Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2019, Article ID 5731830, 6 pages
https://doi.org/10.1155/2019/5731830

Research Article

Directional Convexity of Convolutions of Harmonic Functions

Jay M. Jahangiri [1] **and Raj Kumar Garg** [2]

[1] *Mathematical Sciences, Kent State University, Burton, Ohio 44021-9500, USA*
[2] *Department of Mathematics, DAV University, Jalandhar 144012 (Punjab), India*

Correspondence should be addressed to Jay M. Jahangiri; jjahangi@kent.edu

Received 16 January 2019; Accepted 12 March 2019; Published 1 April 2019

Academic Editor: Teodor Bulboaca

Harmonic functions can be constructed using two analytic functions acting as their analytic and coanalytic parts but the prediction of the behavior of convolution of harmonic functions, unlike the convolution of analytic functions, proved to be challenging. In this paper we use the shear construction of harmonic mappings and introduce dilatation conditions that guarantee the convolution of two harmonic functions to be harmonic and convex in the direction of imaginary axis.

1. Introduction

For u and v real harmonic in the open unit disk $\mathbb{D} \coloneqq \{z : |z| < 1\}$, the continuous complex-valued harmonic function $f = u + iv$ can be expressed as $f = h + \overline{g}$, where h and g are analytic in $\mathbb{E}$. We call h the analytic part and g the coanalytic part of the harmonic function $f = h + \overline{g}$. By a result of Lewy [1] (see [2] or [3]), a necessary and sufficient condition for a harmonic function $f = h + \overline{g}$ to be locally one-to-one and sense-preserving in $\mathbb{D}$ is that its Jacobian $J_f = |h'|^2 - |g'|^2$ is positive in $\mathbb{D}$ or equivalently, if and only if $h'(z) \neq 0$ in $\mathbb{D}$ and the second complex dilatation ω of f satisfies $|\omega| = |g'/h'| < 1$ in $\mathbb{D}$. A simply connected domain $\mathscr{D} \subset \mathbb{C}$ is said to be *convex in the direction* θ, $0 \leq \theta < \pi$ if every line parallel to the line through 0 and $e^{i\theta}$ either misses $\mathscr{D}$, or is contained in $\mathscr{D}$, or its intersection with $\mathscr{D}$ is either a line-segment or a ray. For the open unit disk $\mathbb{D}$, an analytic or harmonic function $f : \mathbb{D} \longrightarrow \mathbb{C}$ is said to be convex in the direction θ if $f(\mathbb{D})$ is convex in the direction θ there. We note that if a mapping is convex in every direction, then it is simply called a convex mapping.

We let $\mathcal{S}_{\mathscr{H}}$ be the class of locally one-to-one and sense-preserving complex-valued harmonic univalent functions $f = h + \overline{g}$ for which $f(0) = f_{\overline{z}}(0) = f_z(0) - 1 = 0$ and $g'(0) = h(0) = h'(0) - 1 = 0$. We also let $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$ be the convolution of two harmonic functions $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$, where the operator $*$ stands for the Hadamard product or convolution of two Taylor power series. Even though the harmonic functions can be constructed using two analytic functions acting as their analytic and coanalytic parts, the prediction of the behavior of convolution of harmonic functions, unlike the convolution of analytic functions, proved to be challenging. In a striking result (see the following Lemma 1), Clunie and Sheil-Small [2] introduced a method of constructing harmonic mappings known as the *shear construction* that produces harmonic functions with a specific dilatation onto a domain convex in one direction.

Lemma 1. *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{E}$ onto a domain convex in the direction $\theta, 0 \leq \theta < \pi$ if and only if $h - e^{2i\theta} g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction θ.*

As a follow-up to the above Lemma 1, Clunie and Sheil-Small [2] provided the following example.

Example 2. Since $z/(1 - z)$ is convex analytic in $\mathbb{D}$, the harmonic function $h + \overline{g}$ defined by

$$h(z) + g(z) = \frac{z}{1 - z}, \qquad g'(z) = -zh'(z) \tag{1}$$

is convex in the direction of imaginary axis.

Along the same line as the above Example 2, Dorff [4] proved the following result.

Theorem 3. *Let $f_1 = h_1 + \overline{g_1} \in \mathcal{S}_{\mathcal{H}}$ and $f_2 = h_2 + \overline{g_2} \in \mathcal{S}_{\mathcal{H}}$ with $h_1(z) + g_1(z) = h_2(z) + g_2(z) = z/(1-z)$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in \mathcal{S}_{\mathcal{H}}$ and is convex in the direction of real axis.*

The second author in his doctoral dissertation [5] proved the following theorem.

Theorem 4. *For $-1 < a < 1$ consider the harmonic function $f_a = h_a + \overline{g_a}$ sheared by $h_a(z) + g_a(z) = z/(1-z)$ with the dilatation $\omega_a(z) = g_a'(z)/h_a'(z) = (a-z)/(1-az)$. If $f = h + \overline{g}$ is the harmonic right half plane mapping given by $h(z)+g(z) = z/(1-z)$ with the dilatation $\omega(z) = g'(z)/h'(z0 = e^{i\theta} z^n$; $\theta \in \mathbb{R}$, $n \in \mathbb{N}$, then the convolution $f_a * f \in \mathcal{S}_{\mathcal{H}}$ and is convex in the horizontal direction for $(n-2)/(n+2) \le a < 1$.*

Since then a number of related articles were published and we refer the readers to three recently published articles ([6–8]) and the citations therein. As an extension to the above Theorem 4, Liu et al. [7] proved the following theorem.

Theorem 5. *Let $f_a = h_a + \overline{g_a}$ be as given in Theorem 4 and let $f_\alpha = h_\alpha + \overline{g_\alpha} \in \mathcal{S}_{\mathcal{H}}$ be a convex harmonic mapping so that f_α maps $\mathbb{D}$ onto the symmetric vertical strip domain*

$$\Omega_\alpha = \left\{\omega : \frac{\alpha - \pi}{2\sin\alpha} < \Re\omega < \frac{\alpha}{2\sin\alpha};\ \frac{\pi}{2} \le \alpha < \pi\right\} \tag{2}$$

with the dilatation $\omega_n(z) = g_\alpha'(z)/h_\alpha'(z) = e^{i\alpha} z^n$, $\theta \in \mathbb{R}$, and $n \in \mathbb{N}^+$. Then $f_a * f_\alpha$ is univalent and convex in the horizontal direction for $(n-2)/(n+2) \le a < 1$.

We remark that for $a = 0$ the function f_a given in Theorems 4 and 5 reduce to the harmonic function $h + \overline{g}$ given in Example 2, where $h(z)+g(z) = z/(1-z)$. Recently, Dorff et al. [9] presented the following Theorem 6 on the directional convexity for the convolution of harmonic functions $h+\overline{g}$ for which $h(z) \pm g(z) \ne z/(1-z)$.

Theorem 6. *For $j = 1, 2$ let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{\mathcal{H}}$, $h_j(z) - g_j(z) = (1/2)\log((1+z)/(1-z))$, and $\omega_j(z) = e^{i\theta_j} z^j$. If $f_1 * f_2$ is locally univalent and sense-preserving in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction of real axis.*

In the following Theorem 7 we improve the shear of the analytic map $h_j(z) - g_j(z) = (1/2)\log((1+z)/(1-z))$ to the general case $h_j(z) + e^{i\theta_j} g_j(z) = (1/2)\log((1+z)/(1-z))$, $j = 1, 2$, and expand the powers of z in the dilatation $\omega_j(z) = e^{i\theta_j} z^j$; $j = 1, 2$ to $\omega_1(z) = \pm e^{-i\theta_1} z^m$ and $\omega_2(z) = \pm e^{-i\theta_2} z^n$, where m and n are arbitrary positive integers. The arguments presented here to prove our Theorem 7 and Example 9 are new and have not yet been used in any of the preceding related articles.

Theorem 7. *For $j = 1, 2$ and for positive integers m and n let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{\mathcal{H}}$ be the shear of the analytic map $h_j(z) + e^{i\theta_j} g_j(z) = (1/2)\log((1+z)/(1-z))$ with the dilatations $\omega_1(z) = g_1'(z)/h_1'(z) = \pm e^{-i\theta_1} z^m$ and $\omega_2(z) = g_2'(z)/h_2'(z) = \pm e^{-i\theta_2} z^n$, where $\theta_1 + \theta_2 = (2k+1)\pi$; $k = 0, 1, 2, 3, \ldots$. If $f_1 * f_2$ is locally univalent and sense-preserving in $\mathbb{D}$, then $f_1 * f_2$ is convex in the direction of imaginary axis.*

2. Preliminaries, Proof and Example

Making use of the fact that a function F is convex in the direction γ if and only if the function $e^{i(\pi/2-\gamma)}F$ is convex in the direction of imaginary axis, in the following we state a lemma that is a variation of a result due to Royster and Ziegler [10].

Lemma 8. *Let F be a nonconstant analytic function in $\mathbb{D}$. The function F maps $\mathbb{D}$ univalently onto a domain convex in the direction γ if and only if there are numbers μ and ν, where $0 \le \mu < 2\pi$ and $0 \le \nu \le \pi$ so that*

$$\Re\left[e^{i(\mu-\gamma)}\left(1 - 2\cos\nu e^{-i\mu} z + e^{-2i\mu} z^2\right) F'(z)\right] \ge 0. \tag{3}$$

Proof of Theorem 7. Adding the identities

$$\begin{aligned}&\left(h_1 + e^{i\theta_1} g_1\right) * \left(h_2 - e^{i\theta_2} g_2\right)\\ &\quad = h_1 * h_2 - e^{i\theta_2}\left(h_1 * g_2\right) + e^{i\theta_1}\left(g_1 * h_2\right) + g_1 * g_2\end{aligned} \tag{4}$$

and

$$\begin{aligned}&\left(h_1 - e^{i\theta_1} g_1\right) * \left(h_2 + e^{i\theta_2} g_2\right)\\ &\quad = h_1 * h_2 + e^{i\theta_2}\left(h_1 * g_2\right) - e^{i\theta_1}\left(g_1 * h_2\right) + g_1 * g_2\end{aligned} \tag{5}$$

we get

$$\begin{aligned}2\left(h_1 * h_2 + g_1 * g_2\right) &= \left(h_2 + e^{i\theta_2} g_2\right) * \left(h_1 - e^{i\theta_1} g_1\right)\\ &\quad + \left(h_1 + e^{i\theta_1} g_1\right)\\ &\quad * \left(h_2 - e^{i\theta_2} g_2\right).\end{aligned} \tag{6}$$

Substituting for $h_j + e^{i\theta_j} g_j = (1/2)\log((1+z)/(1-z))$ yields

$$\begin{aligned}4\left(h_1 * h_2 + g_1 * g_2\right) &= \left(\log\frac{1+z}{1-z}\right) * \left(h_1 - e^{i\theta_1} g_2\right)\\ &\quad + \left(\log\frac{1+z}{1-z}\right)\\ &\quad * \left(h_2 - e^{i\theta_2} g_2\right).\end{aligned} \tag{7}$$

Differentiating

$$F_1(z) = \left(\log\frac{1+z}{1-z}\right) * \left(h_1(z) - e^{i\theta_1} g_1(z)\right) \tag{8}$$

and

$$F_2(z) = \left(\log\frac{1+z}{1-z}\right) * \left(h_2(z) - e^{i\theta_2} g_2(z)\right) \tag{9}$$

we obtain

$$\begin{aligned} F_1'(z) &= \left(\frac{1}{z}\log\frac{1+z}{1-z}\right) * \left(h_1'(z) - e^{i\theta_1} g_1'(z)\right) \\ &= \left(\frac{1}{z}\log\frac{1+z}{1-z}\right) \\ &\quad * \left(h_1'(z) + e^{i\theta_1} g_1'(z)\right) \frac{1 - e^{i\theta_1}\left(g_1'(z)/h_1'(z)\right)}{1 + e^{i\theta_1}\left(g_1'(z)/h_1'(z)\right)} \\ &= \left(\frac{1}{z}\log\frac{1+z}{1-z}\right) * \left(\frac{2}{1-z^2}\right)\left(\frac{1-(\pm z^m)}{1+(\pm z^m)}\right) \end{aligned} \tag{10}$$

and

$$F_2'(z) = \left(\frac{1}{z}\log\frac{1+z}{1-z}\right) * \left(\frac{2}{1-z^2}\right)\left(\frac{1-(\pm z^n)}{1+(\pm z^n)}\right). \tag{11}$$

Since

$$\log\frac{1+z}{1-z} * \psi(z) = \int \frac{\psi(z) - \psi(-z)}{z} dz, \tag{12}$$

$F_1'(z)$ may be written as

$$\begin{aligned} F_1'(z) &= \frac{2}{z}\left[\log\frac{1+z}{1-z} * \left(\frac{z}{1-z^2}\frac{1-(\pm z^m)}{1+(\pm z^m)}\right)\right] \\ &= \frac{2}{z}\int \frac{\left(z/\left(1-z^2\right)\right)\left(\left(1-(\pm z^m)\right)/\left(1+(\pm z^m)\right)\right) - \left(-z/\left(1-z^2\right)\right)\left(\left(1-(\pm(-z)^m)\right)/\left(1+(\pm(-z)^m)\right)\right)}{z} dz \\ &= \frac{2}{z}\int \frac{1}{1-z^2}\left[\frac{1-(\pm z^m)}{1+(\pm z^m)} + \frac{1-(\pm(-z)^m)}{1+(\pm(-z)^m)}\right] dz. \end{aligned} \tag{13}$$

If m is even, then

$$F_1'(z) = \frac{4}{z}\int \frac{1}{1-z^2}\frac{1-(\pm z^m)}{1+(\pm z^m)} dz. \tag{14}$$

If m is odd, then

$$F_1'(z) = \frac{4}{z}\int \frac{1}{1-z^2}\frac{1+z^{2m}}{1-z^{2m}} dz. \tag{15}$$

One can easily verify that

$$P(z) = \frac{1-(\pm z^m)}{1+(\pm z^m)} \tag{16}$$

is a positive real part function in $\mathbb{D}$ with real coefficients. So, by a result of Rogosinski [11] (or see Duren [12] page 56) we conclude that

$$\phi(z) = \frac{z}{1-z^2}P(z) = \frac{z}{1-z^2}\frac{1+(\pm z^m)}{1-(\pm z^m)} \tag{17}$$

is typically real in $\mathbb{D}$ (also see Clunie and Sheil-Small [2] page 22). Therefore the integral function

$$\psi(z) = \int \frac{\phi(z)}{z} dz = \int \frac{1}{1-z^2}\frac{1+(\pm z^m)}{1-(\pm z^m)} dz \tag{18}$$

is also typically real in $\mathbb{D}$ (e.g., see Theorem 2 in Robertson [13] or Duren [12], page 247). Consequently,

$$\frac{1-z^2}{z}\psi(z) = \frac{1-z^2}{z}\int \frac{1}{1-z^2}\frac{1+(\pm z^m)}{1-(\pm z^m)} dz \tag{19}$$

is of positive real part in $\mathbb{D}$ with real coefficients. The argument for m odd would be similar since $(1+z^{2m})/(1-z^{2m})$ is a positive real part function in $\mathbb{D}$ with real coefficients.

Therefore, for any positive integer m we have $\Re[(1-z^2)F_1'(z)] \geq 0$.

Similarly, for any positive integer n we have $\Re[(1-z^2)F_2'(z)] \geq 0$.

So, for all positive integers of m and n, we proved that $\Re(1-z^2)[F_1(z)+F_2(z)]' \geq 0$.

Thus for $\nu = \mu = \gamma = \pi/2$, it follows from Lemma 8 that the function $F_1 + F_2$ or the analytic convolution function $(h_1 * h_2) + (g_1 * g_2)$ is convex in the direction of the imaginary axis. This in conjunction with Lemma 1, for $\theta = \pi/2$ prove that the harmonic convolution function $(f_1 * f_2) = (h_1 * h_2) + \overline{(g_1 * g_2)} \in \mathcal{S}_{\mathcal{H}}$ and is convex in the direction of imaginary axis. □

To demonstrate the beauty of Theorem 7, we give an example of two harmonic functions that satisfy the dilatation stated in Theorem 7 and we then prove that their convolution is locally one-to-one, sense-preserving, and convex in the direction of imaginary axis.

Example 9. For $f_1 = h_1 + \overline{g_1} \in \mathcal{S}_{\mathcal{H}}$ let $h_1(z) - g_1(z) = (1/2)\log((1+z)/(1-z))$ with the dilatation $w_1(z) = g_1'(z)/h_1'(z) = z^2$ and for $f_2 = h_2 + \overline{g_2} \in \mathcal{S}_{\mathcal{H}}$ let $h_2(z) + g_2(z) = (1/2)\log((1+z)/(1-z))$ with the dilatation $w_2(z) = g_2'(z)/h_2'(z) = z^2$. Then $f_1 * f_2$ is locally univalent, sense-preserving, and convex in the direction of imaginary axis.

First we will show that the harmonic convolution function

$$\begin{aligned} (f_1 * f_2)(z) &= (h_1 * h_2)(z) + \overline{(g_1 * g_2)(z)} \\ &= H(z) + \overline{G(z)} \end{aligned} \tag{20}$$

is locally one-to-one and sense-preserving in $\mathbb{D}$, that is, $|W| = |G'/H'| < 1$ in $\mathbb{D}$. Under the hypotheses of Example 9, a simple calculation reveals that

$$h_1(z) = \frac{1}{2}\frac{z}{1-z^2} + \frac{1}{4}\log\frac{1+z}{1-z},$$

$$\begin{aligned} g_1(z) &= \frac{1}{2}\frac{z}{1-z^2} - \frac{1}{4}\log\frac{1+z}{1-z}, \\ h_2(z) &= \frac{1}{2}\tan^{-1}(z) + \frac{1}{4}\log\frac{1+z}{1-z}, \\ g_2(z) &= -\frac{1}{2}\tan^{-1}(z) + \frac{1}{4}\log\frac{1+z}{1-z}. \end{aligned} \tag{21}$$

It is easy to verify that $|G'(0)| = 0 < 1 = |H'(0)|$; therefore we shall take z in D_0, where $D_0 = \{z : 0 < |z| < 1\}$. Now for $z \in D_0$

$$\begin{aligned} W(z) &= \frac{G'(z)}{H'(z)} = \frac{((g_1 * g_2)(z))'}{((h_1 * h_2)(z))'} = \frac{g_1(z) * zg_2'(z)}{h_1(z) * zh_2'(z)} \\ &= \frac{(1/2)\left(z/\left(1-z^2\right) - (1/2)\log((1+z)/(1-z))\right) * zg_2'(z)}{(1/2)\left(z/\left(1-z^2\right) + (1/2)\log((1+z)/(1-z))\right) * zh_2'(z)} \\ &= \frac{z^3/\left(1-z^4\right) + (1/2)\tan^{-1}z - (1/4)\log((1+z)/(1-z))}{z/\left(1-z^4\right) + (1/2)\tan^{-1}z + (1/4)\log((1+z)/(1-z))} \\ &= \frac{z^3/\left(1-z^4\right) - g_2(z)}{z/\left(1-z^4\right) + h_2(z)} = \frac{z^3 - \left(1-z^4\right)g_2}{z + \left(1-z^4\right)h_2}. \end{aligned} \tag{22}$$

In order to prove that $|W| < 1$ in D_0 it suffices to show that $|z + (1-z^4)h_2(z)| > |z^3 - (1-z^4)g_2(z)|$ for all $z \in D_0$ or

$$\begin{aligned} &\left[z + \left(1-z^4\right)h_2(z)\right]\overline{\left[z + \left(1-z^4\right)h_2(z)\right]} \\ &\quad - \left[z^3 - \left(1-z^4\right)g_2(z)\right]\overline{\left[z^3 - \left(1-z^4\right)g_2(z)\right]} \\ &\quad > 0. \end{aligned} \tag{23}$$

The left hand side of the above inequality reduces to

$$\begin{aligned} &|z|^2\left(1-|z|^4\right) + \left|1-z^4\right|^2\left[\left|h_2(z)\right|^2 - \left|g_2(z)\right|^2\right] \\ &\quad + 2\Re\left[\left(1-z^4\right)\left(\overline{z}h_2(z) + \overline{z^3}g_2(z)\right)\right]. \end{aligned} \tag{24}$$

A result of Robinson [14] states that if q and p are analytic in D_0 so that $|q'(z)| < |p'(z)|$ in D_0, then $|q(z)| < |p(z)|$ if p is star-like in D_0. Applying this fact to the functions g_2 and h_2 given in Example 9, we obtain $|h_2(z)|^2 - |g_2(z)|^2 > 0$ since $|g_2'(z)| < |h_2'(z)|$ and h_2 is star-like in D_0.

On the other hand, observe that

$$\begin{aligned} &\Re\left(\left(1-z^4\right)\overline{z}h_2(z)\right) \\ &\quad = |z|^2\,\Re\left(1 - \sum_{k=1}^{\infty}\frac{4}{(4k+1)(4k-3)}z^{4k}\right) \\ &\quad > |z|^2\left(1 - \sum_{k=1}^{\infty}\frac{4}{(4k+1)(4k-3)}|z|^{4k}\right) \\ &\quad > |z|^2\left[1 - \sum_{k=1}^{\infty}\left(\frac{1}{4k-3} - \frac{1}{4k+1}\right)\right] \ge 0. \end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned} &\Re\left[\left(1-z^4\right)\overline{z}^3 g_2(z)\right] \\ &\quad > |z|^6\left[\frac{1}{3} - \sum_{k=1}^{\infty}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)\right] \ge 0. \end{aligned} \tag{26}$$

Thus $|W| = |G'/H'| < 1$ in D_0 and hence $|W| = |G'/H'| < 1$ for all $z \in \mathbb{D}$.

Next we will show that the harmonic convolution function $f_1 * f_2 = H + \overline{G}$ is convex in the direction of imaginary axis. By Lemma 1, it suffices to show that $H+G = h_1 * h_2 + g_1 * g_2$ is convex in the direction $\pi/2$. Equivalently, by Lemma 8, we need to show that

$$\begin{aligned} &\Re\left\{\left(1-z^2\right)[H(z)+G(z)]'\right\} \\ &\quad = \Re\left\{\left(1-z^2\right)[h_1(z) * h_2(z) + g_1(z) * g_2(z)]'\right\} \\ &\quad \ge 0. \end{aligned} \tag{27}$$

We observe that

$$\begin{aligned} [H(z)+G(z)]' &= [h_1(z) * h_2(z) + g_1(z) * g_2(z)]' = \frac{1}{4}\left[\left(\log\frac{1+z}{1-z}\right) * \left(\left(h_1(z) + g_1(z)\right) + \left(h_2(z) - g_2(z)\right)\right)\right]' \\ &= \frac{1}{4}\frac{1}{z}\left(\log\frac{1+z}{1-z}\right) * \left[\left(h_1'(z) + g_1'(z)\right) + \left(h_2'(z) - g_2'(z)\right)\right] \\ &= \frac{1}{4}\frac{1}{z}\left(\log\frac{1+z}{1-z}\right) * \left[\frac{1}{1-z^2}\left(\frac{1+g_1'(z)/h_1'(z)}{1-g_1'(z)/h_1'(z)} + \frac{1-g_2'(z)/h_2'(z)}{1+g_2'(z)/h_2'(z)}\right)\right] \\ &= \frac{1}{4}\frac{1}{z}\left(\log\frac{1+z}{1-z}\right) * \left[\frac{1}{1-z^2}\left(\frac{1+z^2}{1-z^2} + \frac{1-z^2}{1+z^2}\right)\right] = \frac{1}{4}\frac{1}{z}\left[\log\frac{1+z}{1-z} * \left(\frac{z}{1-z^2}\frac{2\left(1+z^4\right)}{1-z^4}\right)\right] \\ &= \frac{1}{2}\frac{1}{z}\int\frac{\left(z/\left(1-z^2\right)\right)\left(\left(1+z^4\right)/\left(1-z^4\right)\right) - \left(-z/\left(1-z^2\right)\right)\left(\left(1+z^4\right)/\left(1-z^4\right)\right)}{z}dz \\ &= \frac{1}{z}\int\left(\frac{1}{1-z^2}\frac{1+z^4}{1-z^4}\right)dz. \end{aligned} \tag{28}$$

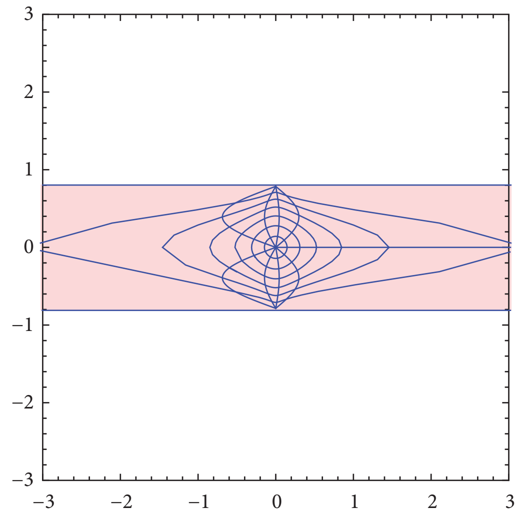

FIGURE 1: Images of $|z| = \ r < 1$ under f_1.

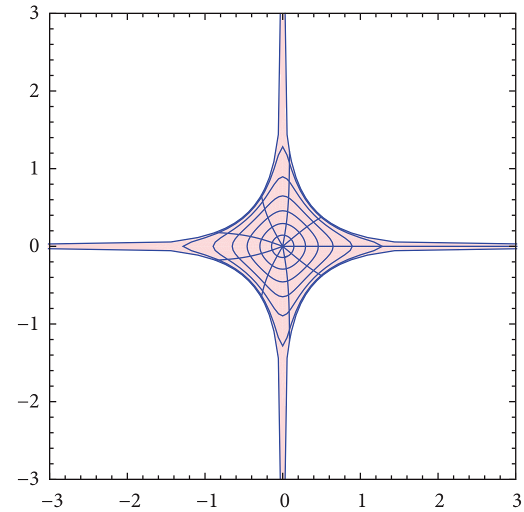

FIGURE 2: Images of $|z| = \ \ r < 1$ under f_2.

The Taylor Expansion of $(1-z^2)[H(z)+G(z)]'$ yields

$$\begin{aligned} P\left(z^2\right) &= \left(1-z^2\right)\left[H(z)+G(z)\right]' \\ &= \frac{1-z^2}{z}\int\left(\frac{1}{1-z^2}\frac{1+z^4}{1-z^4}\right)dz \\ &= 1-\frac{2}{3}z^2+\frac{4}{15}z^4-\frac{6}{35}z^6+\frac{8}{63}z^8-\frac{10}{99}z^{10} \\ &\quad+\frac{12}{143}z^{12}-\cdots. \end{aligned} \tag{29}$$

By a result of Fejér [15] (or see Goodman [16], Chapter 7),

$$\Re\left[P(-z)\right] = \Re\left(c_0+\sum_{k=1}^{\infty}\frac{2k}{(2k-1)(2k+1)}z^k\right) > 0 \tag{30}$$

where $c_0 = 1$ and $c_k = 2k/(2k-1)(2k+1)$; $k = 1, 2, 3 \ldots$ is a convex null sequence.

Therefore, $\Re\{P(z^2)\} > 0$, that is, $\Re(1-z^2)[H(z)+G(z)]' \geq 0$

The images of $|z| = r < 1$ under f_1 and f_2 are shown in Figures 1 and 2, respectively. Figure 3 clearly demonstrates the directional convexity of the convolution $f_1 * f_2$ along the imaginary axis.

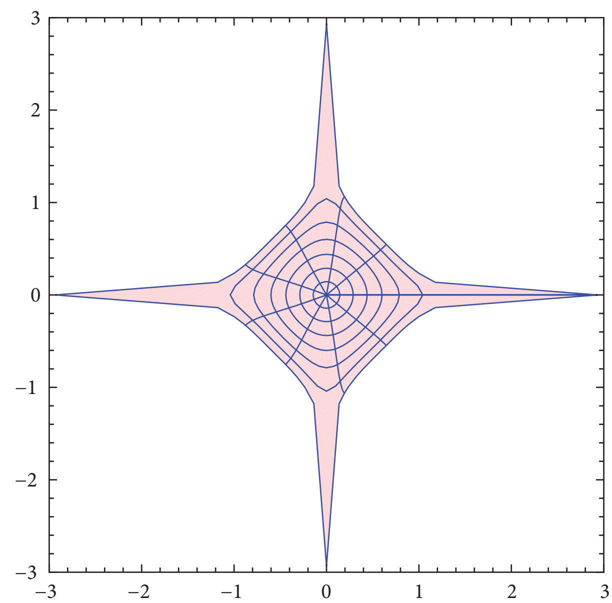

FIGURE 3: Images of $|z| = r < 1$ under $f_1 * f_2$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] H. Lewy, "On the non-vanishing of the Jacobian in a certain one-to-one mappings," *Bulletin of the American Mathematical Society*, vol. 42, no. 10, pp. 689–692, 1936.

[2] J. Clunie and T. Sheil-Small, "Harmonic univalent functions," *Annales Academiae Scientiarum Fennicae. Series A. I. Mathematica*, vol. 9, pp. 3–25, 1984.

[3] P. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, UK, 2004.

[4] M. Dorff, "Convolutions of planar harmonic convex mappings," *Complex Variables, Theory and Application*, vol. 45, no. 3, pp. 263–271, 2001.

[5] R. K. Garg, *Convolution Properties of Some Univalent Harmonic Functions [Ph.D. thesis]*, Sant Longowal Institute of Engineering and Technology, Punjab, India, August 2015.

[6] Z.-H. Liu and S. Ponnusamy, "Univalency of convolutions of harmonic right half-plane mappings," *Computational Methods and Function Theory*, vol. 17, no. 2, pp. 289–302, 2017.

[7] Z.-H. Liu, Z.-G. Wang, A. Rasila et al., "Convolutions of harmonic right half-plane mappings with harmonic strip mappings," *Bulletin of the Malaysian Mathematical Sciences Society*, pp. 1–14, 2018.

[8] L. LI and S. PONNUSAMY, "Note on the convolution of harmonic mappings," *Bulletin of the Australian Mathematical Society*, pp. 1–11, 2019.

[9] M. Dorff, S. G. Hamidi, J. M. Jahangiri, and E. Yasar, "Convolutions with harmonic strip mappings, Preprint: Abstract (1135-30-781) special session on extremal problems in approximations and geometric function theory," in *AMS Joint Mathematics Meeting*, San Diego, Calif, USA, January 2018.

[10] W. C. Royster and M. Ziegler, "Univalent functions convex in one direction," *Publicationes Mathematicae-Debrecen*, vol. 23, no. 3-4, pp. 339–345, 1976.

[11] W. Rogosinski, "Uber positive harmonische Entwicklungen und typisch-reelle Potenzreihen," *Mathematische Zeitschrift*, vol. 35, no. 1, pp. 93–121, 1932.

[12] P. L. Duren, *Univalent Functions, Grundlehren der mathematischen Wissenschaften*, vol. 259, Springer-Verlag, New York, NY, USA, 1983.

[13] M. S. Robertson, "Applications of a lemma of Feje'r to typically-real functions," *Proceedings of the American Mathematical Society*, vol. 1, no. 4, pp. 555–561, 1950.

[14] R. M. Robinson, "Univalent majorants," *Transactions of the American Mathematical Society*, vol. 61, pp. 1–35, 1947.

[15] L. Feje'r, "Uber die positivitat von summen, die nach trigonometrischen order Legendreschen funktionen fortschreiten," *Acta Scientiarum Mathematicarum (Szeged)*, pp. 75–86, 1925.

[16] A. W. Goodman, *Univalent Functions*, vol. I, Mariner Publishing Co., Inc., Tampa, FL, USA, 1983.