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Research Article

A Subclass of Bi-Univalent Functions Defined by Generalized Sãlãgean Operator Related to Shell-Like Curves Connected with Fibonacci Numbers

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The aim of this paper is to study certain subclasses of bi-univalent functions defined by generalized Sãlãgean differential operator related to shell-like curves connected with Fibonacci numbers. We find estimates of the initial coefficients $|a_2|$ and $|a_3|$ and upper bounds for the Fekete-Szegö functional for the functions in this class. The results proved by various authors follow as particular cases.

1. Introduction and Preliminaries

Let A be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$ with normalization $f(0) = f'(0) - 1 = 0$. By S, we denote the class of functions $f(z) \in A$ and univalent in U.

Let us denote by B the class of bounded or Schwarz functions $w(z)$ satisfying $w(0) = 0$ and $|w(z)| \leq 1$ which are analytic in the open unit disc U and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in U. \tag{2}$$

Consider two functions f and g analytic in U. We say that f is subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z) \in B$ for which $f(z) = g(u(z))$. This result is known as principle of subordination.

By S^*, we denote the class of starlike functions $f \in S$ which satisfies the following condition:

$$\begin{gathered} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \\ \text{or } \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, \\ (z \in U). \end{gathered} \tag{3}$$

By K, we denote the class of convex functions $f \in S$ which satisfies the following condition:

$$\begin{gathered} \operatorname{Re}\left(\frac{(zf'(z))'}{f'(z)}\right) > 0 \\ \text{or } \frac{(zf'(z))'}{f'(z)} \prec \frac{1+z}{1-z}, \\ (z \in U). \end{gathered} \tag{4}$$

A function $f \in S$ is said to be α-convex if it satisfies the inequality

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\frac{\left(zf'(z)\right)'}{f'(z)}\right)>0 \quad (0\le\alpha\le 1,\ z\in U). \tag{5}$$

The class of α-convex functions is denoted by $M(\alpha)$ and was introduced by Mocanu [1]. In particular $M(0) \equiv S^*$ and $M(1) \equiv K$.

For $\delta \geq 1$ and $f \in A$, Al-Oboudi [2] introduced the following differential operator:

$$\begin{aligned} D_\delta^0 f(z) &= f(z), \\ D_\delta^1 f(z) &= (1-\delta) f(z) + \delta z f'(z), \end{aligned} \tag{6}$$

and in general,

$$\begin{aligned} D_\delta^n f(z) &= D\left(D_\delta^{n-1} f(z)\right) \\ &= (1-\delta) D_\delta^{n-1} f(z) + \delta z \left(D_\delta^{n-1} f(z)\right)', \quad n\in N \end{aligned} \tag{7}$$

or equivalent to

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1+(k-1)\delta]^n a_k z^k, \quad n\in N_0 = N\cup\{0\} \tag{8}$$

with $D_\delta^n f(0) = 0$. It is obvious that, for $\delta = 1$, the operator $D_\delta^n f(z)$ is equivalent to the Sălăgean operator introduced in [3]. So the operator $D_\delta^n f(z)$ is named as the Generalized Sălăgean operator.

The inverse functions of the functions in the class S may not be defined on the entire unit disc U although the functions in the class S are invertible. However using Koebe-one quarter theorem [4] it is obvious that the image of U under every function $f \in S$ contains a disc of radius 1/4. Hence every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \ (z\in U) \tag{9}$$

and

$$f\left(f^{-1}(w)\right) = w\left(|w| < r_0(f) : r_0(f) \geq \frac{1}{4}\right) \tag{10}$$

where

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ &\quad - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \end{aligned} \tag{11}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U.

By Σ, we denote the class of bi-univalent functions in U defined by (1).

Lewin [5] discussed the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [6] investigated certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients.

Sokól [7] introduced the class SL of shell-like functions $f \in A$ defined as below.

Definition 1. A function $f \in A$ given by (1) is said to be in the class SL of starlike shell-like functions if it satisfies the following condition:

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{12}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

It should be observed that SL is a subclass of the class S^* of starlike functions.

Later Dziok et al. [8] introduced the class KSL of convex functions related to a shell-like curve as below.

Definition 2. A function $f \in A$ given by (1) is said to be in the class KSL of convex shell-like functions if it satisfies the condition that

$$1+\frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{13}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

Again Dziok et al. [9] defined the following class of α-convex shell-like functions.

Definition 3. A function $f \in A$ given by (1) is said to be in the class SLM_α of α-convex shell-like functions if it satisfies the condition that

$$\begin{aligned} (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right) &\prec \tilde{p}(z) \\ &= \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \end{aligned} \tag{14}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

Obviously $SLM_0 \equiv SL$ and $SLM_1 \equiv KSL$.

The function $\tilde{p}$ is not univalent in U, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|}, \tag{15}$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\widetilde{p}$ is a curve described by the equation given by

$$\left(10x - \sqrt{5}\right) y^2 = \left(\sqrt{5} - 2x\right)\left(\sqrt{5}x - 1\right)^2, \tag{16}$$

which is translated and revolved trisectrix of Maclaurin. The curve $\widetilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1+\tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n\tau + u_{n-1}. \tag{17}$$

Also the subclasses of bi-univalent functions related to shell-like curves were studied by various authors [10–12].

The earlier work on bi-univalent functions related to shell-like curves connected with Fibonacci numbers motivated us to define the following subclass.

To avoid repetition, throughout the paper we assume that $0 \leq \alpha \leq 1$, $\tau = (1-\sqrt{5})/2 \approx -0.618$ and $z \in U$.

Definition 4. A function $f \in \Sigma$ given by (1) is said to be in the class $SLM^{\delta}_{\alpha,\Sigma}(n, \widetilde{p}(z))$ if it satisfies the following conditions:

$$(1-\alpha)\frac{D^{n+1}_{\delta} f(z)}{D^{n}_{\delta} f(z)} + \alpha\frac{\left(D^{n+1}_{\delta} f(z)\right)'}{\left(D^{n}_{\delta} f(z)\right)'} \prec \widetilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \tag{18}$$

and

$$(1-\alpha)\frac{D^{n+1}_{\delta} g(w)}{D^{n}_{\delta} g(w)} + \alpha\frac{\left(D^{n+1}_{\delta} g(w)\right)'}{\left(D^{n}_{\delta} g(w)\right)'} \prec \widetilde{p}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \tag{19}$$

where $g = f^{-1}$ and $z, w \in U$.

The following observations are obvious:

(i) $SLM^{1}_{\alpha,\Sigma}(n, \widetilde{p}(z)) \equiv SLM_{\alpha,\Sigma}(n, \widetilde{p}(z))$, the class of bi-univalent functions defined by Sălăgean operator related to shell-like curves.

(ii) $SLM^{1}_{\alpha,\Sigma}(0, \widetilde{p}(z)) \equiv SLM_{\alpha,\Sigma}(\widetilde{p}(z))$, the class of bi-univalent α-convex shell-like functions studied by Güney et al. [13].

(iii) $SLM^{1}_{0,\Sigma}(0, \widetilde{p}(z)) \equiv SL_{\Sigma}(\widetilde{p}(z))$.

(iv) $SLM^{1}_{1,\Sigma}(0, \widetilde{p}(z)) \equiv KSL_{\Sigma}(\widetilde{p}(z))$.

In this paper, we study the class $SLM^{\delta}_{\alpha,\Sigma}(n, \widetilde{p}(z))$ and obtain estimates of the initial coefficients $|a_2|$ and $|a_3|$ and upper bounds for the Fekete-Szegö functional for the functions in this class.

2. Coefficient Bounds for the Function Class $SLM^{\delta}_{\alpha,\Sigma}(n,\widetilde{p}(z))$

For deriving our results, we need the following lemma.

Lemma 5 (see [14]). *If $p \in P$ is family of all functions p analytic in U for which $\mathrm{Re}[p(z)] > 0$ and has the form $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ for $z \in U$, then $|p_n| \leq 2$ for each n.*

Theorem 6. *If $f \in SLM^{\delta}_{\alpha,\Sigma}(n, \widetilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{\delta^2(1+\delta)^{2n}(1+\alpha)^2(1-3\tau) + \tau\delta\left[2(1+2\alpha)(1+2\delta)^n - (1+3\alpha)(1+\delta)^{2n}\right]}} \tag{20}$$

and

$$|a_3| \leq \frac{|\tau|(1+\delta)^{2n}\left[\delta^2(1+\alpha)^2(1-3\tau) - \tau\delta(1+3\alpha)\right]}{2(1+2\alpha)\delta(1+2\delta)^n\left[\delta^2(1+\delta)^{2n}(1+\alpha)^2(1-3\tau) + \tau\delta\left(2(1+2\alpha)(1+2\delta)^n - (1+3\alpha)(1+\delta)^{2n}\right)\right]} \tag{21}$$

Proof. As $f \in SLM^{\delta}_{\alpha,\Sigma}(n, \widetilde{p}(z))$, so by Definition 4 and using the principle of subordination, there exist Schwarz functions $r(z)$ and $s(z)$ such that

$$(1-\alpha)\frac{D^{n+1}_{\delta} f(z)}{D^{n}_{\delta} f(z)} + \alpha\frac{\left(D^{n+1}_{\delta} f(z)\right)'}{\left(D^{n}_{\delta} f(z)\right)'} = \widetilde{p}(r(z)) \tag{22}$$

and

$$(1-\alpha)\frac{D^{n+1}_{\delta} g(w)}{D^{n}_{\delta} g(w)} + \alpha\frac{\left(D^{n+1}_{\delta} g(w)\right)'}{\left(D^{n}_{\delta} g(w)\right)'} = \widetilde{p}(s(w)) \tag{23}$$

where $r(z) = 1+r_1 z+r_2 z^2+\cdots$ and $s(w) = 1+s_1 w+s_2 w^2+\cdots$.

On expanding, it yields

$$(1-\alpha)\frac{D_\delta^{n+1}f(z)}{D_\delta^n f(z)}+\alpha\frac{\left(D_\delta^{n+1}f(z)\right)'}{\left(D_\delta^n f(z)\right)'}=1+(1+\alpha)\cdot\delta(1+\delta)^n a_2 z+\delta\left[2(1+2\alpha)(1+2\delta)^n a_3-(1+3\alpha)(1+\delta)^{2n}a_2^2\right]z^2+\cdots \tag{24}$$

and

$$(1-\alpha)\frac{D_\delta^{n+1}g(w)}{D_\delta^n g(w)}+\alpha\frac{\left(D_\delta^{n+1}g(w)\right)'}{\left(D_\delta^n g(w)\right)'}=1-(1+\alpha)\cdot\delta(1+\delta)^n a_2 w+\delta\left[-2(1+2\alpha)(1+2\delta)^n a_3+\left(4(1+2\alpha)(1+2\delta)^n-(1+3\alpha)(1+\delta)^{2n}\right)a_2^2\right]\cdot w^2+\cdots \tag{25}$$

Again

$$\tilde{p}(r(z))=1+\frac{\tilde{p}_1 c_1 z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{c_1^2}{4}\tilde{p}_2\right]z^2+\cdots \tag{26}$$

and

$$\tilde{p}(s(z))=1+\frac{\tilde{p}_1 d_1 w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)+\frac{d_1^2}{4}\tilde{p}_2\right]w^2+\cdots \tag{27}$$

Using (24) and (26) in (22) and equating the coefficients of z and z^2, we get

$$(1+\alpha)\delta(1+\delta)^n a_2=\frac{c_1\tau}{2} \tag{28}$$

and

$$\delta\left[2(1+2\alpha)(1+2\delta)^n a_3-(1+3\alpha)(1+\delta)^{2n}a_2^2\right]=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{c_1^2}{4}3\tau^2. \tag{29}$$

Again using (25) and (27) in (23) and equating the coefficients of w and w^2, we get

$$-(1+\alpha)\delta(1+\delta)^n a_2=\frac{d_1\tau}{2} \tag{30}$$

and

$$\delta\left[-2(1+2\alpha)(1+2\delta)^n a_3+\left(4(1+2\alpha)(1+2\delta)^n-(1+3\alpha)(1+\delta)^{2n}\right)a_2^2\right]=\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{d_1^2}{4}3\tau^2. \tag{31}$$

From (28) and (30), it is clear that

$$c_1=-d_1 \tag{32}$$

and

$$2a_2^2=\frac{\left(c_1^2+d_1^2\right)\tau^2}{4\delta^2(1+\delta)^{2n}(1+\alpha)^2}. \tag{33}$$

Adding (29) and (31), it yields

$$\delta\left[4(1+2\alpha)(1+2\delta)^n-2(1+3\alpha)(1+\delta)^{2n}\right]a_2^2=\frac{1}{2}(c_2+d_2)\tau-\frac{1}{4}\left(c_1^2+d_1^2\right)\tau+\frac{3}{4}\left(c_1^2+d_1^2\right)\tau^2. \tag{34}$$

Putting (33) in (34), we get

$$\left[\delta\tau\left(4(1+2\alpha)(1+2\delta)^n-2(1+3\alpha)(1+\delta)^{2n}\right)+2(1-3\tau)\delta^2(1+\delta)^{2n}(1+\alpha)^2\right]a_2^2=\frac{1}{2}(c_2+d_2)\cdot\tau^2. \tag{35}$$

Using Lemma 5 and on applying triangle inequality in (35), (20) can be easily obtained.

Now subtracting (31) from (29), we obtain

$$4\delta(1+2\alpha)(1+2\delta)^n a_3-4\delta(1+2\alpha)(1+2\delta)^n a_2^2=\frac{1}{2}(c_2-d_2)\tau. \tag{36}$$

Applying triangle inequality and using Lemma 5 and (35) in (36), it yields

$$4\delta(1+2\alpha)(1+2\delta)^n|a_3|\le 2|\tau|+4\delta(1+2\alpha)(1+2\delta)^n|a_2|^2. \tag{37}$$

From (37), result (21) is obvious. □

For $\delta=1$, Theorem 6 gives the following result.

Corollary 7. *If $f\in SLM_{\alpha,\Sigma}(n,\tilde{p}(z))$, then*

$$|a_2|\le\frac{|\tau|}{\sqrt{4^n(1+\alpha)^2+\left[2(1+2\alpha)3^n-\left(3\alpha^2+9\alpha+4\right)4^n\right]\tau}} \tag{38}$$

and

$$|a_3| \leq \frac{|\tau| 4^n \left[(1+\alpha)^2 - \left(3\alpha^2 + 9\alpha + 4\right)\tau\right]}{2(1+2\alpha)3^n \left[4^n (1+\alpha)^2 + \left(2(1+2\alpha)3^n - \left(3\alpha^2 + 9\alpha + 4\right)4^n\right)\tau\right]} \tag{39}$$

For $\delta = 1$, $n = 0$, Theorem 6 gives the following result due to Güney et al. [13].

Corollary 8. *If $f(z) \in SLM_{\alpha,\Sigma}(\tilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{(1+\alpha)^2 - (1+\alpha)(2+3\alpha)\tau}} \tag{40}$$

and

$$|a_3| \leq \frac{|\tau| \left[(1+\alpha)^2 - \left(3\alpha^2 + 9\alpha + 4\right)\tau\right]}{2(1+2\alpha)(1+\alpha)\left[(1+\alpha) - (2+3\alpha)\tau\right]}. \tag{41}$$

For $\delta = 1, n = 0, \alpha = 0$, Theorem 6 agrees with the following result proved by Güney et al. [13] (Corollary 1).

Corollary 9. *If $f(z) \in SL_{\Sigma}(\tilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{1-2\tau}} \tag{42}$$

and

$$|a_3| \leq \frac{|\tau|(1-4\tau)}{2(1-2\tau)}. \tag{43}$$

For $\delta = 1$, $n = 0$, $\alpha = 1$, Theorem 6 agrees with the following result proved by Güney et al. [13] (Corollary 2).

Corollary 10. *If $f(z) \in KSL_{\Sigma}(\tilde{p}(z))$, then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-10\tau}} \tag{44}$$

and

$$|a_3| \leq \frac{|\tau|(1-4\tau)}{3(2-5\tau)}. \tag{45}$$

3. Fekete-Szegö Inequality for the Function Class $SLM^{\delta}_{\alpha,\Sigma}(n,\tilde{p}(z))$

Theorem 11. *Let $f(z) \in SLM^{\delta}_{\alpha,\Sigma}(n, \tilde{p}(z))$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{2\delta(1+2\alpha)(1+2\delta)^n}, & |\mu - 1| \leq \dfrac{\tau\delta\left[2(1+2\alpha)(1+2\delta)^n - (1+3\alpha)(1+\delta)^{2n}\right] + \delta^2(1+\alpha)^2(1+\delta)^{2n}(1-3\tau)}{2|\tau|\delta(1+2\alpha)(1+2\delta)^n}; \\ \dfrac{|1-\mu|\tau^2}{\tau\delta\left[2(1+2\alpha)(1+2\delta)^n - (1+3\alpha)(1+\delta)^{2n}\right] + \delta^2(1+\alpha)^2(1+\delta)^{2n}(1-3\tau)}, & |\mu - 1| \geq \dfrac{\tau\delta\left[2(1+2\alpha)(1+2\delta)^n - (1+3\alpha)(1+\delta)^{2n}\right] + \delta^2(1+\alpha)^2(1+\delta)^{2n}(1-3\tau)}{2|\tau|\delta(1+2\alpha)(1+2\delta)^n}. \end{cases} \tag{46}$$

Proof. From (35) and (36), it yields

$$a_3 - \mu a_2^2 = (1-\mu)\frac{\tau^2(c_2 + d_2)}{4\tau\delta\left[2(1+2\alpha)(1+2\delta)^n - (1+3\alpha)(1+\delta)^{2n}\right] + 4\delta^2(1+\delta)^{2n}(1+\alpha)^2(1-3\tau)} + \frac{\tau(c_2 - d_2)}{8\delta(1+2\alpha)(1+2\delta)^n}. \tag{47}$$

Equation (47) can be expressed as

$$a_3 - \mu a_2^2 = \left[h(\mu) + \frac{\tau}{8\delta(1+2\alpha)(1+2\delta)^n}\right]c_2 + \left[h(\mu) - \frac{\tau}{8\delta(1+2\alpha)(1+2\delta)^n}\right]d_2 \tag{48}$$

where

$$h(\mu)=\frac{(1-\mu)\tau^2}{4\tau\delta\left[2(1+2\alpha)(1+2\delta)^n-(1+3\alpha)(1+\delta)^{2n}\right]+4\delta^2(1+\delta)^{2n}(1+\alpha)^2(1-3\tau)}. \tag{49}$$

Taking modulus, we obtain

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{2\delta(1+2\alpha)(1+2\delta)^n}, & 0\le |h(\mu)|\le \dfrac{|\tau|}{8\delta(1+2\alpha)(1+2\delta)^n}; \\ 4|h(\mu)|, & |h(\mu)|\ge \dfrac{|\tau|}{8\delta(1+2\alpha)(1+2\delta)^n}. \end{cases} \tag{50}$$

So (46) can be easily obtained from (50). □

For $\delta=1$, Theorem 11 gives the following result.

Corollary 12. *Let $f(z)\in SLM_{\alpha,\Sigma}(n,\widetilde{p}(z))$, then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{2(1+2\alpha)3^n}, & |\mu-1|\le \dfrac{\left[\left(2(1+2\alpha)3^n-\left(3\alpha^2+9\alpha+4\right)4^n\right)\tau+(1+\alpha)^2 4^n\right]}{2|\tau|(1+2\alpha)3^n}; \\ \dfrac{|1-\mu|\tau^2}{\left[\left(2(1+2\alpha)3^n-\left(3\alpha^2+9\alpha+4\right)4^n\right)\tau+(1+\alpha)^2 4^n\right]}, & |\mu-1|\ge \dfrac{\left[\left(2(1+2\alpha)3^n-\left(3\alpha^2+9\alpha+4\right)4^n\right)\tau+(1+\alpha)^2 4^n\right]}{2|\tau|(1+2\alpha)3^n}. \end{cases} \tag{51}$$

For $\delta=1$, $n=0$, Theorem 11 gives the following result due to Güney et al. [13].

Corollary 13. *Let $f(z)\in SLM_{\alpha,\Sigma}(\widetilde{p}(z))$, then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{2(1+2\alpha)}, & |\mu-1|\le \dfrac{(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]}{2|\tau|(1+2\alpha)}; \\ \dfrac{|1-\mu|\tau^2}{(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]}, & |\mu-1|\ge \dfrac{(1+\alpha)\left[(1+\alpha)-(2+3\alpha)\tau\right]}{2|\tau|(1+2\alpha)}. \end{cases} \tag{52}$$

For $\delta=1$, $n=0$, $\alpha=0$, Theorem 11 agrees with the following result proved by Güney et al. [13] (Corollary 4).

Corollary 14. *If $f(z)\in SL_{\Sigma}(\widetilde{p}(z))$, then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{2}, & |\mu-1|\le \dfrac{1-2\tau}{2|\tau|}; \\ \dfrac{|1-\mu|\tau^2}{1-2\tau}, & |\mu-1|\ge \dfrac{1-2\tau}{2|\tau|}. \end{cases} \tag{53}$$

For $\delta=1$, $n=0$, $\alpha=1$, Theorem 11 agrees with the following result proved by Güney et al. [13] (Corollary 5).

Corollary 15. *If $f(z)\in KSL_{\Sigma}(\widetilde{p}(z))$, then*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{|\tau|}{6}, & |\mu-1|\le \dfrac{2-5\tau}{3|\tau|}; \\ \dfrac{|1-\mu|\tau^2}{2(2-5\tau)}, & |\mu-1|\ge \dfrac{2-5\tau}{3|\tau|}. \end{cases} \tag{54}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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