon'$ such that $\gamma', \epsilon' \in R_2$ with $\epsilon'\gamma' = 0$ and $\delta' \in R$. Consider next the product $(\gamma\delta\epsilon)(\gamma'\delta'\epsilon')$. From $\epsilon\gamma = 0$ and $\epsilon'\gamma' = 0$, we obtain that $\epsilon\gamma' = 0$ (then $(\gamma\delta\epsilon)(\gamma'\delta'\epsilon') = 0$) or $\epsilon'\gamma = 0$, by the property (*). Hence, the elements of the preceding form $\gamma\delta\epsilon$ are closed under multiplication.

Next, since R_1 is a domain, we have $U(R_1) = K \setminus \{0\}$ and $U(R_1[x]) = K \setminus \{0\}$. By Lemma 1, $R_2' = N_*(R_2)$; hence, $N_*(R_2[x]) = N_*(R_2)[x] = R_2'[x]$ by [12, Theorem 3]. This yields $(K \setminus \{0\}) + R_2'[x] \subseteq U(R_2[x])$. But $(R_2[x] \setminus R_2'[x]) \cong K[x]$, and so $U(R_2[x]) \subseteq (K \setminus \{0\}) + R_2'[x]$. Thus, $U(R_2[x]) = (K \setminus \{0\}) + R_2'[x]$.

Hence, $\alpha_*\beta_*$ will be a sum of elements in K, R_2 , and elements of $R_2\gamma\delta\epsilon R_2$ where $\gamma, \epsilon \in R_2$ such that $\epsilon\gamma = 0$.

Now, since $uv = 0$ and $\epsilon\gamma = 0$, we have that $u\gamma = 0$ or $\epsilon v = 0$ by the property (*). In either case, $ruR_2\gamma\delta\epsilon R_2vs = 0$ because $uR_2\gamma = 0$ or $\epsilon R_2v = 0$ also by applying the property (*). Therefore, R is Armendariz. $\square$

Data Availability

No data were used to support this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2019R1F1A1A1040405).

References

- [1] R. Antoine, “Nilpotent elements and Armendariz rings,” *Journal of Algebra*, vol. 319, no. 8, pp. 3128–3140, 2008.
- [2] H. K. Kim, T. K. Kwak, Y. Lee, and Y. Seo, “Insertion of units at zero products,” *Journal of Algebra and Its Applications*, vol. 17, no. 3, Article ID 1850043, 2018.

- [3] H. E. Bell, "Near-rings in which each element is a power of itself," *Bulletin of the Australian Mathematical Society*, vol. 2, no. 3, pp. 363–368, 1970.
- [4] L. Motais de Narbonne, "Anneaux semi-commutatifs et unisriels anneaux dont les id aux principaux sont idempotents," in *Proceedings of the 106th National Congress of Learned Societies (Perpignan, 1981)*, pp. 71–73, Paris, France, 1982.
- [5] G. Shin, "Prime ideals and sheaf representation of a pseudo symmetric ring," *Transactions of the American Mathematical Society*, vol. 184, pp. 43–60, 1973.
- [6] Y. Lee, "Structure of unit-IFP rings," *Journal of the Korean Mathematical Society*, vol. 55, pp. 1257–1268, 2018.
- [7] G. M. Bergman, "Modules over coproducts of rings," *Transactions of the American Mathematical Society*, vol. 200, pp. 1–32, 1974.
- [8] C. Huh, Y. Lee, and A. Smoktunowicz, "Armendariz rings and semicommutative rings," *Communications in Algebra*, vol. 30, no. 2, pp. 751–761, 2002.
- [9] M. B. Rege and S. Chhawchharia, "Armendariz rings," *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, vol. 73, no. 1, pp. 14–17, 1997.
- [10] E. P. Armendariz, "A note on extensions of Baer and P. P.-rings," *Journal of the Australian Mathematical Society*, vol. 18, no. 4, pp. 470–473, 1974.
- [11] D. D. Anderson and V. Camillo, "Armendariz rings and Gaussian rings," *Communications in Algebra*, vol. 26, no. 7, pp. 2265–2272, 1998.
- [12] S. A. Amitsur, "Radicals of polynomial rings," *Canadian Journal of Mathematics*, vol. 8, pp. 355–361, 1956.

Submit your manuscripts at
www.hindawi.com