Research Article

Analyzing Some Structural Properties of Topological B -Algebras

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Received 14 February 2019; Accepted 16 June 2019; Published 15 July 2019

Academic Editor: Susana Montes

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In this study, we investigate the topology on B -algebras: an algebraic system of propositional logic. We define here the notion of topological B -algebras (briefly, TB -algebras) and some properties are investigated. A characterization of TB -algebras based on neighborhoods is provided. We also provide a filterbase that generates a unique B -topology, making a TB -algebra in which the filterbase is a neighborhood base of the constant element, provided that the given B -algebra is commutative. Finally, we investigate subalgebras of TB -algebras and introduce the notion of quotient TB -algebras of the given B -algebra.

1. Introduction

J. Neggers and H. S. Kim [1] introduced the concept of B -algebras in 2002. It is an algebraic system which is related to BCK/BCI -algebras but seems to have more profound properties without being excessively complicated. The authors in [2, 3] proved that there is a direct correspondence between B -algebras and groups. Thus, certain properties and results on B -algebras were established via the properties of groups. Readers may refer to [4–8] for some of these “group-alike” properties and results on B -algebras.

The theory of topological groups is one of the already well-defined concepts in both algebra and topology. Thus, in connection to the researches previously mentioned, this study initiates the notion of topological B -algebras. Some fundamental properties of a topological B -algebra that anchors on the basic topological and algebraic concepts will be investigated. This will provide the foundation of future investigations regarding the overall structure of a topological B -algebra.

2. Preliminaries

In this section, we recall some elementary concepts in B -algebras that are necessary in this paper.

A B -algebra [1] is defined as a triple $(X; *, 0)$ where X is a nonempty set with “0” being its constant element

and “ $*$ ” being a binary operation that satisfies the following conditions for all $x, y, z \in X$:

$$(B_1) \quad x * x = 0,$$

$$(B_2) \quad x * 0 = x,$$

$$(B_3) \quad (x * y) * z = x * (z * (0 * y)).$$

In addition, $(X; *, 0)$ is said to be a *commutative B -algebra* if the following condition holds for all $x, y \in X$:

$$(B_4) \quad x * (0 * y) = y * (0 * x).$$

Throughout the remainder of this study, X shall conveniently denote the B -algebra $(X; *, 0)$ unless otherwise specified.

Let X be a B -algebra. A nonempty subset A is said to be a *subalgebra* of X [9] if $x * y \in A$ for all $x, y \in A$. In addition, A is said to be a *normal subset* of X [10] if $(x * a) * (y * b) \in A$ whenever $x * y, a * b \in A$. Accordingly, every normal subset of X is a subalgebra of X . Thus, it is more precise to say that A is a *normal subalgebra* of X if the condition of normality is satisfied.

Some basic notations and properties of (commutative) B -algebras are the following.

Definition 1 (see [7]). Let X be a B -algebra. For each $x \in X$, x^{-1} denotes the expression $0 * x$.

Lemma 2 (see [1]). Let X be a B -algebra. Then, for all $x, y, z \in X$,

$$(i) \quad (x * y) * y^{-1} = x;$$

- (ii) $x * (y * z) = (x * z^{-1}) * y$;
- (iii) $x * y = 0$ if and only if $x = y$.

Lemma 3 (see [11]). *If X is a B -algebra, then $(x^{-1})^{-1} = x$ for all $x \in X$.*

Lemma 4 (see [10]). *Let X be a B -algebra. Then $x * y = (y * x)^{-1}$ for all $x, y \in X$.*

Lemma 5 (see [1]). *Let X be a commutative B -algebra. Then, for every $x, y \in X$, $x^{-1} * y^{-1} = y * x$.*

Theorem 6 (see [12]). *Let X be a B -algebra. Then X is commutative if and only if $x * (x * y) = y$ for all $x, y \in X$.*

On the other hand, the following notions appear in [6]. Let X be a B -algebra and N a normal subalgebra of X . For $x \in X$, the *left coset* and the *right coset* of N in X by x are, respectively, the sets

$$\begin{aligned} x * N &= \{x * n : n \in N\}, \\ N * x &= \{n * x : n \in N\}. \end{aligned} \quad (1)$$

Denote the set of all distinct left cosets of N in X by X/N and define a binary operation “ $\otimes$ ” on X/N as follows: for all $x * N, y * N \in X/N$, $(a * N) \otimes (b * N) = (a * b) * N$. Then $(X/N; \otimes, N)$ is a B -algebra and is called the *quotient B -algebra* of X by N . For simplicity of this study, we shall denote the quotient B -algebra $(X/N; \otimes, N)$ by X/N only, unless otherwise specified.

We finish this section by providing some new classes of sets in B -algebras and their properties that will be needed in this study.

Definition 7. Let A and B be nonempty subsets of a B -algebra X :

- (i) For each $x \in X$, the sets $x * A$ and $A * x$ are called the *left* and *right translates* of A by x , respectively.
- (ii) The *product* of A and B , denoted by $A * B$, is given by

$$A * B = \bigcup_{a \in A} a * B = \bigcup_{b \in B} A * b = \{a * b : a \in A, b \in B\}. \quad (2)$$

We also recall the following notation which has appeared first in [6].

Definition 8. Let A be a nonempty subset of a B -algebra X . The *inverse* of A , denoted by A^{-1} , is the set $\{a^{-1} : a \in A\}$.

Note that if A is a subalgebra of a B -algebra X , then, as explained earlier, the sets $x * A$ and $A * x$ are left and right cosets of A in X , respectively.

Some obvious results on the notations presented above are the following.

Remark 9. Let A and B be nonempty subsets of a B -algebra X :

- (i) $A * 0 = A$ and $0 * A = A^{-1}$.

- (ii) If $0 \in B$, then $A \subseteq A * B$.
- (iii) $A * B \subseteq C * D$ if and only if $A \subseteq C$ and $B \subseteq D$.

The following result, which appears in [6], will be considered in this study.

Corollary 10. *Let N be a subalgebra of a B -algebra X . Then X is a normal subalgebra of X if and only if $x * (x * N) \subseteq N$ for all $x \in X$.*

Proposition 11. *Let A be a nonempty subset of a B -algebra X . Then the following statements hold:*

- (i) $(A^{-1})^{-1} = A$.
- (ii) For all $x, y \in X$, $A * (x * y) = (A * y^{-1}) * x$.
- (iii) If A is a subalgebra of X , then $A * A = A$.
- (iv) If A is a subalgebra of X , then $(x * a) * A = x * A$, for all $x \in X$ and for all $a \in A$.

Proof. Let A be a nonempty subset of X and $x, y \in X$:

- (i) It follows from Lemma 3.
- (ii) It follows from Lemma 2 (ii).
- (iii) It follows from Remark 9 (ii) and from the fact that A is a subalgebra of X .
- (iv) Suppose A is a subalgebra of X . Let $x \in X$ and $a \in A$. Then for $p \in (x * a) * A$, there is an $a' \in A$ such that $p = (x * a) * a'$. By (B_3) , $p = x * (a' * a^{-1})$, with $a' * a^{-1} \in A$. Hence, $p \in x * A$ and so $(x * a) * A \subseteq x * A$. Further, if $q \in x * A$, then there is an $a_1 \in A$ such that $q = x * a_1$. By Lemma 3, Definition 1, and Lemma 4, (B_1) , (B_3) , and (B_2) , we have

$$\begin{aligned} q &= x * a_1 = ((x * a_1)^{-1})^{-1} = 0 * (a_1 * x) \\ &= ((x * a) * (x * a)) * (a_1 * x) \\ &= (x * a) * ((a_1 * x) * (x * a)^{-1}) \\ &= (x * a) * ((a_1 * x) * (a * x)) \\ &= (x * a) * (a_1 * ((a * x) * x^{-1})) \\ &= (x * a) * (a_1 * (a * (x^{-1} * x^{-1}))) \\ &= (x * a) * (a_1 * (a * 0)) = (x * a) * (a_1 * a) \\ &\in (x * a) * A. \end{aligned} \quad (3)$$

Therefore, $x * A \subseteq (x * a) * A$. Consequently, $(x * a) * A = x * A$. □

Meanwhile, we consider the following properties that are present in a commutative B -algebra.

Proposition 12. *Let A and B be nonempty subsets of a commutative B -algebra X . Then the following hold:*

(i) For all $x, y \in X$, $(A * x) * (B * y) = (A * B) * (x * y)$.

(ii) If $(x * A) * x \subseteq B$, for all $x \in X$, then $A \subseteq B^{-1}$.

Proof. (i) It suffices to show that $(a * x) * (b * y) = (a * b) * (x * y)$ for all $x, y, a, b \in X$. Now, by (B_3) , Lemma 5, (B_2) , Lemma 4, and Theorem 6,

$$\begin{aligned}
 (a * x) * (b * y) &= a * ((b * y) * x^{-1}) \\
 &= a * (b * (x^{-1} * y^{-1})) \\
 &= (a * (b * (y * x))) * 0 \\
 &= (a * (b * (y * x))) \\
 &\quad * ((x * y) * (x * y)) \\
 &= ((a * (b * (y * x))) * (x * y)^{-1}) \\
 &\quad * (x * y) \\
 &= ((a * (b * (y * x))) * (y * x)) \\
 &\quad * (x * y) \\
 &= (a * ((y * x) * (b * (y * x))^{-1})) \\
 &\quad * (x * y) \\
 &= (a * ((y * x) * ((y * x) * b))) \\
 &\quad * (x * y) = (a * b) * (x * y).
 \end{aligned} \tag{4}$$

(ii) Let $a \in A$. Then, for each $x \in X$, $(x * a) * x \in (x * A) * x$, and so, by the hypothesis, there is a $b \in B$ such that $b = (x * a) * x$. Observe that by (B_3) , (B_1) , and (B_2) ,

$$\begin{aligned}
 b * x^{-1} &= ((x * a) * x) * x^{-1} = (x * a) * (x^{-1} * x^{-1}) \\
 &= (x * a) * 0 = x * a.
 \end{aligned} \tag{5}$$

By Theorem 6, Lemma 2 (ii), Lemma 3, (B_2) , (B_1) , and Definition 1,

$$\begin{aligned}
 a &= x * (x * a) = x * (b * x^{-1}) \\
 &= (x * ((x^{-1})^{-1})) * b = (x * x) * b = 0 * b = b^{-1} \\
 &\in B^{-1}.
 \end{aligned} \tag{6}$$

Therefore, $A \subseteq B^{-1}$. $\square$

3. Initial Properties of Topological B -Algebras

This section presents the definition of a topological B -algebra as well as some of its fundamental properties. Also, some characterizations of (commutative) topological B -algebras in terms of neighborhood and filters are investigated.

Definition 13. Let X be a B -algebra. A topology τ furnished on X is called a B -topology on X . In addition, (X, τ) is called

TABLE 1: Cayley table for the binary operation “ $*$ ”.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

a *topological B -algebra* (or *TB-algebra*) if τ is a B -topology on X and the binary operation $*$: $X \times X \rightarrow X$ is continuous, where the Cartesian product topology on $X \times X$ is furnished by τ .

Example 14. Consider the group $\mathbb{R}$ of real numbers under the usual addition operation “+”. Then it is well known that $\mathbb{R}$ together with “+” and the Euclidean topology $\mathcal{E}_1$ is a topological group. Let “ $*$ ” be a binary operation on $\mathbb{R}$ defined by $a * b = a - b$ for all $a, b \in \mathbb{R}$. Then it is easy to show that $(\mathbb{R}; *, 0)$ is a B -algebra. Now, one of the properties of the topological group $(\mathbb{R}, \mathcal{E}_1)$ is that the map $(a, b) \mapsto a - b$ is continuous. This implies that $*$: $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous map. Therefore, $(\mathbb{R}, \mathcal{E}_1)$ is also a TB -algebra.

Example 15. Consider the B -algebra $X = \{0, a, b, c\}$ with the binary operation “ $*$ ” defined on the Cayley table provided in Table 1 (see [13]). Let $\tau = \{X, \emptyset, \{0, a\}, \{b, c\}\}$. Then τ is a B -topology on X . By routine calculations, we see that (X, τ) is a TB -algebra.

In view of Definition 7 (ii), the following remark is straightforward.

Remark 16. Let X be a B -algebra. Then $*(A \times B) = A * B$ for any nonempty subsets A and B of X .

On the other hand, the following result provides a necessary and sufficient condition for a TB -algebra.

Theorem 17. Let X be a B -algebra and τ a B -topology on the set X . Then (X, τ) is a TB -algebra if and only if, for all $x, y \in X$ and for every nbd W of $x * y$, there are nbds U and V of x and y , respectively, such that $U * V \subseteq W$.

Proof. Suppose that “ $*$ ” is a continuous map. Let $x, y \in X$ and W a nbd of $x * y$. Then $*^{-1}(W)$ is a nbd of (x, y) in $X \times X$. By the definition of Cartesian product topology, there exist nbds U and V of x and y , respectively, such that $U \times V \subseteq *^{-1}(W)$. By Remark 16, $U * V = *(U \times V)$. Hence, there are nbds U and V of x and y , respectively, such that $U * V \subseteq (*^{-1}(W)) = W$.

Conversely, let $(a, b) \in X \times X$ and W a nbd of $a * b$. Now, by the assumption, there exist nbds U and V of a and b , respectively, such that $U * V \subseteq W$. By the definition of Cartesian product topology, it follows that $U \times V$ is a nbd of (a, b) . Now, $*(U \times V) = U * V$ by Remark 16, and so there is a nbd $U \times V$ of (a, b) such that $*(U \times V) \subseteq W$. Therefore, “ $*$ ” is a continuous function, completing the proof of the theorem. $\square$

Theorem 18. Let (X, τ) be a TB-algebra and a a fixed element of the B-algebra X . Then the following functions are homeomorphisms:

- (i) $R_a : (X, \tau) \longrightarrow (X, \tau)$ defined by $R_a(x) = x * a$.
- (ii) $L_a : (X, \tau) \longrightarrow (X, \tau)$ defined by $L_a(x) = a * x$.
- (iii) $\iota : (X, \tau) \longrightarrow (X, \tau)$ defined by $\iota(x) = x^{-1}$.

Proof. Let a be a fixed element of X . Consider the function $\varphi_a : (X, \tau) \longrightarrow (X, \tau) \times (X, \tau)$ defined by $\varphi_a(x) = (x, a)$. Let $x \in X$ and W a nbd of $\varphi_a(x)$. Then, by the definition of Cartesian product topology, there are τ -open sets U and V in X such that $\varphi_a(x) = (x, a) \in U \times V \subseteq W$. Thus, there is a nbd U of x such that $\varphi_a(U) = U \times \{a\} \subseteq U \times V \subseteq W$. This shows that φ_a is continuous. Similarly, the function $\phi_a : (X, \tau) \longrightarrow (X, \tau) \times (X, \tau)$ defined by $\phi_a(x) = (a, x)$ is continuous. Now, observe that $R_a = * \circ \varphi_a$ and $L_a = * \circ \phi_a$, with “ $*$ ” being a continuous map by Definition 13. Thus, both R_a and L_a are compositions of two continuous functions. Hence, R_a and L_a are continuous.

Next, consider $R_{a^{-1}} : (X, \tau) \longrightarrow (X, \tau)$ defined by $R_{a^{-1}}(x) = x * a^{-1}$. By the previous argument, $R_{a^{-1}}$ is also continuous. By Definition 1 and Lemma 2 (i), we have

$$(R_{a^{-1}} \circ R_a)(x) = R_{a^{-1}}(R_a(x)) = (x * a) * a^{-1} = x. \quad (7)$$

Also, by Definition 1, (B_3) , Lemma 3, (B_1) , and (B_2) , it follows that

$$\begin{aligned} (R_a \circ R_{a^{-1}})(x) &= R_a(R_{a^{-1}}(x)) = (x * a^{-1}) * a \\ &= x * (a * (a^{-1})^{-1}) = x. \end{aligned} \quad (8)$$

Hence, we have $R_{a^{-1}} \circ R_a = 1_X$ and $R_a \circ R_{a^{-1}} = 1_X$. Consequently, R_a is a homeomorphism, proving (i).

On the other hand, for $x \in X$ and W a nbd of $\iota(x) = x^{-1} = 0 * x$, there are nbds U and V of 0 and x , respectively, such that $U * V \subseteq W$. Observe that, by Definition 8 and Remark 9, $\iota(V) = V^{-1} = 0 * V \subseteq U * V$. Thus, there is a nbd V of x such that $\iota(V) \subseteq W$. Therefore, ι is continuous. Next, by Lemma 3, we have $(\iota \circ \iota)(x) = \iota(\iota(x)) = (x^{-1})^{-1} = x$, and so $\iota \circ \iota = 1_X$. Hence, ι is a homeomorphism, proving (iii).

Lastly, by Lemma 4, $L_a(x) = a * x = (x * a)^{-1} = \iota(x * a) = \iota(R_a(x))$ for all $x \in X$. Thus, $L_a = \iota \circ R_a$; that is, L_a is a composition of two homeomorphisms. Consequently, L_a is a homeomorphism, proving (ii). $\square$

Corollary 19. Let (X, τ) be a TB-algebra and $0 \in X$. Then the following statements are equivalent:

- (i) 0 is τ -open in X .
- (ii) 0^{-1} is τ -open in X .
- (iii) $x * 0$ is τ -open in X for all $x \in X$.
- (iv) $0 * x$ is τ -open in X for all $x \in X$.

Meanwhile, the following corollary is immediate from the fact that, in a homeomorphism of topological spaces, the closure of the image of a subset of the domain is equal to the image of the closure of the said subset.

Corollary 20. Let (X, τ) be a TB-algebra. Then the following statements hold:

- (i) $\overline{U^{-1}} = \overline{U}^{-1}$ for all $U \subseteq X$.
- (ii) $\overline{x * U} = x * \overline{U}$ for all $U \subseteq X$ and for all $x \in X$.
- (iii) $\overline{U * x} = \overline{U} * x$ for all $U \subseteq X$ and for all $x \in X$.

We shall now use Theorem 18 in proving some results pertaining to neighborhoods in TB-algebras. Let (X, τ) be a TB-algebra and $x \in X$. Denote the neighborhood filter of x by $\eta_x(X, \tau)$. Also, from Remark 9 (iii), we have $U^2 = U * U$, where $U \subseteq X$.

Theorem 21. Let $\mathcal{U}$ be a neighborhood base of the constant element 0 of a TB-algebra (X, τ) . Then the following conditions hold:

- (i) For every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^2 \subseteq U$.
- (ii) For every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.
- (iii) For every $U \in \mathcal{U}$ and $x \in X$, there is a $V \in \mathcal{U}$ such that $(x * V) * x \subseteq U$.

Proof. Let U be an arbitrary element of $\mathcal{U}$.

- (i) We have $0 * 0 = 0$. By the continuity of “ $*$ ” and by Theorem 17, there are members W_1 and W_2 of $\eta_0(X, \tau)$ such that $W_1 * W_2 \subseteq U$. Thus, there exists $V_1 \in \eta_0(X, \tau)$ such that $V_1 \subseteq W_1 \cap W_2$. Since $\mathcal{U}$ is a neighborhood base of 0 , there is a $V \in \mathcal{U}$ such that $V \subseteq V_1$. Therefore, by Remark 9 (iii) and Remark 16,

$$\begin{aligned} V^2 \subseteq V_1^2 &= * (V_1 \times V_1) \subseteq * (W_1 \times W_2) = W_1 * W_2 \\ &\subseteq U. \end{aligned} \quad (9)$$

- (ii) Similar to (i), but use the continuity of ι in Theorem 18 (iii).
- (iii) Fix $x \in X$ and let $\nu_x : X \longrightarrow X$ be defined by $\nu_x(y) = (x * y) * x$. Then $\nu_x = L_x \circ R_x$, and so, in view of Theorem 18, ν_x is continuous. Now, $\nu_x(0) = (x * 0) * x = 0$ by (B_2) and (B_1) . This means that if $P \in \eta_0(X, \tau)$, then $\nu_x^{-1}(P) \in \eta_0(X, \tau)$. Meanwhile, $U \in \mathcal{U}$ implies $U \in \eta_0(X, \tau)$. Thus, we can choose $V \in \mathcal{U}$ such that $V \subseteq \nu_x^{-1}(U)$. Therefore, we have $(x * V) * x = \nu_x(V) \subseteq \nu_x(\nu_x^{-1}(U)) \subseteq U$. $\square$

On the other hand, we shall construct a B-topology on a B-algebra which is generated by a neighborhood base of the constant element, making the space a TB-algebra provided that the B-algebra is commutative.

Theorem 22. Let X be a B-algebra and $\mathcal{U}$ a filterbase of X which satisfies the statements (i), (ii), and (iii) in Theorem 21. Then there is a unique B-topology τ such that $\mathcal{U}$ is a neighborhood base of the constant element 0 of X in the topology τ . Moreover, if X is commutative, then (X, τ) is a TB-algebra.

Proof. Firstly, we claim that every member of $\mathcal{U}$ contains the constant element 0. Let $U \in \mathcal{U}$. Then, by the hypothesis, there exists $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Let $v \in V$. Then, by (B_1) , $0 = v * v \in V^2$. This proves the claim.

Next, let

$$\tau = \left\{ O \subseteq G : \text{for every } x \in O, \text{ there is a } U \in \mathcal{U} \text{ such that } U * x^{-1} \subseteq O \right\}. \quad (10)$$

We shall show that τ is a B -topology furnished from X . Clearly, $X, \emptyset \in \tau$, and so $(\mathcal{T}_1)$ is satisfied. Let $\mathcal{S}_1 = \{O_\alpha : \alpha \in \mathcal{A}\}$ be a subclass of τ and set $O = \bigcup_\alpha O_\alpha$. Let $x \in O$. Then $x \in O_\alpha$ for some $\alpha \in \mathcal{A}$. Thus, there is a $U \in \mathcal{U}$ such that $U * x^{-1} \subseteq O_\alpha \subseteq O$. Hence, $O \in \tau$. Further, let $\mathcal{S}_2 = \{O_k : k = 1, 2, \dots, n\}$ be a finite subclass of τ and $O' = \bigcap_{k=1}^n O_k$. Let $y \in O'$. Then $y \in O_k$ for all $k = 1, 2, \dots, n$. Thus, for each k , there is a $U_k \in \mathcal{U}$ such that $U_k * y^{-1} \subseteq O_k$. Now, $\mathcal{U}$ a filterbase means that there is a $U \in \mathcal{U}$ such that $U \subseteq U_k$ for all k . Therefore, there is a $U \in \mathcal{U}$ such that $U * y^{-1} \subseteq \bigcap_{k=1}^n U_k * y^{-1} \subseteq O'$. This shows that $O' \in \tau$. Consequently, τ is a B -topology on X .

We will now show that $\mathcal{U}$ is a neighborhood base of 0. We claim first that every member of $\eta_0(X, \tau)$ contains a member of $\mathcal{U}$. Let $N \in \eta_0(X, \tau)$. Then we have $0 \in \text{int}_\tau N \subseteq N$ with $\text{int}_\tau N \in \tau$. From the definition of τ , Remark 9 (i), and by the fact that $0^{-1} = 0$, there is a $U \in \mathcal{U}$ such that $U = U * 0^{-1} \subseteq \text{int}_\tau N \subseteq N$. This proves the claim. Next, we claim that $\mathcal{U} \subseteq \eta_0(X, \tau)$. It suffices to show that every member of $\mathcal{U}$ is a nbd of 0. Let $U \in \mathcal{U}$. Then by the previous argument, $0 \in U$. Now, set

$$B = \{x \in U : \exists V \in \mathcal{U} \text{ such that } V * x^{-1} \subseteq U\}. \quad (11)$$

Then $B \neq \emptyset$ since $0 \in B$. By construction, $B \subseteq U$. Let $w \in B$. Then there is a $V \in \mathcal{U}$ such that $V * w^{-1} \subseteq U$. Also, there is a $P \in \mathcal{U}$ such that $P^2 \subseteq V$. Let $y \in P * w^{-1}$. Then $y = p * w^{-1}$ for some $p \in P$. Thus by (B_2) and by the fact that $0 \in P$, $y = p * w^{-1} = (p * 0) * w^{-1} \in P^2 * w^{-1}$. Hence, by Remark 9 (ii), $P * w^{-1} \subseteq P^2 * w^{-1} \subseteq V * w^{-1} \subseteq U$. This means that $y \in B$, and so there is a $P \in \mathcal{U}$ such that $P * w^{-1} \subseteq B$. We have shown that $B \in \tau$, and so B is τ -open. This proves the second claim; hence, $\mathcal{U}$ is a neighborhood base of 0. Consequently, since a neighborhood base generates one and only one topology on X , τ is unique.

Lastly, suppose X is commutative. We will then prove the continuity of “ $*$ ”. First, we claim that if $O \in \tau$ and $y \in X$, then $O * y \in \tau$. Let $x \in O * y$. Then $x = u * y$ for some $u \in O$. By (B_4) and Lemma 2 (i), $y * x^{-1} = x * y^{-1} = (u * y) * y^{-1} = u \in O$, and so there is a $U \in \mathcal{U}$ such that $U * (y * x^{-1})^{-1} = U * (x^{-1} * y) \subseteq O$ by the definition of τ . By Theorem 6, Proposition 11 (ii), Lemma 3, and Remark 9 (iii), we have

$$\begin{aligned} U * x^{-1} &= U * (y * (y * x^{-1})) \\ &= (U * (y * x^{-1})^{-1}) * y \\ &= (U * (x^{-1} * y)) * y \subseteq O * y. \end{aligned} \quad (12)$$

This proves the claim. Next, let $x, y \in X$ and W a nbd of $x * y$. Then we have $x * y \in O \subseteq W$ for some $O \in \tau$. Thus, by the definition of τ and Lemma 3, $P * (x * y)^{-1} = P * (y * x)$ for some $P \in \mathcal{U}$. Also, $P \in \mathcal{U}$ implies $U^2 \subseteq P$ for some $U \in \mathcal{U}$. So, we have $U^2 * (y * x) \subseteq P * (y * x) \subseteq O \subseteq W$. Further, $U \in \mathcal{U}$ implies there is a $V' \in \mathcal{U}$ such that $(x * V') * x \subseteq U$ for all $x \in X$. But since X is commutative, it follows from Proposition 12 (ii) that $V'^{-1} \subseteq U$. Also, $V' \in \mathcal{U}$ implies there is a $V \in \mathcal{U}$ such that $V \subseteq V'^{-1}$. Now, $x \in U * x^{-1}$ since, by Lemma 3 and Definition 1, we have $x = 0 * (0 * x) = 0 * x^{-1}$ and $0 \in U$. Similarly, $y \in V * y^{-1}$. Since $U, V \in \eta_0(X, \tau)$, it follows that $U * x^{-1}$ and $V * y^{-1}$ are nbds of x and y , respectively. By Proposition 12 (i), Lemma 5, and Remark 9 (iii)

$$\begin{aligned} (U * x^{-1}) * (V * y^{-1}) &= (U * V) * (x^{-1} * y^{-1}) \\ &= (U * V) * (y * x) \subseteq (U * V'^{-1}) * (y * x) \\ &\subseteq U^2 * (y * x) \subseteq W. \end{aligned} \quad (13)$$

Therefore, by Theorem 17, (X, τ) is a TB -algebra. $\square$

4. Subalgebras and Quotients of TB -Algebras

In this section, we provide some properties of TB -algebras in terms of its (normal) subalgebras. Also, we consider the construction of quotient TB -algebras determined by a normal subalgebra of a B -algebra.

Proposition 23. *Let (X, τ) be a TB -algebra. Then (X, τ) is Hausdorff if and only if the trivial subalgebra $\{0\}$ is a closed subset of X .*

Proof. Suppose (X, τ) is Hausdorff. Let $x \in X$ such that $x \neq 0$ and $\mathcal{S}_x$ the subclass of $\mathcal{P}(X)$ containing all the open neighborhoods of x . Also, set

$$\mathcal{S} = \bigcup_{x \in X \setminus \{0\}} \mathcal{S}_x. \quad (14)$$

Then

$$X \setminus \{0\} = \bigcup_{S \in \mathcal{S}} S \quad (15)$$

is an open subset of X which does not contain 0. Hence, $\{0\} = (X \setminus \{0\})^c$ is closed in X .

Conversely, assume that $\{0\}$ is a closed subset of X . Observe that the binary operation “ $*$ ” is continuous. Thus, $*^{-1}(\{0\})$ is closed in $X \times X$. But $*^{-1}(\{0\}) = \{x * x : x \in X\} = \Delta$ by (B_1) and Lemma 2 (iii). Therefore, (X, τ) is Hausdorff. $\square$

Proposition 24. *Let (X, τ) be a TB -algebra and Y a subalgebra of X .*

(i) *The closure $\overline{Y}$ of Y is a subalgebra of X .*

(ii) *If Y is a normal subalgebra of X , then so does $\overline{Y}$.*

(iii) *If (X, τ) is Hausdorff and Y is commutative, then $\overline{Y}$ is commutative.*

Proof. Let Y be a subalgebra of X .

- (i) We will first prove that, for every nonempty subsets A and B of X , $\overline{A * B} \subseteq \overline{A} * \overline{B}$. Indeed, since by Remark 16 and by the continuity of “ $*$ ”, $\overline{A} * \overline{B} = \overline{*(A \times B)} \subseteq \overline{*(A \times B)} = \overline{A * B}$. Now, suppose $x, y \in \overline{Y}$. Observe that by Proposition 11, $\overline{Y} * \overline{Y} \subseteq \overline{Y * Y} = \overline{Y^2} = \overline{Y}$. Hence, we see that $x * y \in \overline{Y} * \overline{Y} \subseteq \overline{Y}$. Therefore, $\overline{Y}$ is a subalgebra of X .
- (ii) Suppose Y is normal in X . Then, by Corollary 10, $x * (x * Y) \subseteq Y$ for all $x \in X$. By Corollary 20 (ii), $x * (x * \overline{Y}) = \overline{x * (x * Y)} \subseteq \overline{Y}$ for all $x \in X$. Hence, by Corollary 10, $\overline{Y}$ is a normal subalgebra of X .
- (iii) Consider the function $\alpha : X \times X \rightarrow X$ defined by $\alpha(x, y) = x * (y * (y * x))$. Then α is continuous since “ $*$ ” is a continuous map. Now, (X, τ) being a Hausdorff space implies $\{0\}$ is closed in X by Proposition 23. Hence, $\alpha^{-1}(\{0\})$ is closed in $X \times X$. Since Y is a commutative B -algebra, it follows that $x * (y * (y * x)) = x * x = 0$ for all $x, y \in Y$ by Theorem 6 and (B_1) . Thus, $Y \times Y \subseteq \alpha^{-1}(\{0\})$; whence $\overline{Y} \times \overline{Y} = \overline{Y \times Y} \subseteq \overline{\alpha^{-1}(\{0\})}$. Note that $(0, 0) \in \overline{Y} \times \overline{Y}$. Therefore, $\{0\} \subseteq \alpha(\overline{Y} \times \overline{Y}) \subseteq \alpha(\alpha^{-1}(\{0\})) \subseteq \{0\}$, and so $\alpha(\overline{Y} \times \overline{Y}) = \{0\}$. This means that $\alpha(x, y) = x * (y * (y * x)) = 0$ for all $x, y \in \overline{Y}$. Now, by Lemma 2 (ii) and Lemma 4,

$$\begin{aligned} 0 &= x * (y * (y * x)) = (x * (y * x))^{-1} * y \\ &= (x * (x * y)) * y. \end{aligned} \quad (16)$$

Thus, by Lemma 2 (iii), $x * (x * y) = y$ for all $x, y \in \overline{Y}$. Consequently, by Theorem 6, $\overline{Y}$ is commutative. $\square$

We shall now consider the construction of quotient TB -algebras by a normal subalgebra of the given B -algebra. Recall first that a function $f : X \rightarrow Y$ where X and X' are B -algebras is said to be a B -homomorphism [9] if $f(x *_X y) = f(x) *_Y f(y)$ for all $x, y \in X$, with “ $_X$ ” and “ $_Y$ ” being the binary operations on X and Y , respectively. In addition, f is a B -epimorphism if f is surjective.

Let X be a B -algebra and N a subalgebra of X . Denote $X(\equiv_N)$ as the collection of all left cosets of X by N . Such members of $X(\equiv_N)$ are distinct, since these are the equivalence classes of an equivalence relation on X ([6], Theorem 3). Clearly, the mapping $\pi : X \rightarrow X(\equiv_N)$ is surjective. If N is normal in X , then $X(\equiv_N)$ coincides with the quotient B -algebra X/N . Moreover, π is a B -epimorphism [9] and is called the *canonical projection* of B -algebras.

Proposition 25. Let (X, τ) be a TB -algebra, N a subalgebra of X , and A a nonempty subset of X . Then $\pi^{-1}(\pi(A)) = A * N$.

Proof. Suppose A is a nonempty subset of X . Let $p \in \pi^{-1}(\pi(A))$. Then

$$p * N = \pi(p) \in \pi(A) = \{a * N : a \in A\}. \quad (17)$$

This means that there is an $a_1 \in A$ such that $p * N = a_1 * N$. Thus, there are $n_1, n_2 \in N$ such that $p * n_1 = a_1 * n_2$. Since N is a subalgebra of X , it follows that $0 \in N$ and $n^{-1} = 0 * n \in N$ for every $n \in N$. Now, by Lemma 2 (i) and (B_3) ,

$$\begin{aligned} p &= (p * n_1) * n_1^{-1} = (a_1 * n_2) * n_1^{-1} \\ &= a * (n_1^{-1} * n_2^{-1}) \in a * N. \end{aligned} \quad (18)$$

This shows that $\pi^{-1}(\pi(A)) \subseteq A * N$. Next, suppose $q \in A * N$. Then there exist $a \in A$ and $n \in N$ such that $q = a * n$. By Proposition 11 (iv), we have $\pi(q) = \pi(a * n) = (a * n) * N = a * N$. Hence, by Definition 7 (ii),

$$q \in \pi^{-1}(a * N) \subseteq \pi^{-1}\left(\bigcup_{a \in A} a * N\right) = \pi^{-1}(\pi(A)). \quad (19)$$

Therefore, $A * N \subseteq \pi^{-1}(\pi(A))$. Consequently, $\pi^{-1}(\pi(A)) = A * N$. $\square$

Theorem 26. Let (X, τ) be a TB -algebra and N a subalgebra of X . For the class $X(\equiv_N)$, define $\tau_{X(\equiv_N)} \subseteq \mathcal{P}(X(\equiv_N))$ by

$$\tau_{X(\equiv_N)} = \{O \subseteq X(\equiv_N) : \pi^{-1}(O) \text{ is open in } X\}. \quad (20)$$

Then $\tau_{X(\equiv_N)}$ is a topology on $X(\equiv_N)$.

Proof. It is straightforward. $\square$

Theorem 27. Let (X, τ) be a TB -algebra and N a subalgebra of X . Then, with respect to the topological space $(X(\equiv_N), \tau_{X(\equiv_N)})$, the surjective map π is continuous and open.

Proof. The continuity of π follows directly from the definition of $\tau_{X(\equiv_N)}$. It remains to show that π is an open map. From the definition of $\tau_{X(\equiv_N)}$, it suffices to show that $\pi^{-1}(\pi(U)) \in \tau$ whenever $U \in \tau$. So let U be τ -open in X . By Proposition 25 and Definition 7 (ii),

$$\pi^{-1}(\pi(U)) = U * N = \bigcup_{n \in N} U * n. \quad (21)$$

By Corollary 19 (iv), for each $n \in N$, $U * n$ is τ -open in X , and so $\pi^{-1}(\pi(U))$ is a union of some τ -open sets in X . Therefore, $\pi^{-1}(\pi(U)) \in \tau$. $\square$

Finally, we introduce the notion of *quotient TB -algebras* of B -algebras which is stipulated in the result below.

Theorem 28. Let (X, τ) be a TB -algebra. If N is a normal subalgebra of X , then

$$(X(\equiv_N), \tau_{X(\equiv_N)}) = (X/N, \tau_{X/N}) \quad (22)$$

is a TB -algebra.

Proof. Let $a * N, b * N \in X/N$ and $\mathcal{W} \in \tau_{X/N}$ such that $(a * N) \oplus (b * N) \in \mathcal{W}$. Now, π a B -epimorphism implies there exist $x, y \in X$ such that $\pi(x) = a * N$ and $\pi(y) = b * N$. Moreover, $\pi(x * y) = \pi(x) \odot \pi(y) = (a * N) \oplus (b * N) \in \mathcal{W}$, and

so $x * y \in \pi^{-1}(\mathcal{W})$. By Theorem 26, $\pi^{-1}(\mathcal{W}) \in \tau$. Since (X, τ) is a TB -algebra, there are $U, V \in \tau$ such that $x \in U$, $y \in V$, and $U * V \subseteq \pi^{-1}(\mathcal{W})$ by virtue of Theorem 17. Also, we have $\pi(U * V) = \pi(U) \otimes \pi(V)$ since π is a B -homomorphism. Note that $a * N \in \pi(U)$ and $b * N \in \pi(V)$. Now,

$$\pi(U) \otimes \pi(V) = \pi(U * V) \subseteq \pi(\pi^{-1}(\mathcal{W})) \subseteq \mathcal{W}. \quad (23)$$

Since π is an open map by Theorem 27, $\pi(U), \pi(V) \in \tau_{X/N}$. Hence, there are $\tau_{X/N}$ -open sets $\mathcal{S} = \pi(U)$ and $\mathcal{T} = \pi(V)$ in X/N such that $a * N \in \mathcal{S}$, $b * N \in \mathcal{T}$, and $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{W}$. Therefore, by Theorem 17, $(X/N, \tau_{X/N})$ is a TB -algebra. $\square$

5. Conclusion

We have presented the notion of TB -algebras and provided some of its initial properties. We showed that, in a B -algebra, a neighborhood base of the constant element generates a unique B -topology, making a TB -algebra provided that the given B -algebra is commutative. Lastly, we provided some properties of (normal) subalgebras of TB -algebras and investigated the notion of quotient TB -algebras. As a consequence of this study, it is best recommended to explore the properties of TB -algebras with respect to some known topological invariants (e.g., separability and compactness).

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

The researcher gratefully acknowledges the support from the Research Unit of Bukidnon State University, Malaybalay City 8700, Philippines, last year 2018.

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