Research Article

Common Fixed Point Theorems on Generalized Ratio Contraction Mapping in Extended Rectangular b -Metric Spaces

Budi Nurwahyu

Department of Mathematics, Hasanuddin University, Tamalanrea KM 10, Makassar, Indonesia

Correspondence should be addressed to Budi Nurwahyu; budinurwahyu@unhas.ac.id

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In this paper, we propose and prove the common fixed point theorems on generalized contraction mappings in extended rectangular b -metric spaces by utilizing the weakly compatible function property.

1. Introduction

The notion of b -metric space was introduced by Czerwik in 1993 [1], as a generalization of a metric space, by modifying the third condition in the metric space. Mohanta [2] and Husain et al. [3] utilized that space to prove existence and uniqueness of common fixed points. Recently, Kamran et al. [4] in 2017 generalized b -metric to become extended b -metric for utilizing in fixed point results and Alqahtani et al. [5] in 2018 utilized extended b -metric to prove the common fixed point. In 2015, George et al. [6] introduced the notion of rectangular b -metric space as a generalization of rectangular metric space. Kadelburg and Radenović [7] and Mitrovic [8] established and proved the common fixed point in such space. Very recently, Asim et al. [9] in 2019 generalized the rectangular b -metric and termed it the extended rectangular b -metric. By utilizing this space, Mustafa et al. [10] proved some fixed point results for contracting mappings.

Inspired of some fixed point results in extended rectangular b -metric, in the main results, we establish and prove existence and uniqueness of the common fixed point in the extended rectangular b -metric and furnish examples to clarify the results.

2. Preliminaries

In the following section, we need some definitions to govern our theorems.

Definition 1 (see [1]). Let X be a nonempty set. A mapping $d_b : X \times X \rightarrow [0, \infty)$ is called a b -metric, if there exists $b \geq 1$ such that d_b satisfies the following conditions:

- (1) $d_b(x, y) = 0$, if and only if $x = y$
- (2) $d_b(x, y) = d_b(y, x)$
- (3) $d_b(x, y) \leq b[d_b(x, s) + d_b(s, y)]$

for all $x, y, s \in X$.

The pair (X, d_b) is called a b -metric space.

Definition 2 (see [4]). Let X be a nonempty set. A mapping $d_b : X \times X \rightarrow [0, \infty)$ is called an extended b -metric, if there exists a function $b : X \times X \rightarrow [1, \infty)$ such that d_b satisfies the following conditions:

- (1) $d_b(x, y) = 0$, if and only if $x = y$
- (2) $d_b(x, y) = d_b(y, x)$
- (3) $d_b(x, y) \leq b(x, y)[d_b(x, s) + d_b(s, y)]$

for all $x, y, s \in X$.

The pair (X, d_b) is called an extended b -metric space.

Definition 3 (see [6]). Let X be a nonempty set. A mapping $d_b : X \times X \rightarrow [0, \infty)$ is called rectangular b -metric, if there exists $b \geq 1$ such that d_b satisfies the following conditions:

- (1) $d_b(x, y) = 0$, if and only if $x = y$
- (2) $d_b(x, y) = d_b(y, x)$

$$(3) d_b(x, y) \leq b[d_b(x, s) + d_b(s, t) + d_b(t, y)]$$

for all $x, y \in X$ and $s, t \in X \setminus \{x, y\}$.

The pair (X, d_b) is called a *rectangular b-metric space*.

Definition 4 (see [9]). Let X be a nonempty set. A mapping $d_b : X \times X \rightarrow [0, \infty)$ is called an *extended rectangular b-metric*, if there exists a function $b : X \times X \rightarrow [1, \infty)$ such that d_b satisfies the following conditions:

- (A1) $d_b(x, y) = 0$, if and only if $x = y$,
 - (A2) $d_b(x, y) = d_b(y, x)$,
 - (A3) $d_b(x, y) \leq b(x, y)[d_b(x, s) + d_b(s, t) + d_b(t, y)]$.
- (1)

for all $x, y \in X$ and $s, t \in X \setminus \{x, y\}$.

The pair (X, d_b) is called an *extended rectangular b-metric space*.

Example 1. Let $X = [0, \infty)$ and $d_b(x, y) = (x - y)^p$, and $p > 1$ with $b(x, y) = x + y + 3^{p-1}$.

It is obvious for condition A1 and A2. For condition A3, we consider from the Jensen inequality.

Thus, we have $d_b(x, y) = (x - y)^2 \leq 3^{p-1}[(x - s)^2 + (s - t)^2 + (t - y)^2] \leq (x + y + 3^{p-1})[(x - s)^2 + (s - t)^2 + (t - y)^2] = b(x, y)[d_b(x, s) + d_b(s, t) + d_b(t, y)]$ for all $x, y, s, t \in X$.

So, this shows that $d_b(x, y) = (x - y)^p$ is an extended rectangular b-metric with $b(x, y) = x + y + 3^{p-1}$. However, in general $d_b(x, y)$ is not continuous.

Example 2 (see [7]). Let $X = \{0, 2\} \cup \{1/n \mid n \in \mathbb{N}\}$, and choose a function $b(x, y) = x + y + 2$, for $x, y \in X$. Define $d_b : X \times X \rightarrow [0, \infty)$ a function as follows:

$$d_b(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, x, y \in \{0, 2\} \text{ or } x, y \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}, \\ y, & x \in \{0, 2\}, y \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}, \\ x, & x \in \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}, y \in \{0, 2\}. \end{cases} \quad (2)$$

It is easy to show that d_b is the complete extended rectangular b-metric on X with $b(x, y) = x + y + 2$.

If we choose a sequence $\{1/n\}_{n \in \mathbb{N}}$, it is easy to show that sequence $\{1/n\}_{n \in \mathbb{N}}$ is convergent to 0 and 2, so it is not a

Cauchy sequence. Moreover, d_b is not continuous, since $\lim_{n \rightarrow \infty} (1/n) = 0$ and $1 = \lim_{n \rightarrow \infty} d_b(1/n, 1/3) \neq d_b(0, (1/3)) = 1/3$. Thus, (X, d_b) is not metrizable because there exists a sequence in (X, d_b) that is not convergent to unique point in X .

Our proofs in the main results need uniqueness of the limit point of sequence in (X, d_b) as given in the following lemma.

Lemma 1 (see [9, 11]). Let (X, d_b) be a complete extended rectangular b-metric space and $\{y_n\}$ is a Cauchy sequence. If $y_n \neq y_m$ for all $n \neq m$, then limit point of $\{y_n\}$ is unique.

Definition 5 (see [12]). Let X be a nonempty set and $T_1, T_2 : X \rightarrow X$ be self-mapping. If $T_1x = T_2x = y$ for some $x \in X$, then y is called a point of coincidence of T_1 and T_2 and x is called a coincidence point of T_1 and T_2 .

Definition 6 (see [13]). Let X be a nonempty set and $T_1, T_2 : X \rightarrow X$ be self-mappings. $\{T_1, T_2\}$ is called weakly compatible. For every $x \in X$, if $T_1x = T_2x$, then $T_2T_1x = T_1T_2x$.

Definition 7 (see [9]). Let (X, d_b) be an extended rectangular b-metric space, then sequence

- (a) $\{x_n\}$ is called convergent if and only if there exists $x \in X$ such that $d_b(x_n, x) \rightarrow 0$, as $n \rightarrow \infty$.
- (b) $\{x_n\}$ is called Cauchy if and only if $d_b(x_n, x_m) \rightarrow 0$, as $n, m \rightarrow \infty$.

In addition, X is called complete if every Cauchy sequence in X is convergent in X .

3. Main Results

In this section, we establish and prove the common fixed point theorems in complete extended rectangular b-metric space and furnish some examples to clarify the theorem.

Lemma 2. Let (X, d_b) be an extended rectangular b-metric space and $\{y_n\}$ be a sequence in X .

If $d_b(y_n, y_{n+1}) \leq K d_b(y_{n-1}, y_n)$, $0 < K < 1$, and $\lim_{n, m \rightarrow \infty} b(y_n, y_m) < (1/K)$, then $\{y_n\}$ is a Cauchy.

Proof. Let $m \in \mathbb{N}$, $m > n$. □

Case 1. For $m - n$ is odd, let $m - n = 2k + 1$.

Consider that from rectangular condition in (1) we have

$$\begin{aligned}
 d_b(y_n, y_m) &\leq b(y_n, y_m)[d_b(y_n, y_{n+1}) + d_b(y_{n+1}, y_{n+2}) + d_b(y_{n+2}, y_m)] \\
 &\leq b(y_n, y_m)[d_b(y_n, y_{n+1}) + d_b(y_{n+1}, y_{n+2})] + b(y_n, y_m)d_b(y_{n+2}, y_m) \\
 &\leq b(y_n, y_m)[K^n d_b(y_0, y_1) + K^{n+1} d_b(y_0, y_1)] + b(y_n, y_m)d_b(y_{n+2}, y_m) \\
 &\leq (K^n + K^{n+1})d_b(y_0, y_1)b(y_n, y_m) + (K^{n+2} + K^{n+3})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) \\
 &\quad + b(y_n, y_m)b(y_{n+2}, y_m)d_b(y_{n+4}, y_m) \\
 &\leq (K^n + K^{n+1})d_b(y_0, y_1)b(y_n, y_m) \\
 &\quad + (K^{n+2} + K^{n+3})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) + \cdots \\
 &\quad + (K^{n+2k-2} + K^{n+2k-1})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) \cdots b(y_{n+2k-2}, y_m) + b(y_n, y_m)b(y_{n+2}, y_m) \cdots b \\
 &\quad \cdot (y_{n+2k-2}, y_m)d_b(y_{n+2k}, y_m) \\
 &\quad + (K^{n+2k-2} + K^{n+2k-1})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) \cdots b(y_{n+2k-2}, y_m) + b(y_n, y_m)b(y_{n+2}, y_m) \cdots b \\
 &\quad \cdot (y_{n+2k-2}, y_m)d_b(y_{n+2k}, y_m) \\
 &\leq (K^n + K^{n+1})d_b(y_0, y_1)b(y_n, y_m) \\
 &\quad + (K^{n+2} + K^{n+3})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) + \cdots \\
 &\quad + (K^{n+2k-2} + K^{n+2k-1})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) \cdots b(y_{n+2k-2}, y_m) + \cdots + b(y_n, y_m)b(y_{n+2}, y_m) \cdots b \\
 &\quad \cdot (y_{n+2k-2}, y_m)K^{n+2k}d_b(y_0, y_1) \\
 &= (1 + K)d_b(y_0, y_1)K^n \sum_{j=0}^{k-1} K^{2j} \prod_{i=0}^j b(y_{n+2i}, y_m) + b(y_n, y_m)b(y_{n+2}, y_m) \cdots b(y_{n+2k-2}, y_m)K^{n+2k}d_b(y_0, y_1).
 \end{aligned} \tag{3}$$

Since $b(y_i, y_j) \geq 1$, using (3) we have that

$$\begin{aligned}
 d_b(y_n, y_m) &\leq (1 + K)d_b(y_0, y_1) \left\{ K^n \left(\sum_{j=0}^{k-1} K^{2j} \prod_{i=0}^j b(y_{n+2i}, y_m) + b(y_n, y_m)b(y_{n+2}, y_m) \cdots b(y_{n+2k-2}, y_m)K^{2k} \right) \right\} \\
 &\leq (1 + K)d_b(y_0, y_1) \left\{ K^n \left(\sum_{j=n}^{m-1} K^{j-n} \prod_{i=1}^j b(y_i, y_m) \right) \right\} = (1 + K)d_b(y_0, y_1) \left(\sum_{j=n}^{m-1} K^j \prod_{i=1}^j b(y_i, y_m) \right) \\
 &= (1 + K)d_b(y_0, y_1)(S_{m-1} - S_{n-1}),
 \end{aligned} \tag{4}$$

where $S_n = \sum_{j=0}^n K^j \prod_{i=1}^j b(y_i, y_m)$.

Let $a_j = K^j \prod_{i=1}^j b(y_i, y_m)$, so by applying ratio criteria and as known that $\lim_{n,m \rightarrow \infty} b(y_n, y_m) < (1/K)$, then we have $\lim_{j \rightarrow \infty} (a_{j+1}/a_j) = \lim_{j \rightarrow \infty} Kb(y_j, y_m) < 1$ for each $m \in N$.

This implies that $S_n = \sum_{j=1}^n K^j \prod_{i=1}^j b(y_i, y_m)$ is convergent for each $m \in N$.

Hence, for $n, m \rightarrow \infty$ in (4) we get $d_b(y_n, y_m) \rightarrow 0$; thus, $\{y_n\}$ is the Cauchy sequence in X .

Case 2. For $m - n$ is even, let $m - n = 2k$.

$$\begin{aligned}
 d_b(y_n, y_m) &\leq b(y_n, y_m)[d_b(y_n, y_{n+1}) + d_b(y_{n+1}, y_{n+2}) + d_b(y_{n+2}, y_m)] \\
 &\leq b(y_n, y_m)[d_b(y_n, y_{n+1}) + d_b(y_{n+1}, y_{n+2})] + b(y_n, y_m)d_b(y_{n+2}, y_m) \\
 &\leq b(y_n, y_m)[K^n d_b(y_0, y_1) + K^{n+1} d_b(y_0, y_1)] + b(y_n, y_m)d_b(y_{n+2}, y_m) \\
 &\leq (K^n + K^{n+1})d_b(y_0, y_1)b(y_n, y_m) + b(y_n, y_m)d_b(y_{n+2}, y_m) \\
 &\leq (K^n + K^{n+1})d_b(y_0, y_1)b(y_n, y_m) \\
 &\quad + b(y_n, y_m)b(y_{n+2}, y_m)[d_b(y_{n+2}, y_{n+3}) + d_b(y_{n+3}, y_{n+4})] \\
 &\quad + b(y_n, y_{n+2k})b(y_{n+2}, y_m)d_b(y_{n+4}, y_m) \\
 &\leq (K^n + K^{n+1})d_b(y_0, y_1)b(y_n, y_m) \\
 &\quad + b(y_n, y_m)b(y_{n+2}, y_m)[K^{n+2}d_b(y_0, y_1) + K^{n+3}d_b(y_0, y_1)] \\
 &\quad + b(y_n, y_m)b(y_{n+2}, y_m)d_b(y_{n+4}, y_m) \\
 &\leq (K^n + K^{n+1})d_b(y_0, y_1)b(y_n, y_m) \\
 &\quad + (K^{n+2} + K^{n+3})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) + \cdots \\
 &\quad + (K^{n+2k-2} + K^{n+2k-1})d_b(y_0, y_1)b(y_n, y_m)b(y_{n+2}, y_m) \cdots b(y_{m-2}, y_m) \\
 &\quad + K^{n+2k-2}d_b(y_0, y_2)b(y_n, y_m)b(y_{n+2}, y_m) \cdots b(y_{m-2}, y_m) \\
 &= (1 + K)d_b(y_0, y_1)K^n \sum_{j=1}^{k-1} K^{2j} \prod_{i=0}^j b(y_{n+2i}, y_m) + K^{m-2}d_b(y_0, y_2) \prod_{i=0}^{k-1} b(y_{n+2i}, y_m).
 \end{aligned} \tag{5}$$

Let $L = \max\{d_b(y_0, y_1), d_b(y_0, y_2)\}$, so by using (5) we have

$$\begin{aligned}
 d_b(y_n, y_m) &\leq (1 + K)L \left(K^n \sum_{j=1}^{k-1} K^{2j} \prod_{i=0}^j b(y_{n+2i}, y_m) + K^{m-2} \prod_{i=0}^{k-1} b(y_{n+2i}, y_m) \right) \\
 &= (1 + K)L \left(K^n \sum_{j=1}^{k-1} K^{2j} \prod_{i=0}^j b(y_{n+2i}, y_m) + K^{n+2k-2} \prod_{i=0}^{k-1} b(y_{n+2i}, y_m) \right) \\
 &= (1 + K)L \left(K^n \left[\sum_{j=1}^{k-1} K^{2j} \prod_{i=0}^j b(y_{n+2i}, y_m) + K^{2k-2} \prod_{i=0}^{k-1} b(y_{n+2i}, y_m) \right] \right) \\
 &\leq (1 + K)L \left(K^n \left(\sum_{j=n}^{m-1} K^{j-n} \prod_{i=1}^j b(y_i, y_m) \right) \right) = (1 + K)L \left(\sum_{j=n}^{m-1} K^j \prod_{i=1}^j b(y_i, y_m) \right) \\
 &= (1 + K)L \left(\sum_{j=0}^{m-1} K^j \prod_{i=1}^j b(y_i, y_m) - \sum_{j=0}^{n-1} K^j \prod_{i=1}^j b(y_i, y_m) \right) = (1 + K)L(S_{m-1} - S_{n-1}).
 \end{aligned} \tag{6}$$

This implies $d_b(y_n, y_m) \rightarrow 0$ as $m, n \rightarrow \infty$, thus $\{y_n\}$ is Cauchy in X .

Theorem 1. Let (X, d_b) be a complete extended rectangular b -metric space and let $f, g : X \rightarrow X$ be self-mappings such that $f(X) \subseteq g(X)$ and $g(X)$ closed. If

$$b(x, y)d_b(fx, fy) \leq \frac{pd_b(gx, gy) + qd_b(gx, fx) + r(gy, fy) + s\beta d_b(fx, gy)d_b(fy, gx)}{\lambda b(fx, gx) + \beta(d_b(fx, gy) + d_b(fy, gx))}, \tag{7}$$

holds for all $x, y \in X$, $\beta > 0$, $0 < s < 1$, $0 < p + q + r < \lambda$, $(p/1 - s) < \lambda$, and $\lim_{n,m \rightarrow \infty} b(y_n, y_m) < (\lambda - r/p + q)$, then f and g have a unique coincidence in X .

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ as an initial point, define sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = f(x_n) = g(x_{n+1})$, $n = 0, 1, 2, \dots$. Suppose that $y_n \neq y_{n+1}$, $n = 0, 1, 2, \dots$, because if $y_n = y_{n+1}$ for some n , then $g(x_{n+1}) = y_n = y_{n+1} = f(x_{n+1})$, so f and g have a coincidence point.

Considering (7), we have

$$\begin{aligned} b(x_n, x_{n+1})d_b(y_n, y_{n+1}) &= b(x_n, x_{n+1})d_b(fx_n, fx_{n+1}) \\ &\leq \frac{pd_b(gx_n, gx_{n+1}) + qd_b(gx_n, fx_n) + rd_b(gx_{n+1}, fx_{n+1}) + s\beta d_b(fx_n, gx_{n+1})d_b(fx_{n+1}, gx_n)}{\lambda b(fx_n, gx_n) + \beta(d_b(fx_n, gx_{n+1}) + d_b(fx_{n+1}, gx_n))} \\ &= \frac{pd_b(y_{n-1}, y_n) + qd_b(y_{n-1}, y_n) + rd_b(y_n, y_{n+1}) + s\beta d_b(y_n, y_n)d_b(y_{n+1}, y_n)}{\lambda b(y_n, y_{n-1}) + \beta(d_b(y_n, y_n) + d_b(y_{n+1}, y_n))} \\ &= \frac{(p+q)d_b(y_{n-1}, y_n) + rd_b(y_n, y_{n+1})}{\lambda b(y_n, y_{n-1}) + \beta d_b(y_{n+1}, y_n)} \\ &\leq \frac{(p+q)d_b(y_{n-1}, y_n) + rd_b(y_n, y_{n+1})}{\lambda b(y_n, y_{n-1})}. \end{aligned} \quad (8)$$

Since $(x_n, x_{n+1}) \geq 1$, then we have

$$\begin{aligned} d_b(y_n, y_{n+1}) &\leq \frac{(p+q)d_b(y_{n-1}, y_n) + rd_b(y_n, y_{n+1})}{\lambda b(x_n, x_{n+1})b(y_n, y_{n-1})} \\ &\leq \frac{(p+q)d_b(y_{n-1}, y_n) + rd_b(y_n, y_{n+1})}{\lambda} \end{aligned} \quad (9)$$

Thus, we get

$$d_b(y_n, y_{n+1}) \leq \delta d_b(y_{n-1}, y_n), \quad (10)$$

where $\delta = p + q/\lambda - r$. Since $p + q + r < \lambda$, we have $0 < \delta < 1$, and as known that $\lim_{n,m \rightarrow \infty} b(y_n, y_m) < 1/\delta$, then by using Lemma 2, we get $\{y_n\}$ is a Cauchy sequence in gX .

Since X is complete, then there exists $u^* \in X$ such that

$$d_b(y_n, u^*) \longrightarrow 0, \quad (11)$$

and since $y_n = g(x_{n+1}) \in gX$ and gX is a closed set, then $u^* \in gX$ and $u^* = g(x^*)$ for some $x^* \in X$.

Furthermore, we will show that f and g have a point coincidence as follows.

From (1) and (7) we have

$$\begin{aligned} d_b(fx^*, gx^*) &\leq b(fx^*, gx^*)[d_b(fx^*, y_n) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, gx^*)] \\ &= b(fx^*, gx^*)[d_b(fx^*, fx_n) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*)] \\ &\leq b(fx^*, gx^*) \left[\frac{1}{b(x^*, x_n)} \left(\frac{pd_b(gx^*, gx_n) + qd_b(gx^*, fx^*) + rd_b(gx_n, fx_n) + s\beta d_b(fx^*, gx_n)d_b(fx_n, gx^*)}{\lambda b(fx^*, gx^*) + \beta(d_b(fx^*, gx_n) + d_b(fx_n, gx^*))} \right) \right. \\ &\quad \left. + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*) \right] \\ &\leq b(fx^*, gx^*) \left[\left(\frac{pd_b(u^*, y_{n-1}) + qd_b(u^*, fx^*) + rd_b(y_{n-1}, y_n) + s\beta d_b(fx^*, y_{n-1})d_b(y_n, u^*)}{\lambda b(fx^*, u^*) + \beta(d_b(fx^*, y_{n-1}) + d_b(y_n, u^*))} \right) \right. \\ &\quad \left. + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*) \right] \\ &\leq b(fx^*, gx^*) \left[\left(\frac{pd_b(u^*, y_{n-1}) + qd_b(u^*, fx^*) + rd_b(y_{n-1}, y_n)}{\lambda b(fx^*, u^*) + \beta(d_b(fx^*, y_{n-1}) + d_b(y_n, u^*))} + sd_b(y_n, u^*) \right) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*) \right] \\ &\leq b(fx^*, gx^*) \left[\left(\frac{pd_b(u^*, y_{n-1}) + qd_b(u^*, fx^*) + rd_b(y_{n-1}, y_n)}{\lambda b(fx^*, u^*)} + sd_b(y_n, u^*) \right) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*) \right]. \end{aligned} \quad (12)$$

It follows from (11) to (12) that if $n \rightarrow \infty$, then we get

$$d_b(fx^*, gx^*) \leq \frac{b(fx^*, gx^*)d_b(u^*, fx^*)}{\lambda b(fx^*, gx^*)} \leq \frac{d_b(gx^*, fx^*)}{\lambda}. \quad (13)$$

This implies that $d_b(fx^*, gx^*) = 0$, so then we get $fx^* = gx^* = u^*$. Hence, u^* is a coincidence point of f and g .

To show uniqueness of the coincidence point of f and g , we suppose there exists $w^* \in X$ such that $fy^* = gy^* = w^*$.

From (1) and (7), we have

$$\begin{aligned} d_b(u^*, w^*) &= d_b((fx^*, fy^*)) \\ &\leq \frac{1}{b(x^*, y^*)} \left(\frac{pd_b(u^*, w^*) + qd_b(u^*, u^*) + rd_b(w^*, w^*) + s\beta d_b(u^*, w^*)d_b(w^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, w^*) + d_b(w^*, u^*))} \right) \\ &\leq \left(\frac{pd_b(u^*, w^*) + s\beta d_b(u^*, w^*)d_b(w^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, w^*) + d_b(w^*, u^*))} \right) = \left(\frac{pd_b(u^*, w^*) + s\beta d_b(u^*, w^*)d_b(w^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, w^*) + d_b(w^*, u^*))} \right) \\ &= \frac{pd_b(u^*, w^*)}{\lambda} + sd_b(w^*, u^*). \end{aligned} \quad (14)$$

Thus, we get

$$\left(1 - \frac{p}{\lambda} - s\right)d_b(w^*, u^*) \leq 0. \quad (15)$$

Since $(p/1 - s) < \lambda$, we have $1 - (p/\lambda) - s > 0$ and thus we obtain $d_b(u^*, w^*) = 0$; hence, $u^* = w^*$.

Furthermore, we have to show that u^* is a common unique fixed point of f and g .

From (1) and (7), we have

$$\begin{aligned} d_b(u^*, fu^*) &= d_b(fx^*, fu^*) \leq \\ &\leq \frac{1}{b(x^*, u^*)} \left(\frac{pd_b(gx^*, gu^*) + qd_b(gx^*, fx^*) + rd_b(gu^*, fu^*) + s\beta d_b(fx^*, gu^*)d_b(fu^*, gx^*)}{\lambda b(fx^*, gx^*) + \beta(d_b(fx^*, gu^*) + d_b(fu^*, gx^*))} \right) \\ &\leq \frac{pd_b(gx^*, gu^*) + qd_b(gx^*, fx^*) + rd_b(gu^*, fu^*) + s\beta d_b(fx^*, gu^*)d_b(fu^*, gx^*)}{\lambda b(fx^*, gx^*) + \beta(d_b(fx^*, gu^*) + d_b(fu^*, gx^*))}. \end{aligned} \quad (16)$$

Since f and g are weakly compatible and $fx^* = gx^* = u^*$, so we have $gfx^* = fgx^* = gu^* = fu^*$. Thus, from (16) we have

$$\begin{aligned} d_b(u^*, fu^*) &= d_b(fx^*, fu^*) \\ &\leq \frac{pd_b(u^*, gu^*) + qd_b(u^*, u^*) + rd_b(gu^*, fu^*) + s\beta d_b(u^*, gu^*)d_b(fu^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, gu^*) + d_b(fu^*, u^*))} \\ &= \frac{pd_b(u^*, fu^*) + qd_b(u^*, u^*) + rd_b(fu^*, fu^*) + s\beta d_b(u^*, fu^*)d_b(fu^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, gu^*) + d_b(fu^*, u^*))} \\ &= \frac{pd_b(u^*, fu^*) + s\beta d_b(u^*, fu^*)d_b(fu^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, gu^*) + d_b(fu^*, u^*))} \\ &\leq \frac{pd_b(u^*, fu^*)}{\lambda b(u^*, u^*)} + sd_b(u^*, fu^*) \leq \frac{pd_b(u^*, fu^*)}{\lambda} + sd_b(u^*, fu^*). \end{aligned} \quad (17)$$

Thus, we get

$$\left(1 - \frac{p}{\lambda} - s\right)d_b(u^*, fu^*) \leq 0. \quad (18)$$

Since $(p/1 - s) < \lambda$, we have $1 - (p/\lambda) - s > 0$ and thus we obtain $d_b(u^*, fu^*) = 0$; hence, $fu^* = u^*$. This implies that $u^* = fu^* = gu^*$.

Hence, u^* is a common fixed point of f and g .

Uniqueness of the common fixed point of f and g are shown as follows.

Suppose w^* is another common fixed point of f and g ,

$$\begin{aligned} d_b(u^*, w^*) &= d_b(fu^*, fw^*) \\ &\leq \frac{1}{b(u^*, w^*)} \left(\frac{pd_b(gu^*, gw^*) + bq(gu^*, fu^*) + rd_b(gw^*, fw^*) + s\beta d_b(fu^*, gw^*)d_b(fw^*, gu^*)}{\lambda b(fu^*, gu^*) + \beta(d_b(fu^*, gw^*) + d_b(fw^*, gu^*))} \right) \\ &\leq \left(\frac{pd_b(u^*, w^*) + bq(u^*, u^*) + rd_b(w^*, w^*) + s\beta d_b(u^*, w^*)d_b(w^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, w^*) + d_b(w^*, u^*))} \right) \\ &= \left(\frac{pd_b(u^*, w^*) + s\beta d_b(u^*, w^*)d_b(w^*, u^*)}{\lambda b(u^*, u^*) + \beta(d_b(u^*, w^*) + d_b(w^*, u^*))} \right) \\ &\leq \frac{pd_b(u^*, w^*)}{\lambda} + sd_b(u^*, w^*). \end{aligned} \quad (19)$$

Thus, we get

$$\left(1 - \frac{p}{\lambda} - s\right)d_b(u^*, w^*) \leq 0. \quad (20)$$

Since $(p/1 - s) < \lambda$, we have $1 - (p/\lambda) - s > 0$ and thus we obtain $d_b(u^*, w^*) = 0$; hence, $u^* = w^*$.

Hence, f and g have a unique common fixed point in X .

By taking $gx = x$ for all $x \in X$ in Theorem 1, we obtain the following. $\square$

Corollary 1. Let (X, d_b) be a complete extended rectangular b -metric space, and let $f : X \rightarrow X$ be self-mappings such that

$$b(x, y)d_b(fx, fy) \leq \frac{pd_b(gx, y) + qd_b(x, fx) + r(y, fy) + s\beta d_b(fx, y)d_b(fy, x)}{\lambda b(fx, x) + \beta(d_b(fx, y) + d_b(fy, x))}, \quad (21)$$

holds for all $x, y \in X$, $\beta > 0$, $0 < s < 1$, $0 < p + q + r < \lambda$, $(p/1 - s) < \lambda$, and $\lim_{n, m \rightarrow \infty} b(y_n, y_m) < (\lambda - r/p + q)$, then f has a unique fixed point in X .

Example 3. Let $A = [(2/3), 1]$ and $B = [0, (1/2)]$. Let $X = A \cup B$, and define $b(x, y) = 2(x + y) + 1$ for all $x, y \in X$, $\lambda = .9$, and define $d_b : X \times X \rightarrow [0, \infty)$ as follows:

$$d_b(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \frac{1}{2}, & \text{if } x, y \in A, \\ 1, & \text{if } x, y \in B, \\ (x - y)^2, & \text{otherwise.} \end{cases} \quad (22)$$

It is easy that (X, d_b) is an extended rectangular space with $b(x, y) = 0.2(x + y) + 1$.

Let $f(x) = x^4/4$ and $g(x) = x^2/2$, for all $x \in X$. It is clear that $fX \subseteq gX$.

By using $y_n = f(x_n) = g(x_{n+1})$, $f(x) = x^4/2^4$, and $g(x) = x^2/2^2$, we have

$$y_0 = f(x_0) = \frac{x^4}{2^4} = g(x_1) \text{ thus } x_1 = \frac{x^2}{2},$$

$$y_1 = f(x_1) = f\left(\frac{x^2}{2}\right) = \frac{x^8}{2^8} = g(x_2) \text{ thus } x_2 = \frac{x^4}{2^3}, \quad (23)$$

$$y_2 = f(x_2) = f\left(\frac{x^4}{2^3}\right) = \frac{x^{16}}{2^{16}} = g(x_3) \text{ thus } x_3 = \frac{x^8}{2^7}.$$

Generally, we have

$$y_n = \frac{x^{2^{n+2}}}{2^{n+2}},$$

$$x_n = \frac{x^{2^n}}{2^{2^n-1}},$$

$$\lim_{n, m \rightarrow \infty} b(y_n, y_m) = \lim_{n, m \rightarrow \infty} 2 + y_n + y_m$$

$$= \lim_{n, m \rightarrow \infty} 1 + \frac{x^{2^{n+2}}}{2^{n+2}} + \frac{x^{2^{m+2}}}{2^{m+2}}$$

$$= 1 < \frac{\lambda - r}{p + q} = \frac{0.9 - 0.4}{0.1 + 0.3} = \frac{0.5}{0.4} = 1.25.$$

(24)

Since $x \in [0, 1]$, the sequence $\{y_n\} = \{x^{2^{n+2}}/2^{n+2}\}$ is convergent to 0. Indeed, $\lim_{n \rightarrow \infty} d_b(x_n, 0) = 0$.

- (1) For $x, y \in A = [(2/3), 1]$ or $x, y \in B = [0, (1/2)]$, it is clear that $b(x, y)d_b(fx, fy) = 0$
 (2) For $x \in A = [(2/3), 1]$ and $y \in B = [0, (1/2)]$

$$\begin{aligned}
 &\text{For } x \in A = \left[\frac{2}{3}, 1\right] \text{ then } f(x) = \frac{x^4}{4} \in \left[\frac{16}{324}, \frac{1}{4}\right] \subseteq B = \left[0, \frac{1}{2}\right], \\
 &\text{For } x \in A = \left[\frac{2}{3}, 1\right] \text{ then } g(x) = \frac{x^2}{2} \in \left[\frac{4}{18}, \frac{1}{2}\right] \subseteq B = \left[0, \frac{1}{2}\right], \\
 &\text{For } y \in B = \left[0, \frac{1}{2}\right] \text{ then } f(y) = \frac{y^4}{4} \in \left[0, \frac{1}{32}\right] \subseteq B = \left[0, \frac{1}{2}\right], \\
 &\text{For } y \in B = \left[0, \frac{1}{2}\right] \text{ then } g(y) = \frac{y^2}{2} \in \left[0, \frac{1}{4}\right] \subseteq B = \left[0, \frac{1}{2}\right], \\
 &\frac{pd_b(gx, gy) + qd_b(gx, fx) + r(gy, fy) + s\beta d_b(fx, gy)d_b(fy, gx)}{\lambda b(fx, gx) + \beta(d_b(fx, gy) + d_b(fy, gx))} \\
 &= \frac{0.1 d_b(x^2/2, y^2/2) + 0.3 d_b(x^2/2, x^4/4) + 0.4(y^2/2, y^4/4) + 0.5(0.1)d_b(x^4/4, y^2/2)d_b(y^4/4, x^2/2)}{0.9b(x^4/4, x^2/2) + 0.1(d_b(x^4/4, y^2/2) + d_b(y^4/4, x^2/2))} \\
 &= \frac{0.1 + 0.3 + 0.4 + 0.5(0.1)}{0.9(0.2)(x^4/4, x^2/2 + 1) + 0.1(1 + 1)} = \frac{0.85}{0.18(x^4/4, x^2/2 + 1) + 0.2} \\
 &b(x, y)d_b(fx, fy) = (0.2(x^4/4, x^2/2) + 1)d_b(x^4/4, y^4/4) = (0.2(x^4/4, x^2/2) + 1) \leq \frac{0.85}{0.18(x^4/4, x^2/2 + 1) + 0.1}.
 \end{aligned} \tag{25}$$

The largest value of $b(x, y)d_b(fx, fy)$ occurs if $x = 1$ and $y = 1/2$. So, we have

$$\begin{aligned}
 b(x, y)d_b(fx, fy) &= \left(0.2\left(\frac{x^4}{4} + \frac{x^2}{2}\right) + 1\right) = \left(0.2\left(\frac{1}{4} + \frac{1}{8}\right) + 1\right) = 1.275 \\
 &\leq \frac{pd_b(gx, gy) + qd_b(gx, fx) + r(gy, fy) + s\beta d_b(fx, gy)d_b(fy, gx)}{\lambda b(fx, gx) + \beta(d_b(fx, gy) + d_b(fy, gx))} \\
 &\leq \frac{0.85}{0.18(11/8) + 0.2} = 1.899.
 \end{aligned} \tag{27}$$

- (3) For $x \in B = [0, (1/2)]$ and $y \in A = [(2/3), 1]$,

$$\begin{aligned}
 &\text{For } x \in B = \left[0, \frac{1}{2}\right] \text{ then } f(x) = \frac{x^4}{4} \in \left[0, \frac{1}{32}\right] \subseteq B = \left[0, \frac{1}{2}\right], \\
 &\text{For } x \in B = \left[0, \frac{1}{2}\right] \text{ then } g(x) = \frac{x^2}{2} \in \left[\frac{0}{8}, \frac{1}{2}\right] \subseteq B = \left[0, \frac{1}{2}\right], \\
 &\text{For } y \in A = \left[\frac{2}{3}, 1\right] \text{ then } f(y) = \frac{y^4}{4} \in \left[\frac{16}{324}, \frac{1}{4}\right] \subseteq B = \left[0, \frac{1}{2}\right], \\
 &\text{For } y \in A = \left[\frac{2}{3}, 1\right] \text{ then } g(y) = \frac{y^2}{2} \in \left[\frac{4}{18}, \frac{1}{2}\right] \subseteq B = \left[0, \frac{1}{2}\right],
 \end{aligned} \tag{28}$$

$$\begin{aligned}
& \frac{pd_b(gx, gy) + qd_b(gx, fx) + r(gy, fy) + s\beta d_b(fx, gy)d_b(fy, gx)}{\lambda b(fx, gx) + \beta(d_b(fx, gy) + d_b(fy, gx))} \\
&= \frac{0.1 d_b(x^2/2, y^2/2) + 0.3 d_b(x^2/2, x^4/4) + 0.4(y^2/2, y^4/4) + 0.5(0.1)d_b(x^4/4, y^2/2)d_b(y^4/4, x^2/2)}{0.9 b(x^4/4, x^2/2) + 0.1(d_b(x^4/4, y^2/2) + d_b(y^4/4, x^2/2))} \\
&= \frac{0.1 + 0.3 + 0.4 + 0.5(0.1)}{0.9(0.2)(x^4/4 + x^2/2 + 1) + 0.1(1 + 1)} = \frac{0.85}{0.18(x^4/4 + x^2/2 + 1) + 0.2}, \\
& b(x, y)d_b(fx, fy) = \left(0.2\left(\frac{x^4}{4} + \frac{x^2}{2}\right) + 1\right) d_b\left(\frac{x^4}{4}, \frac{y^4}{4}\right) = \left(0.2\left(\frac{x^4}{4} + \frac{x^2}{2}\right) + 1\right) \leq \frac{0.85}{0.18(x^4/4 + x^2/2 + 1) + 0.2}.
\end{aligned} \tag{29}$$

The largest value of $b(x, y)d_b(fx, fy)$ occurs if $x = 1/2$ and $y = 1$. So, we have

$$\begin{aligned}
b(x, y)d_b(fx, fy) &= \left(0.2\left(\frac{x^4}{4} + \frac{x^2}{2}\right) + 1\right) d_b\left(\frac{x^4}{4}, \frac{y^4}{4}\right) = \left(0.2\left(\frac{1}{4} + \frac{1}{8}\right) + 1\right) = 1.275 \\
&\leq \frac{pd_b(gx, gy) + qd_b(gx, fx) + r(gy, fy) + s\beta d_b(fx, gy)d_b(fy, gx)}{\lambda b(fx, gx) + \beta(d_b(fx, gy) + d_b(fy, gx))} \\
&\leq \frac{0.85}{0.18(11/8) + 0.2} = 1.899.
\end{aligned} \tag{30}$$

Hence, we get that condition

$$b(x, y)d_b(fx, fy) \leq \frac{pd_b(gx, gy) + qd_b(gx, fx) + r(gy, fy) + s\beta d_b(fx, gy)d_b(fy, gx)}{\lambda b(fx, gx) + \beta(d_b(fx, gy) + d_b(fy, gx))}, \tag{31}$$

is hold.

Hence, from Theorem 1, we have $x = 0$ is a unique common fixed point of f and g .

Theorem 2. Let (X, d_b) be a complete extended rectangular b -metric space, and let $f, g : X \rightarrow X$ be self-mappings such that $f(X) \subseteq g(X)$ and $g(X)$ closed. If the conditions

$$\begin{aligned}
b(x, y)d_b(fx, fy) &\leq pd_b(gx, gy) + \frac{q}{b(fx, gx)} d_b(gx, fx) \\
&+ rd_b(gy, fy) \\
&+ \frac{sd_b(fx, gy)d_b(fy, gx)}{1 + d_b(fx, gy) + d_b(fy, gx)},
\end{aligned} \tag{32}$$

hold for all $x, y \in X$, where $0 < q, r < 1$, $0 < p + s < 1$, $q, r < 1$, $0 < p + s < 1$, $0 < (p + q/1 - r) < 1$, and $\lim_{n, m \rightarrow \infty} b(x_n, y_m) < (1 - r/p + q)$, and then f and g have a unique coincidence in X .

Moreover, if f and g have weakly compatible property, then f and g have a unique fixed point.

Proof. Let taking $x_0 \in X$ as initial point define sequence $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = f(x_n) = g(x_{n+1})$, $n = 0, 1, 2, \dots$. Suppose that $y_n \neq y_{n+1}$, $n = 0, 1, 2, \dots$, because if $y_n = y_{n+1}$ for some n , then $g(x_{n+1}) = y_n = y_{n+1} = f(x_{n+1})$, so f and g have a coincidence point.

Consider from (1) and (32), we have

$$\begin{aligned}
b(x_n, x_{n+1})d_b(y_n, y_{n+1}) &= b(x_n, x_{n+1})d_b(fx_n, fx_{n+1}) \\
&\leq pd_b(gx_n, gx_{n+1}) + \frac{q}{b(fx_n, gx_n)}d_b(gx_n, fx_n) + rd_b(gx_{n+1}, fx_{n+1}) \\
&\quad + \frac{sd_b(fx_n, gx_{n+1})d_b(fx_{n+1}, gx_n)}{1 + d_b(fx_n, gx_{n+1}) + d_b(fx_{n+1}, gx_n)} \\
&= pd_b(y_{n-1}, y_n) + \frac{q}{b(y_n, y_{n-1})}d_b(y_{n-1}, y_n) + rd_b(y_n, y_{n+1}) + \frac{sd_b(y_n, y_n)d_b(y_{n+1}, y_{n-1})}{1 + d_b(y_n, y_n) + d_b(y_{n+1}, y_{n-1})} \\
&\leq (p+q)d_b(y_{n-1}, y_n) + rd_b(y_n, y_{n+1}).
\end{aligned} \tag{33}$$

Since $b(x_n, x_{n+1}) \geq 1$, we get

$$\begin{aligned}
d_b(y_n, y_{n+1}) &\leq \frac{p+q}{b(x_n, x_{n+1}) - r}d_b(y_{n-1}, y_n) \\
&\leq \frac{p+q}{1-r}d_b(y_{n-1}, y_n).
\end{aligned} \tag{34}$$

So, we get

$$d_b(y_n, y_{n+1}) \leq \beta d_b(y_{n-1}, y_n). \tag{35}$$

where $\beta = p+q/1-r$. Since as known $0 < \beta = (p+q/1-r) < 1$ and $\lim_{n,m \rightarrow \infty} b(y_n, y_m) < (1/\beta)$, then by using Lemma 2, we get $\{y_n\}$ is a Cauchy sequence in gX .

Since X is complete, then there exists $u^* \in X$ such that

$$d_b(y_n, u^*) \longrightarrow 0. \tag{36}$$

Since $y_n = g(x_{n+1}) \in X$ and gX closed, then $u^* \in gX$ and $u^* = g(x^*)$ for some $x^* \in X$.

Hence, for $n, m \rightarrow \infty$, in (16) we get $d_b(y_n, y_m) \rightarrow 0$; thus, $\{y_n\}$ is the Cauchy sequence in gX .

Furthermore, we will show that f and g have a point coincidence, as follows.

From (1) and (32), we have

$$\begin{aligned}
d_b(fx^*, gx^*) &\leq b(fx^*, gx^*)[d_b(fx^*, y_n) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, gx^*)] \\
&= b(x^*, x^*)[d_b(fx^*, fx_n) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*)] \\
&\leq b(fx^*, gx^*) \left[\frac{1}{b(x^*, x_n)} \left(pd_b(gx^*, gx_n) + \frac{q}{b(fx^*, gx^*)}d_b(gx^*, fx^*) + rd_b(gx_n, fx_n) \right. \right. \\
&\quad \left. \left. + \frac{sd_b(fx^*, gx_n)d_b(fx_n, gx^*)}{1 + d_b(fx^*, gx_n) + d_b(fx_n, gx^*)} \right) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*) \right] \\
&\leq b(fx^*, gx^*) \left[\frac{1}{b(x^*, x_n)} \left(pd_b(u^*, y_{n-1}) + \frac{q}{b(fx^*, gx^*)}d_b(u^*, fx^*) + rd_b(y_{n-1}, y_n) \right. \right. \\
&\quad \left. \left. + \frac{sd_b(fx^*, y_{n-1})d_b(y_n, u^*)}{1 + d_b(fx^*, y_{n-1}) + d_b(y_n, u^*)} \right) + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*) \right] \\
&\leq b(fx^*, gx^*) \left[\frac{1}{b(x^*, x_n)} \left(pd_b(u^*, y_{n-1}) + \frac{q}{b(fx^*, gx^*)}d_b(u^*, fx^*) + rd_b(y_{n-1}, y_n) + sd_b(y_n, u^*) \right) \right. \\
&\quad \left. + d_b(y_n, y_{n+1}) + d_b(y_{n+1}, u^*) \right].
\end{aligned} \tag{37}$$

By using (36) and (37), if $n \rightarrow \infty$, then we get

$$\begin{aligned}
d_b(fx^*, gx^*) &\leq \frac{qd_b(u^*, fx^*)}{b(x^*, x_n)} = \frac{qd_b(gx^*, fx^*)}{b(x^*, x_n)} \\
&\leq qd_b(gx^*, fx^*).
\end{aligned} \tag{38}$$

Since $0 < q < 1$, this implies that $d_b(fx^*, gx^*) = 0$, so then we get $fx^* = gx^* = u^*$. Hence, u^* is a coincidence point of f and g .

To show uniqueness of the coincidence point of f and g , we suppose there exists $w^* \in X$ such that $fy^* = gy^* = w^*$:

$$\begin{aligned}
d_b(u^*, w^*) &= d_b(fx^*, fy^*) \\
&\leq \frac{1}{b(x^*, y^*)} \left[pd_b(gx^*, gy^*) + \frac{q}{b(fx^*, gx^*)} d_b(gx^*, fx^*) + rd_b(gy^*, fy^*) + \frac{sd_b(fx^*, gy^*)d_b(fy^*, gx^*)}{1 + d_b(fx^*, gy^*) + d_b(fy^*, gx^*)} \right] \\
&\leq pd_b(u^*, w^*) + \frac{q}{b(u^*, u^*)} d_b(u^*, u^*) + rd_b(w^*, w^*) + \frac{sd_b(u^*, w^*)d_b(w^*, u^*)}{1 + d_b(u^*, w^*) + d_b(w^*, u^*)} \\
&\leq pd_b(u^*, w^*) + \frac{q}{b(u^*, u^*)} d_b(u^*, u^*) + rd_b(w^*, w^*) + sd_b(u^*, w^*) \\
&\leq pd_b(u^*, w^*) + sd_b(u^*, w^*) = (p + s)d_b(u^*, w^*).
\end{aligned} \tag{39}$$

Thus, we get

$$(1 - p - s)d_b(u^*, w^*) \leq 0. \tag{40}$$

Since $0 < p + s < 1$, then we get $d_b(u^*, w^*) = 0$; hence, $u^* = w^*$.

Next, we have to show that u^* is a common fixed point of f and g . Consider that

$$\begin{aligned}
d_b(u^*, fu^*) &= d_b(fx^*, fu^*) \\
&\leq \frac{1}{b(x^*, u^*)} \left[pd_b(gx^*, gu^*) + \frac{q}{b(fx^*, gx^*)} d_b(gx^*, fx^*) + rd_b(gu^*, fu^*) + \frac{sd_b(fx^*, gu^*)d_b(fu^*, gx^*)}{1 + d_b(fx^*, gu^*) + d_b(fu^*, gx^*)} \right].
\end{aligned} \tag{41}$$

Since f and g are weakly compatible and $fx^* = gx^* = u^*$, we have $gfx^* = fgx^* = gu^* = fu^*$. Thus, from (41) we get

$$\begin{aligned}
d_b(u^*, fu^*) &= d_b(fx^*, fu^*) \\
&\leq \frac{1}{b(x^*, u^*)} \left[pd_b(gx^*, gu^*) + \frac{q}{b(fx^*, gx^*)} d_b(gx^*, fx^*) + rd_b(gu^*, fu^*) + \frac{sd_b(fx^*, gu^*)d_b(fu^*, gx^*)}{1 + d_b(fx^*, gu^*) + d_b(fu^*, gx^*)} \right] \\
&\leq \frac{1}{b(x^*, u^*)} \left[pd_b(u^*, fu^*) + \frac{q}{b(u^*, u^*)} d_b(u^*, u^*) + rd_b(fu^*, fu^*) + \frac{sd_b(u^*, fu^*)d_b(fu^*, u^*)}{1 + d_b(u^*, fu^*) + d_b(fu^*, u^*)} \right] \\
&\leq \frac{p}{b(x^*, u^*)} d_b(u^*, fu^*) + sd_b(u^*, fu^*) \leq pd_b(u^*, fu^*) + sd_b(u^*, fu^*) \\
&\leq (p + s)d_b(u^*, fu^*).
\end{aligned} \tag{42}$$

Thus, we obtain $(1 - p - s)d_b(u^*, fu^*) \leq 0$.

Since $p + s < 1$, we obtain that $d_b(u^*, fu^*) = 0$. This implies that $u^* = fu^* = gu^*$.

Hence, u^* is a common fixed point of f and g .

Uniqueness of the common fixed point of f and g are shown as follows.

Suppose w^* is another common fixed point of f and g , that is, $w^* = fw^* = gw^*$.

$$\begin{aligned}
d_b(u^*, w^*) &= d_b(fu^*, fw^*) \\
&\leq \frac{1}{b(u^*, w^*)} \left[pd_b(gu^*, gw^*) + \frac{q}{b(fu^*, gu^*)} d_b(gu^*, fu^*) + rd_b(gw^*, fw^*) + \frac{sd_b(fu^*, gw^*)d_b(fw^*, fu^*)}{1 + d_b(fu^*, gw^*) + d_b(fw^*, fu^*)} \right] \\
&\leq \frac{1}{b(u^*, w^*)} \left[pd_b(u^*, w^*) + \frac{q}{b(u^*, u^*)} d_b(u^*, u^*) + rd_b(w^*, w^*) + \frac{sd_b(u^*, w^*)d_b(w^*, u^*)}{1 + d_b(u^*, w^*) + d_b(w^*, u^*)} \right] \\
&= \frac{pd_b(u^*, w^*)}{b(u^*, w^*)} + sd_b(u^*, w^*) \leq (p + s)d_b(u^*, w^*).
\end{aligned} \tag{43}$$

Since $0 < p + s < 1$, we obtain that $d_b(u^*, w^*) = 0$, and it implies that $u^* = w^*$.

Hence, f and g have a unique common fixed point in X .

By taking $gx = x$ in Theorem 2 we obtain $\square$

Corollary 2. Let (X, d_b) be a complete extended rectangular b -metric space, and let $f : X \rightarrow X$ be a self-mapping such that

$$\begin{aligned}
b(x, y)d_b(fx, fy) &\leq pd_b(x, y) + \frac{q}{b(fx, x)}d_b(x, fx) \\
&\quad + rd_b(y, fy) \\
&\quad + \frac{sd_b(fx, y)d_b(fy, x)}{1 + d_b(fx, y) + d_b(fy, x)},
\end{aligned} \tag{44}$$

holds for all $x, y \in X$, where $0 < q, r < 1$, $0 < p + s < 1$, $q, r < 1$, $0 < p + s < 1$, $0 < (p + q/1 - r) < 1$ and $\lim_{n, m \rightarrow \infty} b(x_n, y_m) < (1 - r/p + q)$, and then f has a unique fixed point in X .

Example 4. We use $f(x) = x^4/4$, $g(x) = x^2/2$ and $d_b(x, y)$ in Example 3, and taking $b(x, y) = 0.1(x + y) + 1$, $p = 0.2$, $q = 0.3$, $r = 0.45$, and $s = 0.7$. These parameters satisfy for conditions

$$\begin{aligned}
0 < r < 1, \quad 0 < p + s < 1, \quad 0 < \frac{p + q}{1 - r} < 1, \\
\lim_{n, m \rightarrow \infty} b(x_n, y_m) < \frac{1 - r}{p + q}.
\end{aligned} \tag{45}$$

From Example 3 we have

$$\begin{aligned}
y_n &= \frac{x^{2^{n+2}}}{2^{n+2}}, \\
x_n &= \frac{x^{2^n}}{2^{2^n - 1}}.
\end{aligned} \tag{46}$$

Thus, we have

$$\begin{aligned}
\lim_{n, m \rightarrow \infty} b(y_n, y_m) &= \lim_{n, m \rightarrow \infty} 0.1(y_n + y_m) + 1 \\
&= \lim_{n, m \rightarrow \infty} 0.1 \left(\frac{x^{2^{n+2}}}{2^{n+2}} + \frac{x^{2^{m+2}}}{2^{m+2}} \right) + 1 \\
&= 1 < \frac{1 - r}{p + q} = \frac{1 - 0.4}{0.2 + 0.3} = \frac{0.6}{0.5} = 1.2.
\end{aligned} \tag{47}$$

Since $0 \leq x \leq 1$, the sequence $\{y_n\} = \{x^{2^{n+2}}/2^{n+2}\}$ is convergent to 0. Indeed, $\lim_{n \rightarrow \infty} d_b(y_n, 0) = 0$.

(1) For $x, y \in A = [(2/3), 1]$ or $x, y \in B = [0, (1/2)]$, it is clear that $b(x, y)d_b(fx, fy) = 0$

(2) For $x \in A = [(2/3), 1]$ and $y \in B = [0, (1/2)]$

$$\begin{aligned}
\text{For } x \in A = \left[\frac{2}{3}, 1 \right] \text{ then } f(x) &= \frac{x^4}{4} \in \left[\frac{16}{324}, \frac{1}{4} \right] \subseteq B = \left[0, \frac{1}{2} \right], \\
\text{For } x \in A = \left[\frac{2}{3}, 1 \right] \text{ then } g(x) &= \frac{x^2}{2} \in \left[\frac{4}{18}, \frac{1}{2} \right] \subseteq B = \left[0, \frac{1}{2} \right], \\
\text{For } y \in B = \left[0, \frac{1}{2} \right] \text{ then } f(y) &= \frac{y^4}{4} \in \left[0, \frac{1}{32} \right] \subseteq B = \left[0, \frac{1}{2} \right], \\
\text{For } y \in B = \left[0, \frac{1}{2} \right] \text{ then } g(y) &= \frac{y^2}{2} \in \left[0, \frac{1}{4} \right] \subseteq B = \left[0, \frac{1}{2} \right].
\end{aligned} \tag{48}$$

$$\begin{aligned}
& b(x, y)d_b(fx, fy) \\
& \leq 0.2d_b(gx, gy) + \frac{0.3}{b(fx, gx)}d_b(gx, fx) + 0.45d_b(gy, fy) + \frac{0.7d_b(fx, gy)d_b(fy, gx)}{1 + d_b(fx, gy) + d_b(fy, gx)} \\
& = 0.2d_b\left(\frac{x^2}{2}, \frac{y^4}{4}\right) + \frac{0.3}{b(x^4/4, x^2/2)}d_b(x^2/2, x^4/4) + 0.45d_b\left(\frac{y^2}{2}, \frac{y^4}{4}\right) + \frac{0.7d_b(y^4/4, y^2/2)d_b(y^4/4, x^2/2)}{1 + d_b(y^4/4, y^2/2) + d_b(y^4/4, x^2/2)} \\
& = 0.2 + \frac{0.3}{0.2(x^4/4 + x^2/2) + 1} + 0.45 + \frac{0.7}{1 + 1 + 1} = 0.65 + \frac{0.3}{0.2(x^4/4 + x^2/2) + 1} + \frac{0.7}{3} \\
& b(x, y)d_b(fx, fy) = (0.2(x + y) + 1)d_b\left(\frac{x^4}{4}, \frac{y^4}{4}\right) = 0.2(x + y) + 1.
\end{aligned} \tag{49}$$

The largest value of $b(x, y)d_b(fx, fy)$ occurs if $x = 1$ and $y = 1/2$. So, we have

$$\begin{aligned}
b(x, y)d_b(fx, fy) &= (0.1(x + y) + 1)d_b\left(\frac{x^4}{4}, \frac{y^4}{4}\right) \\
&= 0.1(x + y) + 1 = 0.1\left(1 + \frac{1}{2}\right) + 1 = 1.1.
\end{aligned} \tag{50}$$

And we have

$$\begin{aligned}
& 2d_b(gx, gy) + \frac{0.3}{b(fx, gx)}d_b(gx, fx) \\
& + 0.45d_b(gy, fy) + \frac{0.7d_b(fx, gy)d_b(fy, gx)}{1 + d_b(fx, gy) + d_b(fy, gx)} \\
& = 0.65 + \frac{0.3}{0.1((x^4/4) + (x^2/2)) + 1} \\
& + \frac{0.7}{3} = 0.65 + \frac{0.3}{0.1((1/4) + (1/2)) + 1} + \frac{0.7}{3} = 1.17.
\end{aligned} \tag{51}$$

So, we get that

$$\begin{aligned}
b(x, y)d_b(fx, fy) &\leq 2d_b(gx, gy) + \frac{0.3}{b(fx, gx)}d_b(gx, fx) \\
& + 0.45d_b(gy, fy) \\
& + \frac{0.7d_b(fx, gy)d_b(fy, gx)}{1 + d_b(fx, gy) + d_b(fy, gx)}.
\end{aligned} \tag{52}$$

(3) For $x \in B = [0, (1/2)]$ and $y \in A = [(2/3), 1]$

Similarly, we also obtain that

$$\begin{aligned}
b(x, y)d_b(fx, fy) &\leq 2d_b(gx, gy) + \frac{0.3}{b(fx, gx)}d_b(gx, fx) \\
& \cdot (gx, fx) + 0.45d_b(gy, fy) \\
& + \frac{0.7d_b(fx, gy)d_b(fy, gx)}{1 + d_b(fx, gy) + d_b(fy, gx)}.
\end{aligned} \tag{53}$$

Hence, we get that the condition

$$\begin{aligned}
b(x, y)d_b(fx, fy) &\leq pd_b(x, y) + \frac{q}{b(fx, x)}d_b(x, fx) \\
& + rd_b(y, fy), \\
& + \frac{sd_b(fx, y)d_b(fy, x)}{1 + d_b(fx, y) + d_b(fy, x)}.
\end{aligned} \tag{54}$$

for all $x, y \in X$ holds.

Hence, from Theorem 2, we can conclude that $x = 0$ is a unique common fixed point of f and g .

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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References

- [1] S. Czerwik, "Contraction mappings in b -metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.
- [2] S. K. Mohanta, "Coincidence points and common fixed points for expansive type mappings in b -metric spaces," *Iranian Journal of Mathematical Sciences and Informatics*, vol. 11, pp. 101–113, 2016.
- [3] N. Hussain, V. Parvaneh, B. A. S. Alamri, and Z. Kadelburg, "F-HR-type contractions on (α, η) -complete rectangular b -metric spaces," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 3, pp. 1030–1043, 2017.
- [4] T. Kamran, M. Samreen, and Q. Ul Ain, "A generalization of b -metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, p. 19, 2017.
- [5] B. Alqahtani, A. Fulga, and E. Karapinar, "Common fixed point results on an extended b -metric space," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 158, 2018.

- [6] R. George, S. Radenović, K. P. Reshma, and S. Shukla, "Rectangular b -metric space and contraction principles," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1005–1013, 2015.
- [7] Z. Kadelburg and S. Radenović, "Pata-type common fixed point results in b -metric and b -rectangular metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 944–954, 2015.
- [8] Z. D. Mitrović, "A note on a Banach's fixed point theorem in b -rectangular metric space and b -metric space," *Mathematica Slovaca*, vol. 68, no. 5, pp. 1113–1116, 2018.
- [9] M. Asim, M. Imdad, and S. Radenovic, "Fixed point results in extended rectangular b -metric spaces with an application," *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics*, vol. 81, no. 2, pp. 43–50, 2019.
- [10] Z. Mustafa, V. Parvaneh, M. M. M. Jaradat, and Z. Kadelburg, "Extended rectangular b -metric spaces and some fixed point theorems for contractive mappings," *Symmetry*, vol. 11, no. 4, p. 594, 2019.
- [11] N. V. Dung, "The metrization of rectangular b -metric spaces," *Topology and its Applications*, vol. 261, pp. 22–28, 2019.
- [12] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.
- [13] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.