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Research Article

Multiplicity of Solutions for a Class of Elliptic Problem of p-Laplacian Type with a p-Gradient Term

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We consider the following problem: $-\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded set in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary, $1 < p < N$, $q > 0$, $\mu \in \mathbb{R}^*$, and c and h belong to $L^k(\Omega)$ for some $k > N/p$. In this paper, we assume that $c \gneqq 0$ a.e. in Ω and h without sign condition and then we prove the existence of at least two bounded solutions under the condition that $\|c\|_k$ and $\|h\|_k$ are suitably small. For this purpose, we use the Mountain Pass theorem, on an equivalent problem to (P) with variational structure. Here, the main difficulty is that the nonlinearity term considered does not satisfy Ambrosetti and Rabinowitz condition. The key idea is to replace the former condition by the *nonquadraticity condition at infinity*.

1. Introduction and Main Result

Let Ω be a bounded set in $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary $\partial\Omega$. In this paper, we are concerned with the following elliptic problem:

$$\begin{aligned} -\Delta_p u = c(x)|u|^{q-1}u + \mu|\nabla u|^p + h(x) \quad & \text{in } \Omega, \\ u = 0 \quad & \text{on } \partial\Omega, \end{aligned} \tag{P}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $1 < p < N$, $q > 0$, $\mu \in \mathbb{R}^*$, and c and h belong to $L^k(\Omega)$ for some $k > N/p$.

In the literature, there are many results concerning the existence, the uniqueness, and the multiplicity of solutions for models like (P) under various assumptions on c and h. At first, it is important to mention that the sign of c plays a crucial role in the problem (P) regarding uniqueness, as well as existence, of bounded solutions. In this setting, we refer to ([1]) for more details. In the coercive case, that is, $c(x) \leq -\alpha_0$ a.e. in Ω for some $\alpha_0 > 0$, Boccardo, Murat, and Puel ([2–4]) proved the existence of bounded solutions for more general divergence form problems with quadratic growth in the gradient by using the sub and supersolution method. Moreover, Barles and Murat ([5]) and Barles et at. ([6]) have treated the uniqueness question for similar problems. Notice that if we allow $c(x) \leq 0$ a.e. in Ω, then Ferone and Murat ([7, 8]) observed that finding solutions to (P) becomes rather complex without imposing some strong regularity conditions on the data. For the particular case $c \equiv 0$, there had been many contributions ([9–11]). However, for $c \leq 0$ that may vanish only on some parts of Ω, the uniqueness of solutions was left open until the recent paper authored by Arcoya et at. ([12]). This last result was proved for $p = 2$ and $q = 1$ and under the following condition:

$$\begin{gathered} c, h \text{ belong to } L^k(\Omega) \text{ for some } k > \frac{N}{2},\ \mu \\ \in L^\infty(\Omega) \text{ and } meas(\Omega \setminus \operatorname{Supp} c) > 0, \\ \inf_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \int_\Omega \left(|\nabla u|^2 - \|\mu^+\|_{L^\infty(\Omega)} h^+(x) u^2\right) > 0, \\ \inf_{u \in W_c, \|u\|_{H_0^1(\Omega)}} \int_\Omega \left(|\nabla u|^2 - \|\mu^-\|_{L^\infty(\Omega)} h^-(x) u^2\right) > 0. \end{gathered} \tag{1}$$

where $W_c := \{w \in H_0^1(\Omega) : c(x)w(x) = 0, \text{ a.e. in } \Omega\}$. For a related uniqueness result see also Arcoya et at. ([13]).

The case where $c(x) \gneqq 0$ a.e. in Ω, the question of nonuniqueness has been being an open problem given by Sirakov ([14]) and it has received considerable attention by many authors. Moreover, it should be pointed out that the sign of h and whether μ is a function or a constant generate additional difficulties for solving (P). In this setting, Jeanjean and Sirakov ([1]) showed the existence of two bounded solutions assuming that $\mu \in \mathbb{R}^*$, c, and h are in $L^k(\Omega)$ for some $k > N/2$ and satisfying

$$\begin{gathered}\left\|[\mu h]^+\right\|_{L^{N/2}(\Omega)} < C_N,\\ \max\left\{\|c\|_{L^k(\Omega)}, \left\|[\mu h]^-\right\|_{L^k(\Omega)}\right\} < \bar{c},\end{gathered} \tag{2}$$

where $\bar{c} > 0$ depends only on $N, k, meas(\Omega), |\mu|, \|[\mu h]^+\|_{L^k(\Omega)}$, and C_N is the optimal constant in Sobolev's inequality. Here, h is allowed to change sign. Shortly after, this result was extended by Coster and Jeanjean ([15]) for μ is a bounded function such that $\mu(x) \geq \mu_1 > 0$ by using the degree topological method.

Finally, in the case where c is allowed to change sign and with $c(x) \gneqq 0$ a.e. in Ω, Jenajean and Quoirin ([16, Theorem 1.1]) showed the existence of two bounded positive solutions when $h \gneqq 0$; μ is a positive constant, and c^+ and μh are suitably small.

We would also like to mention that all the above quoted multiplicity results were restricted to the Laplacian operator with quadratic growth in the gradient, i.e. $p = 2$, and for $q = 1$. Moreover, it is interesting to mention that when c is allowed to change sign the solutions are positive.

In this work, we prove the multiplicity of bounded solutions for the problem (P) by assuming the following assumption:

$$\begin{gathered} c, h \text{ belongs to } L^k(\Omega) \\ \text{for some } k > \frac{N}{p},\ h \text{ is allowed to change sign,} \\ c \gneqq 0 \quad \text{a.e. in } \Omega, q > 0,\ \mu \in \mathbb{R}^*. \end{gathered} \tag{H}$$

Now, we give a brief exposition of the proof of our multiplicity result. At first, without loss of generality, we solve the problem (P) by restricting it to the case μ is a positive constant. For μ is a negative constant, we replace u by $-u$ in (P), then we conclude. Next, we observe that the problems of type (P) do not have a variational formulation due to the presence of the p-gradient term. To overcome this difficulty, we perform the Kazdan-Kramer change of variable, that is, $v = (e^{\mu u/(p-1)} - 1)/\mu$. Thus, we obtain the following equivalent problem (P'):

$$\begin{gathered} -\Delta_p v = c(x) g(v) + h(x) f(v) \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{P'}$$

where

$$\begin{aligned} g(s) = {} & \frac{(p-1)^{q-p+1}}{\mu^q} (1+\mu s)^{p-1} \left|\ln(1+\mu s)\right|^{q-1} \\ & \cdot \ln(1+\mu s), \quad \text{with } s > \frac{-1}{\mu}, \end{aligned} \tag{3}$$

and

$$f(s) = \frac{(1+\mu s)^{p-1}}{(p-1)^{p-1}}. \tag{4}$$

We mean by bounded weak solutions of (P') the functions $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla u = \int_\Omega c(x) g(v) u + \int_\Omega h(x) f(v) u, \tag{5}$$

for any $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Obviously, if $v > -1/\mu$ is a solution of (P'), then $u = ((p-1)/\mu)\ln(1+\mu v)$ is a solution of (P). Hence, the solutions obtained here are not necessarily positive (compare with ([16])).

One of the most fruitful ways to deal with (P') is the variational method, which takes into account that the weak solutions of (P') are critical points in $W_0^{1,p}(\Omega)$ of the C^1-functional

$$I(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega c(x) G(v) - \int_\Omega h(x) F(v), \tag{6}$$

with $G(s) = \int_0^s g(t)dt$ and $F(s) = \int_0^s f(t)dt$.

In this work, to obtain the two critical points for I, we use the Mountain Pass Theorem to show one critical point and the standard lower semicontinuity argument to show the other. For the first one, according to the famous paper by Ambrosetti and Rabinowitz ([17]), the most important step is to show that I satisfies the Palais-Smale condition at the level $\tilde{c}$ (see Definition 3). The fulfillment of this condition relies on the well-known Ambrosetti-Rabinowitz condition ((AR_c) for short), namely,

$$\begin{gathered} \text{there exist } \theta > p \text{ and } s_0 > 0 \text{ such that} \\ 0 < \theta G(s) \leq s g(s), \text{ as } |s| > s_0. \end{gathered} \tag{7}$$

Unfortunately, this condition is somewhat restrictive and not being satisfied by many nonlinearities g. However, many researches have been made to drop the (AR_c). We refer, for instance, to [18–22]. Notice that the nonlinearity g considered here does not satisfy (AR_c). Moreover, since we do not assume any sign condition on h, the fulfillment of the Palais-Smale condition turns out more delicate (see, e.g., [16, 23]). To the best of our knowledge, only Jenajean and Quoirin ([16]), recently, proved the Palais-Smale condition under the assumptions c changes sign and h is positive, and without assuming (AR_c). In their proof, for $p = 2$ and $q = 1$, the authors based one of the arguments on the positivity of h and the explicit determination of a function H;

$$H(s) = g(s)s - 2G(s). \tag{8}$$

In our situation, as h is allowed to change sign and the analog of their function H cannot be computed explicitly, due to our general consideration of p and q ($1 < p < N$ and $q > 0$); hence, their arguments cannot be adapted.

The key point to show the Palais-Smale condition in this paper is to prove that g, among other conditions, satisfies the following (see Lemma 7):

$$H(s) = g(s)s - pG(s) \longrightarrow +\infty, \quad \text{where } s \longrightarrow +\infty. \tag{NQ}$$

The condition (NQ) is a variant of the well-known *nonquadraticity condition at infinity*, which was introduced by Costa and Malgalhães ([18]) and is given as follows:

$$\text{there exist } a > 0,\ \nu \geq \nu_0 > 0 \quad \text{such that } \liminf_{|s|\longrightarrow\infty} \frac{H(s)}{|s|^{\nu}} \geq a. \tag{CM}$$

Observe that since $\nu > 0$, then (NQ) is weaker than (CM). Moreover, it should be noted that (NQ) was considered by Furtado and Silva in their recent paper ([21]). Our result follows by using similar arguments.

Concerning the existence of the second critical point handled by the standard lower semicontinuity argument, we look for a local minimum in $W_0^{1,p}(\Omega)$ for the functional I. Indeed, we observe that I takes positive values in a large sphere, due to its geometrical structure (see Proposition 6), and $I(0) = 0$.

Now we state the main result of this paper.

Theorem 1. *Assume that (H) is satisfied. If $\|c\|_k$ and $\|h\|_k$ are suitably small, then the functional I has at least two critical points. Hence, the problem (P) has at least two bounded weak solutions.*

The paper is organized as follows. In Section 2 we recall some preliminary results and show that the functional I has a geometrical structure. In Section 3 we prove our main result, Theorem 1.

Notation. Through this paper, we use the following notations.

(1) The Lebesgue norm $(\int_\Omega |u|^p)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|.\|_p$ for $p \in [1, +\infty[$. The norm in $L^\infty(\Omega)$ is denoted by $\|u\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x\in\Omega}|u(x)|$. The Hölder conjugate of p is denoted by p'.

(2) The spaces $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ are equipped with Poincaré norm $\|u\| := (\int_\Omega |\nabla u|^p)^{1/p}$ and the dual norm $\|\cdot\|_* := \|\cdot\|_{W^{-1,p'}(\Omega)}$, respectively.

(3) We denote by $B(0, R)$ the ball of radius R centered at 0 in $W_0^{1,p}(\Omega)$ and $\partial B(0, R)$ its boundary.

(4) We denote by C_i, $c_i > 0$ any positive constants that are not essential in the arguments and that may vary from one line to another.

2. Preliminaries and Geometry of the Functional I

In this section, we recall the standard definitions of Palais-Smale sequence at the level $\tilde{c}$ and Palais-Smale condition at the level $\tilde{c}$ for I and we prove that the functional I defined in (6) has a geometrical structure.

Let us define the level at $\tilde{c}$ as follows:

$$\tilde{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)), \tag{9}$$

where $\Gamma = \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\ \gamma(1) = \nu_0\}$ is the set of continuous paths joining 0 and ν_0, where $\nu_0 \in W_0^{1,p}(\Omega)$ is defined in Proposition 6 below.

Definition 2. Let E be a Banach space with dual space E^* and (u_n) is a sequence in E. We say that (u_n) is a Palais-Smale sequence at the level $\tilde{c}$ for I if

$$\begin{gathered} I(u_n) \longrightarrow \tilde{c}, \\ \left\|I'(u_n)\right\|_{E^*} \longrightarrow 0. \end{gathered} \tag{10}$$

Definition 3. We say that I satisfies the Palais-Smale condition at the level $\tilde{c}$ if any Palais-Smale sequence at the level $\tilde{c}$ for I possesses a convergent subsequence.

In order to prove that I has a geometrical structure, we need some properties of g, which we gather in the following lemma without proof.

Lemma 4.

(1) $g(s)/|s|^{p-2}s \longrightarrow c$ *as* $s \longrightarrow 0$, *where* $c = 0$ *if* $q > p-1$ *and* $c = 1$ *if* $q = p - 1$.

(2) $g(s)/|s|^{q-1}s \longrightarrow (p-1)^{q-p+1}$ *as* $s \longrightarrow 0$, *for all* $q > 0$.

(3) $g(s)/s^{p-1} \longrightarrow +\infty$ *and* $G(s)/s^p \longrightarrow +\infty$ *as* $s \longrightarrow +\infty$, *for all* $q > 0$.

Lemma 5.

(1) *If* $q \geq p - 1$, *then we have*

$$\left|g(s)\right| \leq c_0 |s|^r + c_1 |s|^{p-1}, \tag{11}$$

for all $s > -1/\mu$, *and for all* $r \in (p-1, p)$.

(2) *If* $0 < q < p - 1$, *then we have*

$$\left|g(s)\right| \leq c_1 |s|^r + c_2 |s|^q, \tag{12}$$

for all $s > -1/\mu$, *and for all* $r \in (p-1, p)$.

Proof. By using Lemma 4(1), there exists $\eta > 0$ such that for all $|s| < \eta$ we have

$$\left|g(s)\right| \leq c_1 |s|^{p-1}. \tag{13}$$

Let $\delta \in (0,1)$. If $s \geq \eta$, then we have

$$g(s) \leq c_2(\eta, \mu, \delta)\, s^{p-1+\delta}. \tag{14}$$

Moreover, simple calculation yields

$$g'(s) = \frac{(p-1)^{q-p+1}}{\mu^{q-1}}(1+\mu s)^{p-2}\left|\ln(1+\mu s)\right|^{q-1} \cdot \left[(p-1)\ln(1+\mu s)+q\right]. \quad (15)$$

Now, if $-1/\mu < s \le -\eta$, then we have $|g(s)| \le |g(T)|$, where $T = (e^{-q/(p-1)}-1)/\mu$. Hence,

$$\left|g(s)\right| \le c_3(\eta,\mu,\delta)|s|^{p-1+\delta}. \quad (16)$$

By combining (14) and (16), (1) holds. To prove the property (2), we use Lemma 4(2) and the same previous argument. □

Proposition 6. *Assume that (H) holds. If $\|c\|_k$ and $\|h\|_k$ are suitably small, then the functional I has a geometrical structure; that is, I satisfies the following properties:*

(i) *There exists $\rho > 0$ such that for all v in $\partial B(0,\rho)$, $I(v) \ge \beta$, where $\beta > 0$.*

(ii) *There exists $v_0 \in W_0^{1,p}(\Omega)$ such that $\|v_0\| > \rho$ and $I(v_0) \le 0$.*

Proof. (i) To prove this lemma we distinguish two cases on q. Firstly, if $0 < q < p-1$, then by using Lemma 5(2) and Hölder's inequality, we get

$$\int_\Omega c(x)G(v) \le c_1\|c\|_k\left\|v^{r+1}\right\|_{k'} + c_2\|c\|_k\left\|v^{q+1}\right\|_{k'}. \quad (17)$$

We choose $r > p-1$ with r close to $p-1$ such that $(r+1)k' < pN/(N-p)$, which exists due to the assumption $k > N/p$. Obviously, $(q+1)k' < pN/(N-p)$. Thus, by using Sobolev's embedding we get

$$\int_\Omega c(x)G(v) \le C_1\|c\|_k\|v\|^{r+1} + C_2\|c\|_k\|v\|^{q+1}. \quad (18)$$

Moreover, from the definition of the function f in (4), we have

$$\left|f(v)\right| \le c\left(1+|v|^{p-1}\right), \quad \text{for some } c > 0. \quad (19)$$

Using Sobolev's embedding, we get

$$\int_\Omega h(x)F(v) \le C_3\|h\|_k + C_4\|h\|_k\|v\|^p. \quad (20)$$

By the definition of I in (6), we deduce that

$$I(v) \ge \frac{1}{p}\|v\|^p - C_1\|c\|_k\|v\|^{r+1} - C_2\|c\|_k\|v\|^{q+1} - C_3\|h\|_k - C_4\|h\|_k\|v\|^p. \quad (21)$$

Now, let v in $\partial B(0,\rho)$. Then, we have

$$I(v) \ge \frac{1}{p}\rho^p - \|c\|_k\left(C_1\rho^{r+1}+C_2\rho^{q+1}\right) - \|h\|_k\left(C_3+C_4\rho^p\right). \quad (22)$$

We take ρ sufficiently large, such that $\|c\|_k \le \rho^{-r-2+p}$ and $\|h\|_k \le \rho^{-1}$ (which are sufficiently small by hypothesis), then

$$I(v) \ge \frac{1}{p}\rho^p - C\rho^{p-1} \ge \rho^{p-1}\left(\frac{1}{p}\rho - C\right) = \beta_1. \quad (23)$$

Secondly, $q \ge p-1$, we choose again r as above such that $pk' < (r+1)k' < pN/(N-p)$. Then, by using Lemma 5(1) and Sobolev's embedding, we get

$$\int_\Omega c(x)G(v) \le c_1\|c\|_k\|v\|^{r+1} + c_2\|c\|_k\|v\|^p. \quad (24)$$

Now, as the first case, we get

$$I(v) \ge \frac{1}{p}\rho^p - C'\rho^{p-1} \ge \rho^{p-1}\left(\frac{1}{p}\rho - C'\right) = \beta_2. \quad (25)$$

Finally, we summarize the two cases and get

$$I(v) \ge \beta, \quad \text{where } \beta = \min(\beta_1,\beta_2). \quad (26)$$

(ii) To prove the second property, we show that $I(tv) \longrightarrow -\infty$ as $t \longrightarrow +\infty$. For this, let $v \in C_0^\infty(\Omega)$ be a positive function such that $cv \gneq 0$. By the definition of I in (6), we have

$$\begin{aligned} I(tv) &= \frac{t^p}{p}\int_\Omega|\nabla v|^p - \int_\Omega c(x)G(tv) - \int_\Omega h(x)F(tv) \\ &= t^p\left(\frac{1}{p}\int_\Omega|\nabla v|^p - \int_\Omega c(x)\frac{G(tv)}{t^pv^p}v^p \right. \\ &\quad \left. - \int_\Omega h(x)\frac{F(tv)}{t^pv^p}v^p\right). \end{aligned} \quad (27)$$

From inequality (19), we get

$$\int_\Omega\left|h(x)\frac{F(tv)}{t^pv^p}v^p\right| \le c \quad \text{as } t \longrightarrow +\infty. \quad (28)$$

Moerever, by Lemma 4(3), we get

$$\int_\Omega c(x)\frac{G(tv)}{t^pv^p}v^p \longrightarrow +\infty \quad \text{as } t \longrightarrow +\infty. \quad (29)$$

Thus, we deduce the desired result. □

Finally, we stress that since I has a geometrical structure, then the existence of a Palais-Smale sequence at the level $\tilde{c}$ for I is ensured. This can be observed directly from the proof given in ([17]), or alternatively using Ekeland's variational principle ([24]).

3. Proof of Theorem 1

Recall from introduction that the proof of our main result is divided into two steps as follows. In the first step, we show the existence of the first critical point for the C^1-functional I by using the Mountain Pass Theorem due to Ambrosetti-Rabinowitz ([17]). Precisely, we show that the functional I satisfies the Palais-Smale condition at the level $\tilde{c}$. In the second step, we show the existence of the second critical point of I on $B(0,\rho)$ (which is a local minimum) by using the lower semicontinuity argument. Moreover, we are going to see that these critical points are not the same. Finally, we show that any solution of problem (P) is bounded.

3.1. First Critical Point: Palais-Smale Condition. In this subsection, we prove that I satisfies the Palais-Smale condition at the level $\tilde{c}$. Precisely, we show that any Palais-Smale sequence at the level $\tilde{c}$ for I is bounded in $W_0^{1,p}(\Omega)$, and, then, it has a strongly convergent subsequence.

They key point to prove the boundedness of the Palais-Smale sequence at the level $\tilde{c}$ in $W_0^{1,p}(\Omega)$ is to show that g verifies the nonquadraticity condition at infinity (NQ). Indeed, we have the following lemma.

Lemma 7. *The function g defined in (3) verifies the nonquadraticity condition at infinity (NQ).*

Proof. To prove (NQ), we show that H is increasing and unbounded for s sufficiently large. We recall that $H(s) = g(s)s - pG(s)$. Then, by simple calculations, we get

$$H'(s) = C\mu s(1+\mu s)^{p-2}(\ln(1+\mu s))^{q-1} \cdot \left[(1-p)\frac{\ln(1+\mu s)}{\mu s} + q\right], \tag{30}$$

where $C = (p-1)^{q-p+1}/\mu^q$. Thus, H is increasing for s large enough. Moreover, H is unbounded. Indeed, by contradiction, if H is bounded, then there exists a positive constant M such that

$$H(s) \le M, \quad \text{for s large enough.} \tag{31}$$

In addition, from the definition of H and using integration by parts on G, we get

$$H(s) = -C\frac{1}{\mu}(\ln(1+\mu s))^q(1+\mu s)^{p-1} + qC\int_0^s (1+\mu t)^{p-1}(\ln(1+\mu t))^{q-1}\,dt. \tag{32}$$

By choosing $\delta \in (p-1, p)$, we obtain

$$\frac{H(s)}{s^\delta} = -\frac{1}{\mu}\frac{(\ln(1+\mu s))^q(1+\mu s)^{p-1}}{s^\delta} + qC\frac{\int_0^s (1+\mu t)^{p-1}(\ln(1+\mu t))^{q-1}}{s^\delta} \le \frac{M}{s^\delta}. \tag{33}$$

When $s \longrightarrow +\infty$, we obtain $H(s)/s^\delta \longrightarrow +\infty$ and $M/s^\delta \longrightarrow 0$. Hence, we have a contradiction. As a conclusion, the function g verifies (NQ). □

Lemma 8. *Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for I in $W_0^{1,p}(\Omega)$. Then, (u_n) is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\tilde{c}$ for I in $W_0^{1,p}(\Omega)$. We prove by contradiction that (u_n) is bounded in $W_0^{1,p}(\Omega)$. We assume that (u_n) is unbounded in $W_0^{1,p}(\Omega)$, that is, $\|u_n\| \longrightarrow +\infty$.

For all integer $n \ge 0$, we define

$$I(z_n) := \max_{0\le t\le 1} I(tu_n), \quad \text{where } z_n = t_n u_n,\ t_n \in [0,1]. \tag{34}$$

We are going to prove that $I(z_n) \longrightarrow +\infty$ and also $(I(z_n))$ is bounded, which is the desired contradiction.

(a) Showing that $I(z_n) \longrightarrow +\infty$: We set $v_n := u_n/\|u_n\|$; then (v_n) is bounded in $W_0^{1,p}(\Omega)$. Hence, there exists a subsequence denoted again (v_n) such that v_n converges weakly and strongly to v in $W_0^{1,p}(\Omega)$ and in $L^s(\Omega)$ for some $1 \le s < p^*$, respectively. Moreover, v_n also converges to v almost everywhere in Ω. Recall that $p^* := Np/(N-p)$ is Sobolev conjugate.

Now, we claim by contradiction that $v \equiv 0$ a.e. in Ω.

Since (u_n) is Palais-Smale type sequence, then we have

$$I(u_n) \longrightarrow \tilde{c}, \qquad \|I'(u_n)\|_* \longrightarrow 0. \tag{35}$$

Hence,

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi - \int_\Omega c(x)g(u_n)\varphi - \int_\Omega h(x)f(u_n)\varphi = \epsilon_n, \tag{36}$$

for all $\varphi \in W_0^{1,p}(\Omega)$ and for some $\epsilon_n \longrightarrow 0$ as $n \longrightarrow +\infty$. We divide both sides of (36) by $\|u_n\|^{p-1}$, to obtain

$$\int_\Omega c(x)\frac{g(u_n)}{\|u_n\|^{p-1}}\varphi = \frac{\epsilon_n}{\|u_n\|^{p-1}} + \int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla\varphi + \int_\Omega h(x)\frac{f(u_n)}{\|u_n\|^{p-1}}\varphi. \tag{37}$$

On the one hand, since v_n converges weakly to v in $W_0^{1,p}(\Omega)$ and by the inequality (19), then for n large enough the second and the third terms of the right-hand side of (37) are bounded.

On the other hand, if $v \not\equiv 0$ in Ω, then $cv \not\equiv 0$ in Ω. Now, we choose $\varphi \in W_0^{1,p}(\Omega)$ such that $cv\varphi > 0$ in Ω_φ and $cv\varphi \equiv 0$ in $\Omega \setminus \Omega_\varphi$, with $|\Omega_\varphi| > 0$. Since $v_n\|u_n\| = u_n$ in Ω, then by using Lemma 4(3), we obtain

$$\liminf c(x)\frac{g(u_n)}{\|u_n\|^{p-1}}\varphi = \liminf c(x)(v_n)^{p-1}\frac{g(v_n\|u_n\|)}{(v_n\|u_n\|)^{p-1}}\varphi = +\infty \quad \text{in } \Omega_\varphi. \tag{38}$$

Hence, by using Fatou's lemma in (37) we obtain the unbounded term in the left-hand side of (37). Hence, the claim (*i.e* $v \equiv 0$ a.e. in Ω.)

Since $\|u_n\| \longrightarrow +\infty$, then there exists $M > 0$ such that $\|u_n\| > M$, for n large enough. Moreover, we have

$$I(z_n) \ge I\left(M\frac{u_n}{\|u_n\|}\right) = I(Mv_n) = \frac{M^p}{p} - \int_\Omega c(x)G(Mv_n) - \int_\Omega h(x)F(Mv_n). \tag{39}$$

In what follows, we treat only the case $0 < q < p-1$. The other case follows with similar arguments. From Lemma 5(2), we have $|G(s)| \le c_1|s|^{r+1} + c_2|s|^{q+1}$, where $p-1 < r < p$. Since $c \in L^k(\Omega)$, for some $k > N/p$ and v_n converges strongly to v in $L^s(\Omega)$ with $1 \le s < p^*$, then, we obtain

$$\int_\Omega c(x)\, G(Mv_n) \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \tag{40}$$

due to $v \equiv 0$ a.e. in Ω. By Hölder's inequality, we get

$$\int_\Omega h(x)\, F(Mv_n) \le C \quad \text{as } n \longrightarrow +\infty. \tag{41}$$

Hence, by choosing $M > 0$ large enough, we deduce that $I(z_n) \longrightarrow +\infty$, as $n \longrightarrow +\infty$.

(b) Showing that $I(z_n)$ is bounded: To prove that $(I(z_n))$ is bounded, we distinguish two cases: $t_n \le 2/\|u_n\|$ and $t_n > 2/\|u_n\|$.

The Case $t_n \le 2/\|u_n\|$. Here, we only handle the proof for $q \in (0, p-1)$. The other case follows as in the proof of Proposition 6(i). By the definition of (z_n) and $I \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$, we have $\langle I'(t_nu_n), t_nu_n\rangle = 0$, which means that

$$\begin{aligned} t_n^p \|u_n\|^p = \int_\Omega c(x)\, g(t_nu_n)\, t_nu_n \\ + \int_\Omega h(x)\, f(t_nu_n)\, t_nu_n. \end{aligned} \tag{42}$$

By the definition of I in (6), we have

$$\begin{aligned} pI(t_nu_n) &= t_n^p \|u_n\|^p - p\int_\Omega c(x)\, G(t_nu_n) \\ &\quad - p\int_\Omega h(x)\, F(t_nu_n) \\ &= \int_\Omega c(x)\, H(t_nu_n) + \int_\Omega h(x)\, K(t_nu_n), \end{aligned} \tag{43}$$

where the function H is defined in (NQ) and $K(s) := f(s)s - pF(s)$. Moreover, from Lemma 5(2), we have

$$\begin{aligned} \int_\Omega c(x)\, H(t_nu_n) &\le \int_\Omega |c(x)|\, |g(t_nu_n)\, t_nu_n| \\ &\quad + p\int_\Omega |c(x)|\, |G(t_nu_n)| \\ &\le c_1 \int_\Omega |c(x)|\, |t_nu_n|^{r+1} \\ &\quad + c_2 \int_\Omega |c(x)|\, |t_nu_n|^{q+1}. \end{aligned} \tag{44}$$

By choosing r and q as in the proof of Proposition 6(i), we get

$$\begin{aligned} \int_\Omega c(x)\, H(t_nu_n) &\le C_1 \|c\|_k \|t_nu_n\|^{r+1} \\ &\quad + C_2 \|c\|_k \|t_nu_n\|^{q+1}. \end{aligned} \tag{45}$$

By inequality (19) and Sobolev's embedding, we get

$$\begin{aligned} \int_\Omega h(x)\, K(t_nu_n) &\le \int_\Omega |h(x)|\, |f(t_nu_n)\, t_nu_n| \\ &\quad + p\int_\Omega |h(x)|\, |(F(t_nu_n)\, t_nu_n| \\ &\le c_1 \|h\|_k + c_2 \|h\|_k \|t_nu_n\| \\ &\quad + c_3 \|h\|_k \|t_nu_n\|^p. \end{aligned} \tag{46}$$

Then, by (43), (45), and (46), we obtain

$$I(t_nu_n) \le C, \tag{47}$$

for all $n \ge 0$, where C is independent of n. Thus, $(I(z_n))$ is bounded, which contradicts the fact that $(I(z_n))$ is unbounded (see *(a)*).

The Case $t_n > 2/\|u_n\|$. Here, we are proceeding the technique inspired by [21]. To this end, we need the following technical lemma.

Lemma 9. *Let $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ the nonnegative function defined as*

$$\Phi(s) = \begin{cases} e^{-\epsilon/s^2}, & \text{if } s \ne 0, \\ 0, & \text{if } s = 0, \end{cases} \tag{48}$$

with $\epsilon > 0$. Then, we have

(i) $\lim_{s\longrightarrow 0}\Phi(s) = \lim_{s\longrightarrow 0}\Phi'(s) = 0$.

(ii) *for any positive function z in Ω and $p > 1$,*

$$\begin{aligned} \lim_{\epsilon\longrightarrow 0} \int_\Omega \int_s^t \frac{z(x)}{\tau^{p+1}} \left(\frac{1 - \Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p} \right) d\tau\, dx = 0, \\ \text{uniformly in } n \in \mathbb{N}. \end{aligned} \tag{49}$$

Proof. Obviously we have (i). To prove (ii), we follow the same approach given in [21] for the case $p = 2$ and $z(x) = 1$, which can be immediately generalized for any positive function z and $p > 1$. □

Now, we resume the proof of Lemma 8. From Lemma 7, we have $H(s) \ge \sigma$, for s large enough and some $\sigma > 0$ (which will be chosen later). Moreover, if $0 < q < p-1$, then from Lemma 4(2), we have for s sufficiently small,

$$H(s) \ge -C_1 |s|^{q+1}. \tag{50}$$

Then, by the continuity of H, we have for all $s > -1/\mu$

$$H(s) \ge \sigma\Phi_\epsilon(s) - C_2 |s|^{q+1}. \tag{51}$$

Letting $0 < s < t$, then we have

$$\begin{aligned}\frac{I(tu_n)}{t^p\|u_n\|^p}-\frac{I(su_n)}{s^p\|u_n\|^p} &= -\int_\Omega c(x)\left[\frac{G(tu_n)}{t^p\|u_n\|^p}-\frac{G(su_n)}{s^p\|u_n\|^p}\right] \\ &\quad -\int_\Omega h(x)\left[\frac{F(tu_n)}{t^p\|u_n\|^p}-\frac{F(su_n)}{s^p\|u_n\|^p}\right] \\ &= \underbrace{-\int_\Omega c(x)\int_s^t \frac{d}{d\tau}\left(\frac{G(\tau u_n)}{\tau^p\|u_n\|^p}\right)d\tau\,dx}_{A} \\ &\quad \underbrace{+\int_\Omega -h(x)\left[\frac{F(tu_n)}{t^p\|u_n\|^p}-\frac{F(su_n)}{s^p\|u_n\|^p}\right]}_{B}.\end{aligned} \tag{52}$$

Let us handle the two terms A and B, respectively.

$$\begin{aligned}A &= -\int_\Omega\int_s^t c(x)\frac{\tau^p u_n g(\tau u_n)-p\tau^{p-1}G(\tau u_n)}{\tau^{2p}\|u_n\|^p}d\tau\,dx \\ &= -\int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\frac{H(\tau u_n)}{\tau^{p+1}}d\tau\,dx.\end{aligned} \tag{53}$$

By using (51), we get

$$\begin{aligned}A &\le \int_\Omega\int_s^t \frac{c(x)}{\|u_n\|^p}\left(C_2\frac{|u_n|^{q+1}}{\tau^{p-q}}-\sigma\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}\right)d\tau\,dx \\ &\le \int_\Omega \frac{c(x)}{\|u_n\|^p}\left(\frac{C_2}{p-q-1}\frac{|u_n|^{q+1}}{s^{p-q-1}}\right. \\ &\quad \left.-\sigma\int_s^t\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}}d\tau\right)dx.\end{aligned} \tag{54}$$

For the term B, we have

$$\begin{aligned}B &\le C\left(\int_\Omega |h(x)|\frac{(1+|tu_n|)^p}{t^p\|u_n\|^p}\right. \\ &\quad \left.+\int_\Omega|h(x)|\frac{(1+|su_n|)^p}{s^p\|u_n\|^p}\right) \\ &\le C\left(\int_\Omega|h(x)|\left(\frac{1}{t_n\|u_n\|}+\frac{|u_n|}{\|u_n\|}\right)^p\right. \\ &\quad \left.+\int_\Omega|h(x)|\left(\frac{1}{s_n\|u_n\|}+\frac{|u_n|}{\|u_n\|}\right)^p\right).\end{aligned} \tag{55}$$

By setting $s := 1/\|u_n\|$, we obtain

$$\begin{aligned}\frac{I(tu_n)}{t^p\|u_n\|} &\leqslant I(v_n)+\int_\Omega c(x)\left(\frac{C_2}{p-q-1}|v_n|^{q+1}\right. \\ &\quad \left.-\sigma\int_s^t\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}\|u_n\|^p}\right)d\tau\,dx \\ &\quad +C\left(\int_\Omega|h(x)|\left(\frac{1}{2}+|v_n|\right)^p+\int_{\Omega^+}|h(x)|(1\right. \\ &\quad \left.+|v_n|)^p\right) \leqslant I(v_n)+C\left[\int_\Omega c(x)|v_n|^{q+1}\right. \\ &\quad \left.+\int_\Omega|h(x)|+2\int_\Omega|h(x)||v_n|^p\right]-\sigma\int_\Omega\frac{c(x)}{p}\left(1\right. \\ &\quad \left.-\frac{1}{t_n^p\|u_n\|^p}\right)+\sigma\int_\Omega\frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) \\ &\quad -\sigma\int_\Omega\int_s^t c(x)\frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}\|u_n\|^p}d\tau\,dx \leqslant I(v_n) \\ &\quad +C\left[\int_\Omega c(x)|v_n|^{q+1}+\int_\Omega|h(x)|+2\int_\Omega|h(x)|\right. \\ &\quad \left.\cdot|v_n|^p\right]-\sigma\int_\Omega\frac{c(x)}{p}\left(1-\frac{1}{t_n^p\|u_n\|^p}\right) \\ &\quad -\sigma\int_\Omega\int_s^t\frac{c(x)}{\tau^{p+1}}\left(\frac{1-\Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p}\right)d\tau\,dx\end{aligned} \tag{56}$$

By the technical Lemma 9, we have

$$\lim_{\epsilon\longrightarrow 0}\int_\Omega\int_s^t\frac{c(x)}{\tau^{p+1}}\left(\frac{1-\Phi_\epsilon(|\tau u_n|)}{\|u_n\|^p}\right)d\tau\,dx = 0, \quad \text{uniformly in } n\in\mathbb{N}. \tag{57}$$

Then,

$$\begin{aligned}\frac{I(tu_n)}{t^p\|u_n\|^p} &\leqslant \frac{1}{p}-\int_\Omega c(x)G(v_n)-\int_\Omega h(x)F(v_n) \\ &\quad +C\left[\int_\Omega c(x)|v_n|^{q+1}+\int_\Omega|h(x)|\right. \\ &\quad \left.+2\int_\Omega|h(x)||v_n|^p\right]-\sigma\int_\Omega\frac{c(x)}{p}\left(1-\frac{1}{2^p}\right).\end{aligned} \tag{58}$$

We choose σ such that

$$\sigma > \frac{2^p(1+pC\|h\|_k)}{(2^p-1)\int_\Omega c(x)} \tag{59}$$

which gives

$$\frac{1}{p}+C\|h\|_k-\sigma\int_\Omega\frac{c(x)}{p}\left(1-\frac{1}{2^p}\right)dx < 0. \tag{60}$$

Since v_n converges to 0 almost everywhere in Ω, weakly in $W_0^{1,p}(\Omega)$, and strongly in $L^s(\Omega)$ for some $1 \le s < p^*$, then, we have

$$I(t_n u_n) < 0, \quad \text{in } \Omega \text{ for } n \text{ large enough}. \tag{61}$$

Hence, $(I(z_n))$ is bounded. Therefore, this contradicts the fact that $(I(z_n))$ is unbounded (see *(a)*).

Now, If $q \ge p-1$, then from Lemma 4(1) and the continuity of $H(s)$, we have for all $s > -1/\mu$,

$$H(s) \ge \sigma \Phi_\epsilon(s) - C_1 |s|^{p-1}. \tag{62}$$

Following the computations as in (52), we find exactly the same terms A and B. The term B is handled as in (55), whereas A is handled as follows:

$$\begin{aligned} A &\le \int_\Omega \int_s^t \frac{c(x)}{\|u_n\|^p} \left(C_1 \frac{|u_n|^{p-1}}{\tau^2} - \sigma \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}} \right), \\ &\le \int_\Omega \int_s^t \frac{c(x)}{\|u_n\|^p} \left(C_1 \frac{|u_n|^{p-1}}{s} - \sigma \frac{\Phi_\epsilon(|\tau u_n|)}{\tau^{p+1}} \right). \end{aligned} \tag{63}$$

Moreover, since $(p-1)k' < pk' < Np/(N-p)$, then by using Sobolev embedding, the rest of the proof is similar to the case $q \in (0, p-1)$. Hence, we have also the contradiction with the fact that I is unbounded (see *(a)*). □

To finish the proof of the Palais-Smale condition for I, we only need to show the following lemma.

Lemma 10. *Any Palais-Smale sequence at the level $\widetilde{c}$ of $W_0^{1,p}(\Omega)$ has a strongly convergent subsequence.*

Proof. Let (u_n) be a Palais-Smale sequence at the level $\widetilde{c}$ and then $I'(u_n) \longrightarrow 0$ in $W^{-1,p'}(\Omega)$, which means that

$$\begin{aligned} -\Delta_p u_n - c(x) g(u_n) - h(x) f(u_n) \longrightarrow 0 \\ \text{in } W^{-1,p'}(\Omega). \end{aligned} \tag{64}$$

By Lemma 8, (u_n) is bounded in $W_0^{1,p}(\Omega)$. Hence, u_n converges weakly to u in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ for some $1 \le s < p^*$. Therefore,

$$-\Delta_p u_n \longrightarrow c(x) g(u) + h(x) f(u) \quad \text{in } W^{-1,p'}(\Omega). \tag{65}$$

We know that the operator $-\Delta_p : W_0^{1,p}(\Omega) \longmapsto W^{-1,p'}(\Omega)$ is a homeomorphism ([25]). Hence, from (65) we get

$$\begin{aligned} u_n \longrightarrow (-\Delta_p)^{-1} (c(x) g(u) + h(x) f(u)) \\ \text{in } W_0^{1,p}(\Omega). \end{aligned} \tag{66}$$

Therefore, by the uniqueness of the limit we have

$$u_n \longrightarrow u, \quad \text{in } W_0^{1,p}(\Omega). \tag{67}$$

□

3.2. Second Critical Point. In this subsection, we use the geometrical structure of I (see Proposition 6) and the standard lower semicontinuity argument; we show the existence of the second critical point. We state the result as follows.

Theorem 11. *Assume that $\|c\|_k$ and $\|h\|_k$ are suitably small to ensure Proposition 6. Then, the functional I possesses a critical point $v \in B(0,\rho)$ with $I(v) \le 0$.*

Proof. Since $I(0) = 0$, then $\inf_{v \in B(0,\rho)} I(v) \le 0$. Moreover, if $h \not\equiv 0$, then we obtain that $\inf_{v \in B(0,\rho)} I(v) < 0$. Indeed, we choose $v \in C_0^\infty(\Omega)$ a positive function that satisfies $cv > 0$ and $hv > 0$. From the definition of I in (6), we have for $t > 0$

$$\begin{aligned} I(tv) = t^p \Bigg(&\frac{1}{p} \int_\Omega |\nabla v|^p - \int_\Omega c(x) \frac{G(tv)}{t^p v^p} v^p \\ &- \int_\Omega h(x) \frac{F(tv)}{t^p v^p} v^p \Bigg). \end{aligned} \tag{68}$$

If $q \ge p-1$, then from Lemma 4(2), we have $G(s)/s^p \longrightarrow c < +\infty$ as $s \longrightarrow 0^+$. If $0 < q < p-1$, obviously, we have $G(s)/s^p \longrightarrow +\infty$ as $s \longrightarrow 0^+$. In addition, in both cases, we have $F(s)/s^p \longrightarrow +\infty$ as $s \longrightarrow 0^+$. Hence, by using these limits, we get from (68) that $I(tv) < 0$ for $t > 0$ small enough.

Now, we set $m := \inf_{v \in B(0,\rho)} I(v)$. Then, by Proposition 6(i), we have $I(v) \ge \beta > 0$ for $\|v\| = \rho$. Moreover, there exists a sequence $(v_n) \subset B(0,\rho)$ such that $I(v_n)$ converges to m. Since (v_n) is bounded in $W_0^{1,p}(\Omega)$, then there exists a subsequence denoted again (v_n) such that v_n converges to v weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ for some $1 \le s < p^*$ respectively. Hence, we get

$$\begin{aligned} &\int_\Omega h(x) F(v_n) \longrightarrow \int_\Omega h(x) F(v), \\ &\int_\Omega c(x) G(v_n) \longrightarrow \int_\Omega c(x) G(v) \\ &\qquad \text{as } n \longrightarrow +\infty. \end{aligned} \tag{69}$$

In addition, since $\|v\|^p \le \liminf_{n \longrightarrow \infty} \|v_n\|^p$, then $I(v) \le m = \inf_{v \in B(0,\rho)} I(v)$. Hence, we conclude that v is a local minimum of I in $B(0,\rho)$. □

Remark 12. By Section 3.1, I has a critical point at the level $\widetilde{c}$; that is, there exists w in $W_0^{1,p}(\Omega)$ such that $I(w) = \widetilde{c}$ and $I'(w) = 0$. Since $I(w) = \widetilde{c} > 0 \ge I(v)$, where $v \in B(0,\rho)$ is the second critical point given in the previous theorem, then w is different from v. Hence, we have two distinct solutions for the problem (P).

3.3. Boundedness of Solutions. Now, to finish the proof of our main result, it remains to show the boundedness of the solutions. Therefore, we show the following result.

Proposition 13. *Any solution u of the problem (P') belongs to $L^\infty(\Omega)$.*

Proof. If $|u| \leq 1$, it is over. Otherwise, we begin by writing the problem (P') as follows:

$$-\Delta_p u = a(x)\left(1 + |u|^{p-1}\right), \tag{70}$$

where

$$a(x) = \frac{c(x)\, g(u) + h(x)\, f(u)}{1 + |u|^{p-1}}. \tag{71}$$

Then, by Theorem 2.4 in [26], we can deduce the boundedness of u if we show that a belongs to $L^{p/N(1-\epsilon)}(\Omega)$, for some $\epsilon \in]0,1[$. Indeed, from (19) and Lemma 5, we obtain

$$|a(x)| \leq C\left[|c(x)|\left(|u|^{r-p+1} + 1\right) + |h(x)|\right]. \tag{72}$$

Let $m > 1$ and m' its conjugate. By using Hölder's inequality in (72), we obtain

$$\begin{aligned}\int_\Omega & |a(x)|^{p/N(1-\epsilon)} \\ & \leq C\left[\left\|c(x)^{p/N(1-\epsilon)}\right\|_m \left\|u^{(r-p+1)p/N(1-\epsilon)}\right\|_{m'}\right. \\ & \quad \left. + \left\|h^{p/N(1-\epsilon)}\right\|_m + 1\right].\end{aligned} \tag{73}$$

By choosing $0 < \epsilon < 1 - (N-p)(r-p+1)/N^2 - p/kN$, we have

$$\begin{aligned}\frac{p}{N(1-\epsilon)} m &\leq k, \\ (r-p+1)\frac{p}{N(1-\epsilon)} m' &< \frac{Np}{N-p}.\end{aligned} \tag{74}$$

Hence, the terms $\|c(x)^{p/N(1-\epsilon)}\|_m$, $\|h(x)^{p/N(1-\epsilon)}\|_m$, and $\|u^{(r-p+1)p/N(1-\epsilon)}\|_{m'}$ are finite (recall that $c, h \in L^k(\Omega)$ for some $k > N/p$). □

Data Availability

This work is concerned with mathematical analysis, especially, the study of the existence of a multiplicity of solutions. Thus, we have no data or experimental experience.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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