Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2019, Article ID 7146073, 7 pages
https://doi.org/10.1155/2019/7146073

Research Article

Isomorphisms from Extremely Regular Subspaces of $C_0(K)$ into $C_0(S, X)$ Spaces

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Received 30 October 2019; Accepted 3 December 2019; Published 27 December 2019

Academic Editor: Luca Vitagliano

For a locally compact Hausdorff space K and a Banach space X, let $C_0(K, X)$ be the Banach space of all X-valued continuous functions defined on K, which vanish at infinite provided with the sup norm. If X is $\mathbb{R}$, we denote $C_0(K, X)$ as $C_0(K)$. If $\mathscr{A}(K)$ be an extremely regular subspace of $C_0(K)$ and $T: \mathscr{A}(K) \longrightarrow C_0(S, X)$ is an into isomorphism, what can be said about the set-theoretical or topological properties of K and S? Answering the question, we will prove that if X contains no copy of c_0, then the cardinality of K is less than that of S. Moreover, if $\|T\|\|T^{-1}\| < 3$ and $\mathscr{A}(K)$ is also a subalgebra of $C_0(K)$, the cardinality of the αth derivative of K is less than that of the αth derivative of S, for each ordinal α. Finally, if $\lambda(X) > 1$ and $\|T\|\|T^{-1}\| < \lambda(X)$, then K is a continuous image of a subspace of S. Here, $\lambda(X)$ is the geometrical parameter introduced by Jarosz in 1989: $\lambda(X) = \inf\{\max\{\|x + \lambda y\|: |\lambda| = 1\}: \|x\| = \|y\| = 1\}$. As a consequence, we improve classical results about into isomorphisms from extremely regular subspaces already obtained by Cengiz.

1. Introduction

All Banach spaces here are assumed to be real. For a locally compact Hausdorff space K and Banach space X, let $C_0(K, X)$ be the Banach space of all X-valued continuous functions defined in K which vanish at infinite, endowed with the supremum norm. If K is compact, we write $C(K, X)$ instead of $C_0(K, X)$. When X is $\mathbb{R}$, these spaces are denoted by $C_0(K)$ and $C(K)$, respectively. For a set A, we denote its cardinality by $|A|$. We denote $X \sim Y$ if X and Y are isomorphic Banach spaces and $X \hookrightarrow Y$ if Y contains a subspace isomorphic to X. If $X \sim Y$ and $T: X \longrightarrow Y$ is an isomorphism, the number $\|T\|\|T^{-1}\|$ is called distortion of T; if $\|T\|\|T^{-1}\| < \lambda$, $1 < \lambda < +\infty$, we write $X \overset{<\lambda}{\sim} Y$. Other notations and definitions used here can be found in [1].

The classical Banach–Stone theorem established that if $C_0(K)$ and $C_0(S)$ are isometric, then K and S are homeomorphic [2–5]. Amir and Cambern, independently, showed that the conclusion of Banach–Stone theorem holds if we assume that $C_0(K) \overset{<2}{\sim} C_0(S)$ [6, 7]. Moreover, it is well-known that 2 is the best number in this result [8]. On the other hand, the number 2 can be replaced by 3 in the case of countable compact metric spaces [9].

Many authors have obtained generalizations of the Banach–Stone theorem, both in the scalar and vector-valued case (see [10–14]). Among the many generalizations of this theorem, we would like to highlight the following one. First, we recall a definition introduced by Cengiz [15].

Definition 1. A closed subspace $\mathscr{A}(K)$ of $C_0(K)$ is called *extremely regular* if for each $k \in K$, $U \subset K$ open with $k \in U$ and $0 < \varepsilon < 1$, there is $f \in \mathscr{A}(K)$ such that $1 = f(k) = \|f\| > \varepsilon > |f(w)|$ for all $w \in K \backslash U$. An extremely regular subspace $\mathscr{A}(K)$ of $C_0(K)$ is called *extremely regular subalgebra* if $f \cdot g \in \mathscr{A}(K)$ whenever $f, g \in \mathscr{A}(K)$.

For examples of extremely regular subspaces (subalgebras), see [15]. The next result was obtained by Cengiz [16].

Theorem 1. *Let $\mathscr{A}(K)$ and $\mathscr{B}(S)$ be extremely regular subspaces of $C_0(K)$ and $C_0(S)$, respectively. If $A(K) \overset{<2}{\sim} \mathscr{B}(S)$, then K and S are homeomorphic.*

Despite this, the conclusion of the Banach–Stone is far from to be valid for arbitrary isomorphisms. Thus, it is natural asking.

Problem 1. Which set-theoretical and topological properties are preserved by isomorphisms of extremely regular subspaces?

So, the aim of this paper is to give answers to the problem posed above. In this direction, Cengiz showed that, under additional topological hypothesis on K and S, existence of an isomorphism between extremely regular subspaces of $C_0(K)$ and $C_0(S)$, respectively, implies that $|K| = |S|$ [17, 18]. Our first result shows that Cengiz's result holds for arbitrary locally compact Hausdorff spaces. So, we prove the following.

Theorem 2. *Let $\mathscr{A}(K)$ and $\mathscr{B}(S)$ be extremely regular subspaces of $C_0(K)$ and $C_0(S)$, respectively. If $\mathscr{A}(K) \sim \mathscr{B}(S)$, then $|K| = |S|$.*

In order to prove Theorem 2, we establish the following vector-valued result, which generalizes ([19], Theorem 2).

Theorem 3. *Let K be infinite. For an extremely regular subspace $\mathscr{A}(K)$ of $C_0(K)$ and a Banach space X, we have*

$$\mathscr{A}(K) \hookrightarrow C_0(S, X) \Longrightarrow c_0 \hookrightarrow X \text{ or } |K| \leq |S|. \tag{1}$$

Recall that if S is a topological space, the derivative of S, denoted by $S^{(1)}$, is obtained by deleting from S its isolated points. If α is an ordinal, we define the αth derivative $S^{(\alpha)}$ of S as follows: $S^{(0)} = S$, $S^{(\alpha+1)} = (S^{(\alpha)})^{(1)}$, and $S^{(\alpha)} = \cap_{\beta<\alpha} S^{(\beta)}$, in the case where α is a limit ordinal.

The following result is an extension of ([19], Theorem 5) for extremely regular subspaces.

Theorem 4. *Let X be a Banach space containing no copy of c_0. Let $\mathscr{A}(K)$ be an extremely regular subalgebra of $C_0(K)$ and $T\colon \mathscr{A}(K) \longrightarrow C_0(S, X)$ an into isomorphism. For each ordinal α such that $S^{(\alpha)}$ is infinite, we have*

$$\|T\|\left\|T^{-1}\right\| < 3 \Longrightarrow \left|K^{(\alpha)}\right| \leq \left|S^{(\alpha)}\right|. \tag{2}$$

Moreover, if $S^{(\alpha)}$ is finite, then so is $K^{(\alpha)}$.

On the other hand, a well-known generalization of the Banach–Stone theorem was given by Holsztyński who showed that if $C_0(K)$ is isometrically isomorphic to a subspace of $C_0(S)$, then K is a continuous image of a subset of S [20]. Jarosz extended this result for extremely regular subspaces of $C_0(K)$ and into isomorphisms with distortion less than 2 [21]. Moreover, in the setting of metric spaces, Jarosz obtained a vector-valued extension of Holsztyński's theorem for isomorphisms from extremely regular subspaces of $C_0(K)$ into $C_0(S, X)$ spaces [14]. Our last theorem is a vector-valued version of the previously mentioned Jarosz's result which is valid for all locally compact Hausdorff spaces. Before stating it, we recall a parameter introduced by Jarosz [14] for all Banach spaces X:

$$\lambda(X) = \inf\{\max\{\|x + \lambda y\|\colon |\lambda| = 1\}\colon \|x\| = \|y\| = 1\}. \tag{3}$$

Theorem 5. *Let $\mathscr{A}(K)$ be an extremely regular subspace of $C_0(K)$ and X a Banach space satisfying $\lambda(X) > 1$. If $T\colon \mathscr{A}(K) \longrightarrow C_0(S, X)$ is an into isomorphism satisfying $\|T\|\|T^{-1}\| < \lambda(X)$, then K is a continuous image of a subset of S.*

Paper is organized as follows: in the first section, we will establish some properties for extremely regular subspaces of $C_0(K)$. In the remaining sections, we will prove Theorems 2, 3, 4, and 5. The idea of the proof of these theorems is inducing a set-valued map from isomorphisms defined on extremely regular subspaces and then improving the arguments given in [19] and [22] in order to obtain the wanted conclusions.

2. On Extremely Regular Subspaces

In this section, we prove some properties about extremely regular spaces. We start by stating the notation used throughout the paper.

The unit ball of a Banach space X is denoted by B_X. We identify dual space $C_0(K)^*$ with the space of regular countably additive bounded measures and denote it by $M(K)$ ([23], p. 222). The space $M(K)$ will be equipped with the *weak** topology inherited from $C_0(K)^*$. The total variation of a measure $\mu \in M(K)$ on a Borel set E is denoted by $|\mu|(E)$ and its norm by $\|\mu\| = |\mu|(K)$. As usual, δ_k denotes the *Dirac measure* at $k \in K$.

Let $k \in K$ be given and $\mathscr{V}_k$ is a fundamental system of open neighborhoods of k. Consider the set $\mathscr{C}_k = \mathscr{V}_k \times (0, \infty)$ and define a partial order in $\mathscr{C}_k$ as follows: $(U, t) \prec (V, s)$ if and only if $V \subset U$ and $s < t$. Note that $(\mathscr{C}_k)$ is a directed set. Let $\mathscr{A}(K)$ be an extremely regular subspace of $C_0(K)$. It is easy to see that there exists a net $(f_{(U,t)})_{(U,t)\in\mathscr{C}_k}$ in $\mathscr{A}(K)$ satisfying

(1) $\|f_{(U,t)}\| = f_{(U,t)}(k) = 1$

(2) $|f_{(U,t)}(w)| < t$ for all $w \in K \backslash U$

We write $\left\{(U, t), f_{(U,t)}\right\}_{(U,t)\in\mathscr{C}_k} \longleftrightarrow \{k\}$ to indicate that above conditions are satisfied.

The next result is proved in ([24], Lemma 1) for compact spaces. The proof also works in the locally compact case, so we restate it as follows.

Lemma 1. *Let $\mathscr{A}(K)$ be an extremely regular subspace of $C_0(K)$ and $k \in K$ given. Suppose that $\left\{(U, t), f_{(U,t)}\right\}_{(U,t)\in\mathscr{C}_k} \longleftrightarrow \{k\}$. If $\mu \in M(K)$, then*

$$\lim_{(U,t)\in\mathscr{C}_k} \int_K f_{(U,t)} \mathrm{d}\mu = \mu(\{k\}). \tag{4}$$

It is well known that if K is infinite, then $C_0(K)$ contains a subspace isometric to c_0. We will prove an analogous result for extremely regular subspaces of $C_0(K)$.

Proposition 1. *Let K be infinite and $\mathscr{A}(K)$ an extremely regular subspace of $C_0(K)$. Then $\mathscr{A}(K)$ contains a subspace isomorphic to c_0.*

Proof. Let (k_n) be a sequence of distinct elements of K and $\{U_n\colon n \in \mathbb{N}\}$ a sequence of open subsets of K satisfying

$U_n \cap U_m = \varnothing$ if $m \neq n$ and $k_n \in U_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $f_n \in \mathscr{A}(K)$ be such that

(1) $f_n(k_n) = \|f_n\| = 1$;

(2) $|f_n(w)| < 1/2^n$, if $w \in K \backslash U_n$.

We claim that (f_n) is a weakly unconditionally Cauchy sequence. Let (a_n) be a bounded sequence and $m \in \mathbb{N}$ fixed. Then

$$\begin{aligned}\alpha_0 &:= \sup\left\{\left|\sum_{j=1}^{m} a_j f_j(k)\right| : k \in K \backslash \bigcup_{j=1}^{m} U_j\right\} \\ &\leq \sum_{j=1}^{m} \frac{|a_j|}{2^j} \leq \sup_{n\in\mathbb{N}} |a_n|.\end{aligned} \tag{5}$$

On the other hand, if $p \in \{1, \dots, m\}$ is given, we have

$$\begin{aligned}\alpha_p &:= \sup\left\{\left|\sum_{j=1}^{m} a_j f_j(k)\right| : k \in U_p\right\} \\ &\leq |a_p| + \sum_{j=1, j\neq p}^{m} \frac{|a_j|}{2^j} \leq 2 \sup_{n\in\mathbb{N}} |a_n|.\end{aligned} \tag{6}$$

From inequalities (5) and (6), it follows that

$$\begin{aligned}\left\|\sum_{j=1}^{m} a_j f_j\right\| &= \sup\left\{\left|\sum_{j=1}^{m} a_j f_j(k)\right| : k \in K\right\} \\ &= \max\{\alpha_0, \alpha_1, \dots, \alpha_m\} \leq 2 \sup_{n\in\mathbb{N}} |a_n|.\end{aligned} \tag{7}$$

Since $m \in \mathbb{N}$ is arbitrary, we have

$$\sup_{n\in\mathbb{N}} \left\|\sum_{j=1}^{n} a_j f_j\right\| \leq 2 \sup_{n\in\mathbb{N}} |a_n|. \tag{8}$$

So, we prove the claim. Now, observe that $\inf\{\|f_n\| : n \in \mathbb{N}\} > 0$. By the well-known Bessaga–Pełczyński theorem ([1], Theorem 2.4.11), we conclude that c_0 embeds in $\mathscr{A}(K)$. □

3. On Isomorphisms from Extremely Regular Spaces into $C_0(S, X)$ Spaces

The aim of this section is proving Theorem 3, and as a consequence, we give a proof of Theorem 2 which was proved by Cengiz under additional topological hypothesis. First, we state two lemmas. A proof of the following one can be found in ([25], Proposition 5.1).

Lemma 2. *Let X be a Banach space. If $x, y \in X$ satisfy $\min\{\|x\|, \|y\|\} \geq \eta$ for some $\eta > 0$, then there are $a_1, a_2 \in \mathbb{K}$ with $\max\{|a_1|, |a_2|\} \leq 1$ such that $\|a_1 x + a_2 y\| \geq \eta\lambda(X)$.*

The following result was proved for compact spaces in ([24], Theorem 2.4). By modifying slightly the proof showed in [24], we see that result also holds in the locally compact case.

Theorem 6. *Let $\mathscr{A}(K)$ be an extremely regular subspace of $C_0(K)$ and $T\colon \mathscr{A}(K) \longrightarrow C_0(S)$ be an into isomorphism. For each $k \in K$, we have*

$$\sup\left\{\left|T^* \delta_s(\{k\})\right| : s \in S\right\} \geq \frac{1}{\|T\|\|T^{-1}\|}. \tag{9}$$

To go on, we introduce a set-valued map which plays a fundamental role in our proofs.

Notation 1. Let $\mathscr{A}(K)$ be an extremely regular space and $T\colon \mathscr{A}(K) \longrightarrow C_0(S, X)$ an into isomorphism. For any $r > 0$, we define a set-valued map from S to K as follows:

$$\Omega(k, r) = \left\{s \in S : \left|T^*(x^* \cdot \delta_s)(\{k\})\right| \geq r \text{ for some } x^* \in B_{X^*}\right\}, \tag{10}$$

where $x^* \cdot \delta_s \in C_0(S, X)^*$ is defined by $(x^* \cdot \delta_s)(g) = x^*(g(s))$ for all $g \in C_0(S, X)$.

Given a set-valued map Λ from K to S, for any $F \subset S$, we define

$$\Lambda^{-1}(F) = \{k \in K : \Lambda(k) \cap F \neq \varnothing\}. \tag{11}$$

In the case where $F = \{s\}$ is a singleton, we write $\Lambda^{-1}(s)$ instead of $\Lambda^{-1}(\{s\})$.

Lemma 3. *Let X be a Banach space containing no copy of c_0 and $\mathscr{A}(K)$ an extremely regular space of $C_0(K)$. Let $T\colon \mathscr{A}(K) \longrightarrow C_0(S, X)$ be an into isomorphism. For any $r > 0$ and $s \in S$, $\Omega_r^{-1}(s)$ is finite.*

Proof. Suppose that there is $s_0 \in S$ such that $\Omega_r^{-1}(s_0)$ is infinite and take $\{k_n : n \in \mathbb{N}\}$ a sequence of distinct elements of $\Omega_r^{-1}(s_0)$. For each $n \in \mathbb{N}$, there is $x_n^* \in B_{X^*}$ such that

$$\left|T^*\left(x_n^* \cdot \delta_{s_0}\right)(\{k_n\})\right| \geq r. \tag{12}$$

Fix $a \in \mathbb{R}$ with $0 < a < r$. By Lemma 1, if $\left\{(U, t), f_{(U,t)}\right\}_{(U,t)\in\mathscr{C}_{k_1}} \longleftrightarrow \{k_1\}$ we have

$$\left|\lim_{(U,t)\in\mathscr{C}_{k_1}} \int_K f_{(U,t)} \mathrm{d}T^*\left(x_1^* \cdot \delta_{s_0}\right)\right| = \left|T^*\left(x_1^* \cdot \delta_{s_0}\right)(\{k_1\})\right| \geq r > a. \tag{13}$$

So, there exists $(U^1, t_1) \in \mathscr{C}_{k_1}$ such that if $(U, t) \in \mathscr{C}_{k_1}$ and $(U^1, t_1) \prec (U, t)$ then

$$\left|\int_K f_{(U,t)} \mathrm{d}T^*\left(x_1^* \cdot \delta_{s_0}\right)\right| > a. \tag{14}$$

Set $U_1 := U^1$ and choose $n_1 \in \mathbb{N}$ satisfying $1/2^{n_1} < t_1$. Thus, $(U^1, t_1) \prec (U_1, 1/2^{n_1})$ and

$$\begin{aligned}\left\|T f_{(U_1,1/2^{n_1})}(s_0)\right\| &\geq \left|x_1^*\left(T f_{(U_1,1/2^{n_1})}(s_0)\right)\right| \\ &= \left|\int_K f_{(U_1,1/2^{n_1})} \mathrm{d}T^*\left(x_1^* \cdot \delta_{s_0}\right)\right| > a.\end{aligned} \tag{15}$$

By setting $f_1 := f_{(U_1,1/2^{n_1})}$, we have

(1) $\|f_1\| = f_1(k_1) = 1$

(2) $|f_1(w)| < 1/2^{n_1}$ for all $w \in K \backslash U_1$

(3) $\|Tf_1(s_0)\| > a$

Now by induction, assume that there are disjoint open subsets $U_1, \ldots, U_m$ of K, $f_1, \ldots, f_m \in \mathscr{A}(K)$, and $n_1, n_2, \ldots, n_m \in \mathbb{N}$ with $n_1 < n_2 < \cdots < n_m$ satisfying

(1) $\|f_j\| = f_1(k_j) = 1$

(2) $|f_j(w)| < 1/2^{n_j}$ for all $w \in K \backslash U_j$

(3) $\|Tf_j(s_0)\| > a$

for all $j \in \{1, \ldots, m\}$. Hence, by Lemma 1, if $\{(U,t), f_{(U,t)}\}_{(U,t)\in\mathscr{C}_{k_{m+1}}} \longleftrightarrow \{k_{m+1}\}$, we have

$$\left|\lim_{(U,t)\in\mathscr{C}_{k_{m+1}}} \int_K f_{(U,t)} \mathrm{d}T^*\left(x^*_{m+1} \cdot \delta_{s_0}\right)\right| = \left|T^*\left(x^*_{m+1} \cdot \delta_{s_0}\right)(\{k_{m+1}\})\right| \geq r > a. \tag{16}$$

Thus, there exists $(U^{m+1}, t_{m+1}) \in \mathscr{C}_{k_{m+1}}$ such that if $(U,t) \in \mathscr{C}_{k_{m+1}}$ and $(U^{m+1}, t_{m+1}) \prec (U,t)$, then

$$\left|\int_K f_{(U,t)} \mathrm{d}T^*\left(x^*_{m+1} \cdot \delta_{s_0}\right)\right| > a. \tag{17}$$

Let $V_{m+1} \subset K$ be open with $k_{m+1} \in V_{m+1}$ and $V_{m+1} \cap U_j = \varnothing$ for all $j \in \{1, \ldots, m\}$ and set $U_{m+1} = V_{m+1} \cap U^{m+1}$. If $n_{m+1} \in \mathbb{N}$ satisfies $n_m < n_{m+1}$ and $1/2^{n_{m+1}} < t_{m+1}$, then $(U^{m+1}, t_{m+1}) \prec (U_{m+1}, 1/2^{n_{m+1}})$ and

$$\begin{aligned}\left\|Tf_{(U_{m+1},1/2^{n_{m+1}})}(s_0)\right\| &\geq \left|x^*_{m+1}\left(Tf_{(U_{m+1},1/2^{n_{m+1}})}(s_0)\right)\right| \\ &= \left|\int_K f_{(U_{m+1},1/2^{n_{m+1}})} \mathrm{d}T^*\left(x^*_{m+1} \cdot \delta_{s_0}\right)\right| > a.\end{aligned} \tag{18}$$

Finally, we set $f_{m+1} = f_{(U_{m+1},1/2^{n_{m+1}})}$. Thus

(1) $\|f_{m+1}\| = f_1(k_{m+1}) = 1$

(2) $|f_{m+1}(w)| < 1/2^{n_{m+1}}$ for all $w \in K \backslash U_{m+1}$

(3) $\|Tf_{m+1}(s_0)\| > a$

Hence, there exists a sequence of mutually disjoint open subsets $\{U_m : m \in \mathbb{N}\}$ of K, a sequence $\{f_m : m \in \mathbb{N}\}$ in $\mathscr{A}(K)$, and a strictly increasing sequence of natural numbers $\{n_m : m \in \mathbb{N}\}$ such that

(1) $\|f_m\| = f_1(k_m) = 1$

(2) $|f_m(w)| < 1/2^{n_m}$ for all $w \in K \backslash U_m$

(3) $\|Tf_m(s_0)\| > a$

for all $m \in \mathbb{N}$. By arguing as in Proposition 1, it is easy to check that $(Tf_n(s_0))$ is a weakly unconditional Cauchy sequence. Since

$$\inf\left\{\left\|Tf_n(s_0)\right\| : n \in \mathbb{N}\right\} \geq a > 0, \tag{19}$$

we conclude that c_0 embeds in X by ([1], Theorem 2.4.11), contradicting the hypothesis. Thus, $\Omega_r^{-1}(s)$ must be finite for each $s \in S$. □

Lemma 4. *If $0 < r < 1/\|T\|\|T^{-1}\|$, then for all $k \in K$, we have $\Omega(k,r) \neq \varnothing$.*

Proof. Notice that $C_0(S,X)$ embeds isometrically in $C_0(S \times B_{X^*})$ via the isomorphism $f \longmapsto \widehat{f}$, where $\widehat{f}(s,x^*) = x^*(f(s))$ for all $(s,x^*) \in S \times B_{X^*}$. Let $\widehat{T} : \mathscr{A}(K) \longrightarrow C_0(S \times B_{X^*})$ be the into isomorphism given by $\widehat{T}(f) = \widehat{Tf}$ for $f \in \mathscr{A}(K)$. By Theorem 6, we have

$$\sup\left\{\left|\widehat{T}^*\delta_{(s,x^*)}(\{k\})\right| : (s,x^*) \in S \times B_{X^*}\right\} \geq \frac{1}{\|\widehat{T}\|\left\|\widehat{T}^{-1}\right\|}. \tag{20}$$

Since $\|\widehat{T}^{-1}\| \leq \|T^{-1}\|$ and $\|T\| = \|\widehat{T}\|$, it follows that

$$\sup\left\{\left|\widehat{T}^*\delta_{(s,x^*)}(\{k\})\right| : (s,x^*) \in S \times B_{X^*}\right\} \geq \frac{1}{\|T\|\|T^{-1}\|}. \tag{21}$$

Thus, there is $(s,x^*) \in S \times B_{X^*}$, satisfying $|\widehat{T}^*\delta_{(s,x^*)}(\{k\})| \geq 1/\|T\|\|T^{-1}\| > r$.

Now, if $\{(U,t), f_{(U,t)}\}_{(U,t)\in\mathscr{C}_k} \longleftrightarrow \{k\}$, from Lemma 1 we obtain

$$\begin{aligned}\widehat{T}^*\delta_{(s,x^*)}(\{k\}) &= \lim_{(U,t)\in\mathscr{C}_k} \widehat{T}^*\delta_{(s,x^*)}\left(f_{(U,t)}\right) \\ &= \lim_{(U,t)\in\mathscr{C}_k} x^*\left(Tf_{(U,t)}(s)\right) \\ &= \lim_{(U,t)\in\mathscr{C}_k} T^*\left(x^* \cdot \delta_s\right)\left(f_{(U,t)}\right) \\ &= T^*\left(x^* \cdot \delta_s\right)(\{k\}).\end{aligned} \tag{22}$$

Whence, there are $s \in S$ and $x^* \in B_{X^*}$ such that $|T^*(x^* \cdot \delta_s)(\{k\})| \geq r$; that is, $\Omega(k,r) \neq \varnothing$. □

Proof of Theorem 1. Let $T : \mathscr{A}(K) \longrightarrow C_0(S,X)$ be an into isomorphism. We claim that S is infinite. Indeed, if S were finite, say $|S| = n$, then $C_0(S,X) \cong X^n$. Since K is infinite, $c_0 \hookrightarrow \mathscr{A}(K)$ by Proposition 1. Whence, c_0 embeds in X^n and by ([26], Theorem 1) we conclude that $c_0 \hookrightarrow X$ which is impossible. So, S must be infinite. Now, let $0 < r < 1/\|T\|\|T^{-1}\|$ be fixed. If $k \in K$ is given, by Lemma 4, we have $\Omega(k,r) \neq \varnothing$. By taking $s_0 \in \Omega(k,r)$, we obtain that $k \in \Omega_r^{-1}(s_0)$. Thus,

$$K = \bigcup_{s\in S} \Omega_r^{-1}(s). \tag{23}$$

On the other hand, by Lemma 3, $\Omega_r^{-1}(s)$ is finite for all $s \in S$. Since S is infinite, the equation above implies that

$$|K| = \left|\bigcup_{s\in S} \Omega_r^{-1}(s)\right| \leq \left(\sup_{s\in S}\left|\Omega_r^{-1}(s)\right|\right)|S| \leq \aleph_0|S| = |S|. \tag{24}$$

Thus, $|K| \leq |S|$. □

Proof of Theorem 2. Suppose that $\mathscr{A}(K) \sim \mathscr{B}(S)$, then $\mathscr{A}(K) \hookrightarrow C_0(S)$ and $\mathscr{B}(S) \hookrightarrow C_0(K)$. If K were finite, then S also must be finite by Theorem 3. By ([15], Main theorem), we have $\mathscr{A}(K) = C_0(K) \cong \mathbb{R}^n$ and $\mathscr{B}(S) = C_0(S) \cong \mathbb{R}^m$, where $|K| = n$ and $|S| = m$. So, $m = n$. If K and S are infinite, then by Theorem 2, we have $|K| \leq |S|$ and $|S| \leq |K|$. Thus, by the Cantor–Schröder–Bernstein theorem, we conclude that $|K| = |S|$. □

4. On Isomorphisms from Extremely Regular Spaces into $C_0(S,X)$ Spaces with Distortion Less than 3

In this section, we prove Theorem 4. We need the following lemma for isomorphisms from extremely regular subalgebras of $C_0(K)$ into $C_0(S,X)$ spaces. The proof follows by adapting the argument used in ([19], Proposition 3.1) for extremely regular subalgebras. First, we state a notation.

Notation 2. For $f \in C_0(K,X)$ and $r>0$, we denote

$$\mathscr{K}(f,r)=\{k \in K\colon \|f(k)\| \geq r\}. \tag{25}$$

Lemma 5. *Let $\mathscr{A}(K)$ be an extremely regular subalgebra of $C_0(K)$. Let $T\colon \mathscr{A}(K) \longrightarrow C_0(S,X)$ be an into isomorphism with $\|T\|\|T^{-1}\|<3$ and fix $\delta \in (0,1)$ such that $\|T\|\|T^{-1}\|<3\delta$. If $h,g \in \mathscr{A}(K)$ satisfy $0 \leq h \leq g \leq 1$ and $\|h|_{K^{(\alpha)}}\|>\delta$, then*

$$\bigcap_{h\leq f\leq g} \mathscr{K}\left(Tf,\eta/\left\|T^{-1}\right\|\right) \cap S^{(\alpha)} \neq \varnothing, \tag{26}$$

where

$$\eta=\frac{3\delta-\|T\|\left\|T^{-1}\right\|}{2}. \tag{27}$$

Proof of Theorem 3. Let α be an ordinal and $0<\delta<1$ satisfying $\|T\|\|T^{-1}\|<3\delta$. Also let η be as in Lemma 5. We shall use the fact that closed subalgebras of $C_0(K)$ are lattices ([23], Lemma 4.48). Consider the collection

$$\mathscr{F}_k=\{f \in \mathscr{A}(K)\colon 0 \leq f \leq 1,\ f(k)>\delta\}, \tag{28}$$

for each $k \in K$ and define

$$\Lambda_k=\bigcap_{f\in\mathscr{F}_k} \mathscr{K}\left(Tf,\frac{\eta}{\|T^{-1}}\right). \tag{29}$$

Observe that

$$\left\{\mathscr{K}\left(Tf,\frac{\eta}{\|T^{-1}}\right)\cap S^{(\alpha)}\colon f \in \mathscr{F}_k\right\}, \quad \text{for } k \in K^{(\alpha)}, \tag{30}$$

is family of compact subsets of S. So, if we prove that this family has the finite intersection property, we will immediately conclude that $\Lambda_k \cap S^{(\alpha)} \neq \varnothing$ for $k \in K^{(\alpha)}$.

Let $f_1,\dots,f_n \in \mathscr{F}_k$ then $h=\min_{1\leq j\leq n} f_j$ and $g=\max_{1\leq j\leq n} f_j$ are in $\mathscr{F}_k$. So, by Lemma 5, we have

$$\bigcap_{j=1}^{n} \mathscr{K}\left(Tf_j,\frac{\eta}{\|T^{-1}}\right)\cap S^{(\alpha)} \bigcap_{h\leq f\leq g} \mathscr{K}\left(Tf,\frac{\eta}{\|T^{-1}\|}\right)\cap S^{(\alpha)} \neq \varnothing. \tag{31}$$

Thus by compactness, the family $\{\mathscr{K}(Tf,\eta/\|T^{-1})\cap S^{(\alpha)}\colon f \in \mathscr{F}_k\}$ has nonempty intersection. It follows that

$$\Lambda_k \cap S^{(\alpha)}=\bigcap_{f\in\mathscr{F}_k} \mathscr{K}\left(Tf,\frac{\eta}{\|T^{-1}}\right)\cap S^{(\alpha)} \neq \varnothing. \tag{32}$$

Now, consider the set-valued map from K to S given by $k \longmapsto \Lambda_k$. We will prove that $\Lambda^{-1}(s)$ is finite for all $s \in S$. If not, there is $s_0 \in S$ such that $\Lambda^{-1}(s_0)$ is infinite. Since K is regular, there is a sequence $\{U_n\colon n \in \mathbb{N}\}$ of open disjoint subsets of K such that $U_n \cap \Lambda^{-1}(s_0) \neq \varnothing$. For each $n \in \mathbb{N}$, pick $k_n \in U_n \cap \Lambda^{-1}(s_0)$. Also, for each $n \in \mathbb{N}$, let $f_n \in \mathscr{A}(K)$ be such that $0 \leq f_n \leq 1$, $f_n(k_n)=1$, and $|f_n(w)|<1/2^n$ for all $w \in K\backslash U_n$. Note that $f_n \in \mathscr{F}_{k_n}$ and in view of $k_n \in \Lambda^{-1}(s_0)$, we have $s_0 \in \Lambda_{k_n}$ for each $n \in \mathbb{N}$. Whence,

$$\inf\left\{\left\|Tf_n(s_0)\right\|\colon n \in \mathbb{N}\|\right\} \geq \frac{\eta}{\|T^{-1}\|}>0. \tag{33}$$

It is not difficult to show that $(Tf_n(s_0))$ is weakly unconditionally convergent. By (33) and ([1], Theorem 2.4.11), we conclude that c_0 embeds in X which is impossible. Henceforth, $\Lambda^{-1}(s)$ is finite for all $s \in S$.

Now, if $K^{(\alpha)} \neq \varnothing$ and $k \in K^{(\alpha)}$, we have $\Lambda_k \cap S^{(\alpha)} \neq \varnothing$ which implies that $k \in \Lambda^{-1}(s)$ where $s \in \Lambda_k \cap S^{(\alpha)}$ Thus,

$$K^{(\alpha)} \subset \bigcup_{s\in S^{(\alpha)}} \Lambda^{-1}(s). \tag{34}$$

Consequently, if $S^{(\alpha)}$ is finite, then so is $K^{(\alpha)}$. Now if $S^{(\alpha)}$ is infinite, from the equation above, we have

$$\left|K^{(\alpha)}\right| \leq \left|\bigcup_{s\in S^{(\alpha)}} \Lambda^{-1}(s)\right| \leq \left(\sup_{s\in S^{(\alpha)}}\left|\Lambda^{-1}(s)\right|\right)\left|S^{(\alpha)}\right| \leq \aleph_0\left|S^{(\alpha)}\right|=\left|S^{(\alpha)}\right|, \tag{35}$$

that is $|K^{(\alpha)}| \leq |S^{(\alpha)}|$, and we are done. □

5. On Isomorphisms from Extremely Regular Spaces into $C_0(S,X)$ Spaces with Distortion Less than $\lambda(X)$

Finally, we prove Theorem 5. We remark that Jarosz obtained the statement of theorem in the metric setting ([14], Theorem 1). By following a new approach, we show that conclusion of theorem holds in the general topological case.

Proof of Theorem 4. Let $T\colon \mathscr{A}(K) \longrightarrow C_0(S,X)$ be a into isomorphism with $\|T\|\|T^{-1}\|<\lambda(X)$. By considering the isomorphism $A=T/\|T\|$, we have $\|A\|=1$ and $\|A^{-1}\|=\|T\|\|T^{-1}\|<\lambda(X)$. So, we may assume that $\|T\|=1$ and $\|T^{-1}\|<\lambda(X)$. Let $r \in \mathbb{R}$ such that

$$\frac{1}{\lambda(X)}<r<\frac{1}{\|T^{-1}\|}. \tag{36}$$

The following claim is a consequence of Lemma 4. □

Claim 1. For each $k \in K$, we have $\Omega(k,r) \neq \varnothing$.

Claim 2. For $k_1,k_2 \in K$ with $k_1 \neq k_2$, we have $\Omega(k_1,r)\cap\Omega(k_2,r)=\varnothing$.

Suppose that $\Omega(k_1,r)\cap\Omega(k_2,r) \neq \varnothing$ and let $s \in \Omega(k_1,r)\cap\Omega(k_2,r)$. There are $x_1^*,x_2^* \in B_{X^*}$ satisfying

$$\begin{aligned}\left|T^*\left(x_1^*\cdot\delta_s\right)(\{k_1\})\right| &\geq r,\\ \left|T^*\left(x_2^*\cdot\delta_s\right)(\{k_2\})\right| &\geq r.\end{aligned} \tag{37}$$

Let $0<\varepsilon<1$ be given, then

$$\begin{aligned}&\left|T^*\left(x_1^*\cdot\delta_s\right)(\{k_1\})\right|>r-\varepsilon,\\&\left|T^*\left(x_2^*\cdot\delta_s\right)(\{k_2\})\right|>r-\varepsilon.\end{aligned}\tag{38}$$

If $\left\{(U^1,t),f_{(U^1,t)}\right\}_{(U^1,t)\in\mathscr{C}_{k_1}}\longleftrightarrow\{k_1\}$ and $\left\{(U^2,t),f_{(U^2,t)}\right\}_{(U^2,t)\in\mathscr{C}_{k_2}}\longleftrightarrow\{k_2\}$, it follows from Lemma 1 that

$$\begin{aligned}&\lim_{(U^1,t)\in\mathscr{C}_{k_1}}\left|T^*\left(x_1^*\cdot\delta_s\right)\left(f_{(U^1,t)}\right)\right|>r-\varepsilon,\\&\lim_{(U^2,t)\in\mathscr{C}_{k_2}}\left|T^*\left(x_2^*\cdot\delta_s\right)\left(f_{(U^2,t)}\right)\right|>r-\varepsilon.\end{aligned}\tag{39}$$

Thus, there is $(U_0^1,t_1)\in\mathscr{C}_{k_1}$ such that if $(U^1,t)\in\mathscr{C}_{k_1}$ and $(U_0^1,t_1)\prec(U^1,t)$, then

$$\left|T^*\left(x_1^*\cdot\delta_s\right)\left(f_{(U^1,t)}\right)\right|>r-\varepsilon.\tag{40}$$

In a similar way, there exists $(U_0^2,t_2)\in\mathscr{C}_{k_2}$ such that if $(U^2,t)\in\mathscr{C}_{k_2}$ and $(U_0^2,t_2)\prec(U^2,t)$, we have

$$\left|T^*\left(x_2^*\cdot\delta_s\right)\left(f_{(U^2,t)}\right)\right|>r-\varepsilon.\tag{41}$$

Choose U_1 and U_2 open subsets of K with $k_1\in U_1\subset U_0^1$, $k_2\in U_2\subset U_0^2$, and $U_1\cap U_2=\varnothing$, and let $t\in(0,\infty)$ be such that $t<\min\{t_1,t_2,\varepsilon\}$. Then, $(U_0^1,t_1)\prec(U_1,t)$ and $(U_0^2,t_2)\prec(U_2,t)$. In view of inequalities (40) and (41), we obtain

$$\begin{aligned}&\left|T^*\left(x_1^*\cdot\delta_s\right)\left(f_{(U_1,t)}\right)\right|>r-\varepsilon,\\&\left|T^*\left(x_2^*\cdot\delta_s\right)\left(f_{(U_2,t)}\right)\right|>r-\varepsilon.\end{aligned}\tag{42}$$

Hence,

$$\begin{aligned}&\left\|T\left(f_{(U_1,t)}\right)(s)\right\|>r-\varepsilon,\\&\left\|T\left(f_{(U_2,t)}\right)(s)\right\|>r-\varepsilon.\end{aligned}\tag{43}$$

By Lemma 2, there are $a_1,a_2\in\mathbb{K}$ with $\max\{|a_1|,|a_2|\}\le1$ satisfying

$$\left\|a_1T\left(f_{(U_1,t)}\right)(s)+a_2T\left(f_{(U_2,t)}\right)(s)\right\|>(r-\varepsilon)\lambda(X).\tag{44}$$

Since $\|a_1f_{(U_1,t)}+a_2f_{(U_2,t)}\|\le1+\varepsilon$ and $\|T\|=1$, we infer that

$$\begin{aligned}1+\varepsilon&\ge\left\|T\left(a_1f_{(U_1,t)}+a_2f_{(U_2,t)}\right)\right\|\\&\ge\left\|a_1T\left(f_{(U_1,t)}\right)(s)+a_2T\left(f_{(U_2,t)}\right)(s)\right\|\\&>(r-\varepsilon)\lambda(X).\end{aligned}\tag{45}$$

Arbitrariness of $\varepsilon>0$ implies that $1\ge r\lambda(X)$ which contradicts (36). So, we are done.

Now, let $S_0=\bigcup_{k\in K}\Omega(k,r)$ and define $\phi\colon S_0\longrightarrow K$ by $\phi(s)=k$ if and only if $s\in\Omega(k,r)$. Claims 1 and 2 show that ϕ is a surjective function.

Claim 3. The function ϕ is continuous.

Let $(s_\gamma)_{\gamma\in\Gamma}$ be a net in S_0 converging to $s\in S_0$ and assume that $\phi(s_\gamma)=k_\gamma\not\longrightarrow\phi(s)=k$. Let $0<\varepsilon<1$ be given. Since $\phi(s)=k$, there is $x^*\in B_{X^*}$ such that

$$\left|T^*\left(x^*\cdot\delta_s\right)(\{k\})\right|\ge r>r-\varepsilon.\tag{46}$$

Now, if $\left\{(U,t),f_{(U,t)}\right\}_{(U,t)\in\mathscr{C}_k}\longleftrightarrow\{k\}$, we have by Lemma 1

$$\lim_{(U,t)\in\mathscr{C}_k}\left|T^*\left(x^*\cdot\delta_s\right)\left(f_{(U,t)}\right)\right|>r-\varepsilon.\tag{47}$$

Since $k_\gamma\not\longrightarrow k$, there is a compact neighborhood $V\subset K$ with $k\in V$ such that for any $\gamma\in\Gamma$, there exists $\gamma'\ge\gamma$ satisfying $k_{\gamma'}\notin V$. In view of (47), there is $U_1\subset K$ open with $k\in U_1\subset V$ and $0<t_1<\varepsilon$ such that

$$\left\|T\left(f_{(U_1,t_1)}\right)(s)\right\|\ge\left|T^*\left(x^*\cdot\delta_s\right)\left(f_{(U_1,t_1)}\right)\right|>r-\varepsilon,\tag{48}$$

whereas $s_\gamma\longrightarrow s$, there is $\gamma_1\in\Gamma$ such that $k_{\gamma_1}\notin V$ and

$$\left\|T\left(f_{(U_1,t_1)}\right)\left(s_{\gamma_1}\right)\right\|>r-\varepsilon.\tag{49}$$

Because of $\phi(s_{\gamma_1})=k_{\gamma_1}$, there is $x_1^*\in B_{X^*}$ such that

$$\left|T^*\left(x_1^*\cdot\delta_{s_{\gamma_1}}\right)\left(\left\{k_{\gamma_1}\right\}\right)\right|\ge r>r-\varepsilon.\tag{50}$$

So, if $\left\{(U^1,t),f_{(U^1,t)}\right\}_{(U^1,t)\in\mathscr{C}_{k_{\gamma_1}}}\longleftrightarrow\left\{k_{\gamma_1}\right\}$, by Lemma 1 we get

$$\lim_{(U^1,t)\in\mathscr{C}_{k_{\gamma_1}}}\left|T^*\left(x_1^*\cdot\delta_{s_{\gamma_1}}\right)\left(f_{(U^1,t)}\right)\right|>r-\varepsilon.\tag{51}$$

From (51), there is $U_2\subset K$ open with $k_{\gamma_1}\in U_2$, $U_2\cap V=\varnothing$, and $0<t_2<\varepsilon$ satisfying

$$\left\|T\left(f_{(U_2,t_2)}\right)\left(s_{\gamma_1}\right)\right\|\ge\left|T^*\left(x_2^*\cdot\delta_{s_{\gamma_1}}\right)\left(f_{(U_2,t_2)}\right)\right|>r-\varepsilon.\tag{52}$$

It follows from (49), (52), and Lemma 2 that there exist $a_1,a_2\in\mathbb{K}$ satisfying $\max\{|a_1|,|a_2|\}\le1$ and

$$\left\|a_1T\left(f_{(U_1,t_1)}\right)\left(s_{\gamma_1}\right)+a_2T\left(f_{(U_2,t_2)}\right)\left(s_{\gamma_1}\right)\right\|>(r-\varepsilon)\lambda(X).\tag{53}$$

Notice that $\|a_1f_{(U_1,t_1)}+a_2f_{(U_2,t_2)}\|\le1+\varepsilon$, so, from the inequality above, we obtain

$$\begin{aligned}1+\varepsilon&\ge\left\|T\left(a_1f_{(U_1,t_1)}+a_2f_{(U_2,t_2)}\right)\right\|\\&\ge\left\|a_1T\left(f_{(U_1,t_1)}\right)\left(s_{\gamma_1}\right)+a_2T\left(f_{(U_2,t_2)}\right)\left(s_{\gamma_1}\right)\right\|\\&>(r-\varepsilon)\lambda(X).\end{aligned}\tag{54}$$

Since $0<\varepsilon<1$ was chosen arbitrarily, we have $1\ge r\lambda(X)$ which is impossible by (36). This completes the proof of Theorem 4.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The second author thanks Programa de Movilidad de la Universidad Industrial de Santander for supporting this work (request n°2958).

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