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Research Article

Asymptotic Exponential Arbitrage in the Schwartz Commodity Futures Model

Tesfamariam Tadesse Welemical,[1]
Jane Akinyi Aduda,[2] **and Martin Le Doux Mbele Bidima**[3]

[1] *Pan African University for Basic Sciences, Technology and Innovation, JKUAT, P.O. Box 6200000200 Nairobi, Kenya*
[2] *Jomo Kenyatta University of Agriculture and Technology, P.O. Box 6200000200 Nairobi, Kenya*
[3] *Department of Mathematics, University of Yaounde I, P.O. Box 823 Yaounde, Cameroon*

Correspondence should be addressed to Tesfamariam Tadesse Welemical; tesfatade@gmail.com

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In this paper, we consider the Schwartz's one-factor model for a storable commodity and a futures contract on that commodity. We introduce the analysis of asymptotic arbitrage in storable commodity models by proving that the futures prices process allows asymptotic exponential arbitrage with geometric decaying failure probability. Next, we find by comparison that, under some similar conditions, our result is a corresponding commodity assets (stronger) version of Föllmer and Schachermayer's result stated in the modeling setting of geometric Ornstein-Uhlenbeck process for financial security assets.

1. Introduction

Classical Arbitrage Theory in financial market models (including both security and commodity models) has been basically concerned with the study of investors' trading opportunities that generate risk-less profit in any finite time horizon $T > 0$. Prohibiting such trading opportunities in any such financial model is characterized in the classical literature by the existence of an equivalent martingale measure (*EMM*), a result celebrated as the "First Fundamental Theorem of Asset Pricing" stated for instance in [1, 2]. It has been proved that most of these market models (financial securities models especially) share the following feature: for any finite time horizon $T > 0$, there is a possibility of excluding arbitrage opportunities from the model by verifying the so-called Novikov condition on the associated "market price of risk" (see for instance [3]). But, when the trading time horizon T is large enough (i.e., when $T \longrightarrow \infty$), one may generate risk-less profits. The name "asymptotic arbitrage" was first given to such long-term risk-less profits by Kabanov and Kramkov in their article "Asymptotic arbitrage in large financial markets" (see [4]). Next, Föllmer and Schachermayer ([5]) in a single financial market (of security assets) discussed this concept of asymptotic arbitrage in a slightly different form first in a typical case of Ornstein-Uhlenbeck process and next in a general continuous-time diffusion setting in which they conjectured the possibility of generating exponential growth profit on investors wealth in long-term (i.e., when $T \longrightarrow \infty$). Mbele Bidima and Rásonyi proved this conjecture in a corresponding discrete-time setting under two different sets of conditions in [6, 7], proposing the name "asymptotic exponential arbitrage" to the literature. Moreover, the authors of [6, 7] clarified the concept by improving the later definition to "asymptotic exponential arbitrage with geometrically/exponentially decaying failure probability" by requiring the control at an exponential/geometrical decaying rate with the probability of failing to achieve such an exponential growth profit in long-term. Next, Du and Neufeld used this later definition in [8] and proved Föllmer and Schachermayer's conjecture in continuous-time setting under stronger conditions using general security models which are not necessarily diffusion processes. And recently, Haba and Jacquier discussed also in [9] the later form of asymptotic arbitrage in the typical case of Heston stochastic volatility model.

We have noticed that those studies on asymptotic arbitrage are carried out since the past decade only for standard security models (equity, bonds, interest rates, etc.) and not yet already for commodity models (oil, gas, coffee, energy, etc.). In this article, given the particularities of commodity modeling (especially for storable commodities), we intend to introduce the analysis of asymptotic (exponential) arbitrage in those models, in the typical case of Schwartz one-factor model of commodity futures, just as Haba and Jacquier discussed it in the special case of Heston model of equities. And we compare our result to a result from our motivating paper [5] of Föllmer and Schachermayer where they treated the special case of geometric Ornstein-Uhlenbeck process for security assets. But first, in Section 2 below, we start with a comparative analysis of (storable) commodity markets with standard security markets, recalling basic definitions and concepts that we use. Next, in Section 3, the key part of this paper, we present the Schwartz's model for storable commodity futures. After verifying absence of arbitrage and model completeness on each finite time horizon, we state and prove our main result. And in Section 4, we end our work with a technical comparative discussion and concluding remarks.

2. Storable Commodities versus Financial Securities

Unlike standard financial securities (such as stock equities, bonds, etc.) and their derivatives which are traded in a single market counter, storable commodities (such as oil, natural gas, coffee, etc.) are related to two interconnected markets: the cash-storage (inventory) market where one (or more) storable commodity is traded at a spot price S_t, $t > 0$, and the futures market in which a futures contract on this underlying commodity, initiated at any time $t \geq 0$ and with delivery date (time horizon) $T > t$, is traded at price $F_t \coloneqq F(t,T)$.

Under the standard assumption of no-arbitrage, the relationship between the spot price and the futures price is given, as stated in [10] or [11, Section 5.6.3], by

$$F(t,T) = S_t e^{r(T-t)}, \tag{1}$$

where r is a constant interest rate on an accompanying risk-free bond paying 1 at time T. This relation is also true in standard security markets. But it is known in the literature that the relationship (1) does not hold in practice for most commodity markets, especially in storable commodity markets, because of the inability of investors and speculators to short the underlying asset, S_t. Hence as indicated in [12], there is an adjustment by the cost of carry in the nonarbitrage pricing formula above which is given by the so-called "(net) convenience yield", a sort of rate of the cost of carrying the physical commodity denoted δ_t, letting the relationship above to become

$$F(t,T) = S_t e^{(r-\delta_t)(T-t)}, \quad \text{for all times } t \in [0,T]. \tag{2}$$

By comparison to financial security markets, as established in [13, p. 37], note that the no-arbitrage price of a forward contract on an underlying security asset (such as stock) with prices process S_t which pays a continuous dividend at (stochastic) rate, say q_t, with the same risk-free interest rate and the same maturity, is mathematically the same as formula (2) above by replacing δ_t with q_t. The only difference resides on the two types of market and the corresponding type of gain on holding or carrying the underlying asset: in financial security markets, an asset may be assumed paying a continuous dividend, and in storable commodity markets, carrying a storable commodity necessarily requires a cost; i.e., the presence of convenience yield is consistent with the theory of storage as explained above.

Next, another distinction of storable commodity markets with security markets is highlighted by Liu and Tang in [14, Lemma 1, Theorem 1] where they proved that the following.

Theorem 1. *(1) If short-selling is allowed in any physical commodity market under no-arbitrage condition, then the convenience yield process δ_t is zero almost surely.*

(2) Under short-selling prohibition, a storable commodity market is arbitrage-free if and only if the convenience yield is always nonnegative i.e., $\delta_t \geq 0$.

The interpretation of these results is as follows: in regulating any storable commodity market with futures contracts, while one should not allow short-selling of physical goods, absence of arbitrage opportunity on any finite trading time period $[0,T]$, $T > 0$, is guaranteed by insuring that the convenience yield is always nonnegative.

This key initial comparative study will lead our choice of realistic assumptions on the study of finite time horizon arbitrage and the investigation of long-term arbitrage in the Stchwartz one-factor model of storable commodity futures below.

3. The Schwartz Commodity Futures Model

For a fixed time horizon $T > 0$, consider a cash-storage market where a single factor, i.e., the only underlying asset, is a single physical commodity with prices process S_t. Besides there is a futures exchange where a single futures contract on this commodity with prices process $F_t \coloneqq F(t,T)$ for all times $t \in [0,T]$. There is also an accompanying bank account or a risk-free bond which serves as numéraire in the no-arbitrage relation (2). It pays 1 at time T with a constant continuously compounded interest rate $r > 0$; hence, its price is $B_t = e^{-rt}$ at any time $t \leq T$.

According to Schwartz in [15], we assume that the spot prices process S_t is governed by the dynamics,

$$\frac{dS_t}{S_t} = \kappa(\mu - \ln S_t)\,dt + \sigma dZ_t \tag{3}$$

or equivalently, the log spot prices process $X_t \coloneqq \ln S_t$ follows a mean-reverting process of Ornstein-Uhlenbeck (OU) type as follows:

$$dX_t = \kappa(\nu - X_t)\,dt + \sigma dZ_t, \tag{4}$$

where $X_0 := \ln S_0$ is an initial real constant, μ, σ (with $\sigma > 0$) are real constants representing respectively the long-term mean and the volatility of the spot prices process S_t, $\kappa > 0$ and $\nu := \mu - \sigma^2/2\kappa$ are respectively the speed and the level of mean-reversion for the OU log prices process X_t, and Z_t is a standard Brownian motion on a modeling probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with respect to which the underlying process S_t (with any other derivative and claim) verifies the usual integrability properties (such as the square-integrability) required in the sequel. We equip this space with the natural filtration $\mathbb{F} := (\mathscr{F}_t)_t$ of the spot prices process S_t, which is the same as the natural filtration of Z_t. $\mathbb{E}$ and $\mathbb{V}ar$ will denote respectively the expectation and the variance with respect to the measure $\mathbb{P}$, and $\mathbb{E}^{\mathbb{Q}}$ will denote the expectation relative to another appropriate measure $\mathbb{Q}$ on the same measurable space $(\Omega, \mathscr{F})$.

We formulate the following set of assumptions in line with the general conditions discussed in the preceding section.

Assumption 2. (1) Short-selling the physical commodity is not allowed, but

(2) one may short-sell any real quantity in the futures contract.

(3) The convenience yield is a constant δ and is strictly positive; i.e., $\delta > 0$.

Itô's Lemma on (2) and (3) implies that the futures prices process F_t, still under the probability measure $\mathbb{P}$, follows the dynamics,

$$dF_t = \sigma F_t \left(\psi_t dt + dZ_t\right), \tag{5}$$

where

$$\psi_t := \frac{\gamma}{\sigma} - \frac{\kappa}{\sigma} X_t, \quad \text{with } \gamma := \kappa\mu + \delta, \tag{6}$$

stands for the "market price of risk" for the futures prices contract. As in financial security markets, its interpretation is straightforward: since the sign of ψ_t at any time t depends on the sign of the log-price $X_t = \ln S_t$ of the commodity, this gives an insight to investors for when to hold negative or positive quantities of the futures contract when building their portfolio in this spot-futures market.

Definition 3. (i) We call the dynamics (5) the Schwartz's model for the commodity futures.

(ii) According to [14] of Liu and Tang, a trading portfolio in this commodity futures model is a pair of $\mathbb{R}$-valued and $\mathbb{F}$-adapted processes $\pi_t := (\alpha_t, \varphi_t)$ representing respectively the investor's position in the risk-free bond of the money market and in the futures contract in the futures exchange.

(iii) The value process of such a trading portfolio (or wealth of the investor) denoted V_t^π is defined by

$$V_t^\pi := \alpha_t B_t + \varphi_t F_t, \quad \text{for all times } t \in [0, T]. \tag{7}$$

(iv) Any such trading opportunity is said to be self-financing if it is a pair of predictable process such that

$$dV_t^\pi := \alpha_t dB_t + \varphi_t dF_t. \tag{8}$$

Assumption 4. For computational simplicity, we also assume that the risk-free interest rate r on the bond is zero, i.e., the bond price is $B_t := 1$ for all times $t \in [0, T]$.

Remark 5. (1) This assumption implies that the discounted futures prices process $\widetilde{F}_t := F_t/B_t$ is just F_t and the discounted value $\widetilde{V}_t^\pi := V_t^\pi/B_t$ of any portfolio is again V_t^π.

(2) Similar to the restriction assumed in [12, Section 2.2], the assumption above hints that we may restrict the definition of self-financing portfolios only to the part φ_t of investment in the futures contract. Denoting the value process now $V_t^\pi \equiv V_t^\varphi$, the self-financing relation above becomes

$$\begin{aligned} dV_t^\varphi &= \varphi_t dF_t \; \textit{i.e.}, \\ V_t^\varphi &= V_0^\varphi + \int_0^t \varphi_s dF_s \\ &\qquad \text{for all times } t \in [0, T]. \end{aligned} \tag{9}$$

Definition 6 (admissible strategy). A self-financing trading portfolio φ_t is called an admissible strategy, if there exists a constant $a \geq 0$ such that its corresponding value process V_t^φ verifies the constraint $V_t^\varphi \geq -a, \mathbb{P} - a.s.$ for all $t \leq T$.

Let $\mathscr{H}$ be the set of admissible self-financing strategies in this model. For any time $t > 0$ (in particular for fixed time horizon $T > 0$), denote the set of value processes for all admissible self-financing strategies φ_s up to time t with $V_0^\varphi = 0$ as $\mathscr{K}_t$; i.e.,

$$\mathscr{K}_t := \left\{ \int_0^t \varphi_s dF_s : \varphi_s \in \mathscr{H} \text{ with } V_0^\varphi = 0 \right\}. \tag{10}$$

3.1. No Finite Horizon Arbitrage and Model Completeness. Recall the following definitions from [16, 17].

Definition 7 (finite horizon arbitrage). (i) For any finite time horizon $T > 0$, a (self-financing) portfolio φ_t is an arbitrage opportunity in this futures market model if its corresponding value process $V_t^\varphi, t \in [0, T]$, satisfies the following conditions.

$$\begin{aligned} &V_0^\varphi = 0 \\ &\text{and } V_T^\varphi \geq 0, \\ &\qquad \mathbb{P} - a.s. \text{ with } \mathbb{P}\left(V_T^\varphi > 0\right) > 0. \end{aligned} \tag{11}$$

(ii) The model is said to be arbitrage-free for any finite time horizon $T > 0$ if there is no arbitrage opportunity in the sense above.

Definition 8 (model completeness). A market model is said to be complete if every contingent claim is attainable; i.e., there is a self-financing strategy whose final value equals the payoff of the claim.

Theorem 1 (2) (recalled from [14]) applied on our Assumption 2 (3) insures there is no arbitrage opportunity in the Schwartz one-factor model of storable commodity (3).

We check in the result below, using the "First Fundamental Theorem of Asset Pricing", that the Schwartz commodity futures model (5) is arbitrage-free on any finite time horizon.

Proposition 9. *The Schwartz's model of commodity futures (5) admits an equivalent martingale measure; hence, it is arbitrage-free, for any finite time horizon $T > 0$.*

Proof. Solving the stochastic differential equation (4) we get for all $t \in [0, T]$,

$$X_t = e^{-\kappa t} X_0 + \nu \left[1 - e^{-\kappa t} \right] + \sigma \int_0^t e^{-\kappa(t-s)} dZ_s. \quad (12)$$

This (as argued for instance in [13, pp. 65-66]) implies that the random variable X_t is normally distributed as

$$X_t \sim \mathcal{N}\left(e^{-\kappa t} X_0 + \nu \left[1 - e^{-\kappa t} \right], \frac{\sigma^2}{2\kappa} \left(1 - e^{-2\kappa t} \right) \right). \quad (13)$$

From this we get $\mathbb{E}(X_t) < \infty$ and $\mathbb{E}(X_t^2) = \mathbb{V}ar(X_t) + [\mathbb{E}(X_t)]^2 < \infty$ for all times $t \in [0, T]$, and next we obtain that

$$\begin{aligned} \mathbb{E}\left[\psi_t^2\right] &= \mathbb{E}\left[\left(\frac{\gamma}{\sigma} - \frac{\kappa}{\sigma} X_t \right)^2 \right] \\ &= \mathbb{E}\left[\frac{\gamma^2}{\sigma^2} - \frac{2\gamma\kappa}{\sigma^2} X_t + \frac{\kappa^2}{\sigma^2} X_t^2 \right] \\ &= \frac{\gamma^2}{\sigma^2} - \frac{2\gamma\kappa}{\sigma^2} \mathbb{E}\left[X_t\right] + \frac{\kappa^2}{\sigma^2} \mathbb{E}\left[X_t^2\right] < \infty. \end{aligned} \quad (14)$$

Hence $\mathbb{E}[\exp((1/2)\int_0^t \psi_s^2 ds)] < \infty$, showing that the market price of risk $\psi_t = \gamma/\sigma - (\kappa/\sigma) X_t$ for the futures prices satisfies the Novikov condition. It follows by Radon-Nikodym Theorem that the strictly positive and $\mathbb{P}$-integrable random variable $L_T : \Omega \longrightarrow \mathbb{R}_+$ defined as

$$L_T \coloneqq \exp\left(-\int_0^T \psi_t dZ_t - \frac{1}{2} \int_0^T \psi_t^2 dt \right) \quad (15)$$

defines an equivalent probability measure $\mathbb{Q}$ by setting $d\mathbb{Q}/d\mathbb{P} \coloneqq L_T$. Applying Girsanov's Theorem (see [17, Theorem 4.5.1]), this measure is an equivalent martingale measure, i.e., under which the (discounted) futures prices process F_t is a martingale and is written as

$$dF_t = \sigma F_t dW_t, \quad (16)$$

where $W_t \coloneqq Z_t + \int_0^t \psi_s ds$, $t \in [0, T]$, is a Brownian motion under $\mathbb{Q}$, showing the result, as required. ☐

Next, we state and prove the following result.

Proposition 10. *The Schwartz commodity futures model (5) is complete.*

Proof. For any fixed time horizon $T > 0$, we have to show that for any contingent claim with payoff a (square integrable) random variable C_T, there is a self-financing portfolio φ_t such that $V_T^\varphi = C_T$. The self-financing condition (9) means, we seek a predictable process φ_t such that

$$dV_t^\varphi = \sigma \varphi_t F_t dW_t \quad \text{with } V_T^\varphi = C_T, \quad (17)$$

under the equivalent martingale measure $\mathbb{Q}$.

Noting that the filtration $(\mathcal{F}_t)_t$ is also generated by the Brownian motion W_t, the (discounted) portfolio value process V_t^φ is a $\mathbb{Q}$-martingale if and only if

$$V_t^\varphi = \mathbb{E}^{\mathbb{Q}}\left[V_T^\varphi \mid \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[C_T \mid \mathcal{F}_t \right] \quad \text{for all times } t \in [0, T], \quad (18)$$

and the process $M_t \coloneqq \mathbb{E}^{\mathbb{Q}}[C_T \mid \mathcal{F}_t]$ is a $\mathbb{Q}$-martingale. But it is easy to check that the process M_t is a $\mathbb{Q}$-martingale since C_T is square $\mathbb{Q}$-integrable hence integrable because it is square $\mathbb{P}$-integrable (from the model set-up assumptions) and $\mathbb{Q}$ is equivalent to $\mathbb{P}$. Therefore, the Itô martingale representation theorem (in [17, Theorem 4.6.2]) entitles that there is an $\mathbb{F}$-predictable process θ_t such that

$$\begin{aligned} dM_t &= \theta_t dW_t \text{ i.e.,} \\ dV_t^\varphi &= \theta_t dW_t. \end{aligned} \quad (19)$$

Hence it is enough in (17) to take $\varphi_t \coloneqq \theta_t / \sigma F_t$ for all times $t \in [0, T]$, ending the proof. ☐

Remark 11. Denote $\mathcal{M}_T^e(F)$ the set of equivalent local-martingale measures (which are equivalent martingale measures since F_t is a diffusion) in the model up to time $T > 0$. By the "Second Fundamental Theorem of Asset Pricing" (see [11, Theorem 5.4.9]), the completeness of this model entitles that $\mathcal{M}_T^e(F) = \{\mathbb{Q}\}$ where $\mathbb{Q}$ is the above equivalent martingale probability measure. We will need this fact as one key argument in proving the main result of this paper in the subsection below.

3.2. Existence of Asymptotic Exponential Arbitrage. Recall under Assumptions 2 and 4 that the futures prices of the underlying commodity in (2) is $F_t \equiv F(t, T) \coloneqq S_t e^{-\delta(T-t)}$ for all times $t \in [0, T]$, where $T > 0$ is a fixed time horizon, and that for any admissible self-financing portfolio φ_t in $\mathcal{H}$, the investors' wealth in the futures contract which we considered in (9) is $V_t^\varphi = V_0^\varphi + \int_0^t \varphi_s dF_s$ at any time $t \in [0, T]$.

We see that, unlike in financial security models, since the futures price depends on two time parameters t and T, to discuss the asymptotic behavior of the wealth process V_t^φ, i.e., from long-term trading in the futures contract when the delivery date T becomes larger and larger, it is enough to do it when the running time t is getting larger and larger.

Next, Föllmer and Schachermayer introduced the following form of asymptotic arbitrage, which we adapt here in our present modeling setting of commodity futures.

Definition 12 (Definition 1.1 in [5]). The futures price process F_t is said to allow (strong) asymptotic arbitrage if, for any threshold $\varepsilon > 0$, there are $t < \infty$ and $V_t^{\varphi} \in \mathcal{K}_t$ such that

$$\begin{gathered}\text{(i) } V_t^{\varphi} \geq -\varepsilon, \quad \mathbb{P}-a.s., \\ \text{(ii) } \mathbb{P}\left[V_t^{\varphi} \geq \varepsilon^{-1}\right] \geq 1-\varepsilon.\end{gathered} \tag{20}$$

And they stated the following general result to prove existence of (strong) asymptotic arbitrage in a general semimartingale setting.

Proposition 13 (Proposition 2.1 in [5]). *For any locally bounded semimartingale S_t such that the set $\mathcal{M}_t^e(S)$ of equivalent local martingales, for any time $t > 0$, is not empty, then for any constants $0 < \varepsilon_1, \varepsilon_2 < 1$, for any time $t > 0$, the following are equivalent:*

(i) There is $V_t^{\varphi} \in \mathcal{K}_t$ such that $V_t^{\varphi} \geq -\varepsilon_2$, $\mathbb{P}-a.s.$ and $\mathbb{P}[V_t^{\varphi} \geq 1-\varepsilon_2] \geq 1-\varepsilon_1$.

(ii) There is $A_t \in \mathcal{F}_t$ with $\mathbb{P}[A_t] \leq \varepsilon_1$ such that for each $\widetilde{\mathbb{P}} \in \mathcal{M}_t^e(S)$ we have $\widetilde{\mathbb{P}}[A_t] \leq 1-\varepsilon_2$.

In our present paper we use this result in proving that the futures prices process F_t in (5) allows asymptotic exponential arbitrage in the sense below, defined by Mbele Bidima and Rásonyi, and we compare our result to their result in a special case of standard geometric Ornstein-Uhlenbeck process for security assets.

Definition 14 (from Definition 1.1 in [6]). We say that the futures prices process F_t generates asymptotic exponential arbitrage (AEA) with geometrically decaying failure probability (GDP-F), if there are constants $C, \lambda_1, \lambda_2 > 0$ and a trading strategy $\varphi_t \in \mathcal{H}$ such that the value process V_t^{φ} satisfies the following conditions:

(i) $V_t^{\varphi} \geq -e^{-\lambda_1 t}$, $\mathbb{P}-$a.s.,

(ii) and $\mathbb{P}[V_t^{\varphi} \geq e^{\lambda_1 t}] \geq 1 - Ce^{-\lambda_2 t}$ or equivalently,

(ii)' $\mathbb{P}[V_t^{\varphi} < e^{\lambda_1 t}] \leq Ce^{-\lambda_2 t}$, for large time $t > 0$.

This definition presents a better economic interpretation than Föllmer and Schachermayer's definition. Indeed, Definition 12 above means that while the maximum loss an investor may achieve is bounded by ε, its wealth up to time t is at least ε^{-1} with probability larger than $1-\varepsilon$. But there is no clear relationship between the threshold ε and the time t from which one realizes a potential profit. This drawback is solved in the preceding definition which says that the maximal loss of the investor's wealth at any time t is exponentially bounded in (i) by $e^{-\lambda_1 t}$ and (ii) means that, even from zero initial capital, an investor may generate a profit that grows exponentially fast in time with probability converging to 1 exponentially fast. And (ii)' states that failing to achieve such a growth profit can be controlled in time by a probability converging to 0 exponentially fast.

Now we state below the main theorem of this paper.

Theorem 15. *Consider the Schwartz commodity futures model (5). Under Assumptions 2 and 4, there exists a trading strategy $\varphi_t \in \mathcal{H}$ such that its corresponding value process V_t^{φ} (with zero initial value i.e., $V_0^{\varphi} = 0$) satisfies, for some positive constants $c, \lambda \in \mathbb{R}$ with $c < \lambda \in (0, \kappa/2 + \gamma^2/2\sigma^2)$,*

$$\begin{gathered}(i)\; V_t^{\varphi} \geq -e^{-ct}, \quad \mathbb{P}-a.s., \\ (ii)\; \mathbb{P}\left[V_t^{\varphi} \geq e^{ct}\right] \geq 1-e^{-bt},\end{gathered} \tag{21}$$

for large time $t > 0$, where $b := (\gamma^2/2\sigma^2 + \kappa/4 - \lambda)^2/(\gamma^2/2\sigma^2 + \kappa/2 - \lambda) > 0$.

The proof of this result relies on the application of Proposition 13 above, an appropriate use of Large Deviation Principle in the following preliminary result, which we prove analogously to that derived in [5].

Indeed, consider, for any time $t > 0$ and for any constant $\lambda > 0$, the event,

$$A_t := \left[L_t e^{(Y_t - Y_0)\gamma/\sigma} \geq e^{-\lambda t}\right], \tag{22}$$

where ψ_t is the market price of risk process for the futures prices process F_t, $L_t := \exp(-\int_0^t \psi_s dZ_s - (1/2)\int_0^t \psi_s^2 ds)$ is the Radon-Nikodym density process similarly defined as in equation (15) and $Y_t := (1/\sigma)X_t$ is the log spot prices process weighted by the volatility σ. Then, we have the following.

Lemma 16. *Assume for computational simplicity that the level of mean-reversion of log spot prices in (4) is 0; i.e., $\nu := \mu - \sigma^2/2\kappa = 0$. Then, for any $\lambda \in (0, \kappa/2 + \gamma^2/2\sigma^2)$, we have*

$$\limsup_{t \to \infty} \frac{1}{t} \ln \mathbb{P}\left[A_t\right] \leq -b, \tag{23}$$

where b is defined as above.

Proof. Let us mention first that our proof below is analogue (but not exactly) to part of that of Proposition 4.1 in [5] from which we avoid a complicating used argument on ergodic properties of the process X_t by defining our A_t above differently.

Next, since we assumed $\nu := \mu - \sigma^2/2\kappa = 0$; i.e., $\mu = \sigma^2/2\kappa$, then (4) becomes

$$\begin{gathered}dX_t = -\kappa X_t dt + \sigma dZ_t \\ \text{or equivalently } dY_t = -\kappa Y_t dt + dZ_t.\end{gathered} \tag{24}$$

For all time $t > 0$ we have $\ln(L_t e^{(Y_t - Y_0)\gamma/\sigma}) = \ln L_t + (Y_t - Y_0)\gamma/\sigma$, and

$$\begin{aligned}\ln L_t &= -\int_0^t \psi_s dZ_s - \frac{1}{2}\int_0^t \psi_s^2 ds \\ &= -\int_0^t \left(\frac{\gamma}{\sigma} - \frac{\kappa}{\sigma}X_s\right) dZ_s - \frac{1}{2}\int_0^t \left(\frac{\gamma}{\sigma} - \frac{\kappa}{\sigma}X_s\right)^2 ds \\ &= -\int_0^t \left(\frac{\gamma}{\sigma} - \kappa Y_s\right)(dY_s + \kappa Y_s ds) \\ &\quad - \frac{1}{2}\int_0^t \left(\frac{\gamma}{\sigma} - \kappa Y_s\right)^2 ds\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\gamma}{\sigma}\int_0^t \left(dY_s + \kappa Y_s ds\right) + \kappa \int_0^t Y_s \left(dY_s + \kappa Y_s ds\right) \\
&\quad - \frac{1}{2}\int_0^t \left(\frac{\gamma^2}{\sigma^2} - 2\frac{\gamma\kappa}{\sigma}Y_s + \kappa^2 Y_s^2\right) ds \\
&= -\frac{\gamma}{\sigma}\left(Y_t - Y_0\right) + \kappa \int_0^t Y_s dY_s + \kappa^2 \int_0^t Y_s^2 ds \\
&\quad - \frac{\kappa^2}{2}\int_0^t Y_s^2 ds - \frac{\gamma^2}{2\sigma^2}t \\
&= -\frac{\gamma}{\sigma}\left(Y_t - Y_0\right) + \kappa \int_0^t Y_s dY_s + \frac{\kappa^2}{2}\int_0^t Y_s^2 ds \\
&\quad - \frac{\gamma^2}{2\sigma^2}t \\
&= \frac{\gamma}{\sigma}\left(Y_0 - Y_t\right) + \kappa \int_0^t Y_s dY_s + \frac{\kappa^2}{2}\int_0^t Y_s^2 ds \\
&\quad - \frac{\gamma^2}{2\sigma^2}t \\
&= \kappa \left(\int_0^t Y_s dY_s + \frac{\kappa}{2}\int_0^t Y_s^2 ds\right) + \frac{\gamma}{\sigma}\left(Y_0 - Y_t\right) \\
&\quad - \frac{\gamma^2}{2\sigma^2}t.
\end{aligned} \tag{25}
$$

So for all times $t > 0$,

$$
\ln\left(L_t e^{(Y_t - Y_0)\gamma/\sigma}\right) = \kappa\Gamma_t - \frac{\gamma^2}{2\sigma^2}t, \qquad \text{where } \Gamma_t := \int_0^t Y_s dY_s + \frac{\kappa}{2}\int_0^t Y_s^2 ds \tag{26}
$$

Applying Theorem 2.2 of [18], the stochastic process Γ_t/t satisfies a large deviations principle with good rate function I; i.e., (see for instance [19])

$$
\begin{aligned}
-\inf_{x\in U} I(x) &\le \liminf_{t\longrightarrow\infty} \frac{1}{t}\ln \mathbb{P}\left[\frac{\Gamma_t}{t} \in U\right] \\
&\le \limsup_{t\longrightarrow\infty} \frac{1}{t}\ln \mathbb{P}\left[\frac{\Gamma_t}{t} \in B\right] \le -\inf_{x\in B} I(x),
\end{aligned} \tag{27}
$$

for any open set U and any closed set B in the natural topology on $\mathbb{R}$, with the function I defined by

$$
I(x) = \begin{cases} \dfrac{2\kappa (x + 1/4)^2}{2x+1}, & \text{if } x > -\dfrac{1}{2} \\ \infty, & \text{if } x \le -\dfrac{1}{2}. \end{cases} \tag{28}
$$

Since this function is strictly increasing on $(-1/2, \infty)$, the large deviations upper bound above implies for $B := [x, \infty]$ that $\lim_{t\longrightarrow\infty}(1/t)\ln \mathbb{P}[\Gamma_t/t \ge x] \le -I(x)$, for any $x > -1/2$. Next, for all times $t > 0$, we obtain from (26) that

$$
\begin{aligned}
\mathbb{P}\left[A_t\right] &= \mathbb{P}\left[L_t e^{(Y_t - Y_0)\gamma/\sigma} \ge e^{-\lambda t}\right] \\
&= \mathbb{P}\left[\ln\left(L_t e^{(Y_t - Y_0)\gamma/\sigma}\right) \ge -\lambda t\right] \\
&= \mathbb{P}\left[\frac{\Gamma_t}{t} \ge \frac{\gamma^2}{2\kappa\sigma^2} - \frac{1}{\kappa}\lambda\right].
\end{aligned} \tag{29}
$$

Hence,

$$
\begin{aligned}
&\limsup_{t\longrightarrow\infty} \frac{1}{t}\ln \mathbb{P}\left[A_t\right] \\
&\quad \le \limsup_{t\longrightarrow\infty} \frac{1}{t}\ln \mathbb{P}\left[\frac{\Gamma_t}{t} \ge \frac{\gamma^2}{2\kappa\sigma^2} - \frac{1}{\kappa}\lambda\right] \\
&\quad \le -I\left(\frac{\gamma^2}{2\kappa\sigma^2} - \frac{1}{\kappa}\lambda\right) = -b,
\end{aligned} \tag{30}
$$

showing the lemma, as required. □

Next, we have the following.

Proof of Theorem 15. The preceding lemma implies that $\mathbb{P}[A_t] \le e^{-bt}$ for all time $t > 0$. Next, by definition of the Radon-Nikodym density L_t, we have $\mathbb{Q}[\Omega \setminus A_t] = \int_{\Omega\setminus A_t} L_t d\mathbb{P} \le e^{-\lambda t}$. Hence $\mathbb{Q}[A_t] \ge 1 - e^{-\lambda t}$ for all times $t > 0$. Now, since the futures prices process F_t is a locally bounded semimartingale, then Proposition 13 and Remark 11 (on the uniqueness of the *EMM* $\mathbb{Q}$) imply (with $\varepsilon_1 := e^{-bt}$ and $\varepsilon_2 := e^{-\lambda t}$) that for large time $t > 0$, there is a value process $V_t^{\varphi'} \in \mathscr{K}_t$; i.e., there exists a trading strategy $\varphi_t' \in \mathscr{H}$ such that $V_t^{\varphi'} \ge -e^{-\lambda t}$, $\mathbb{P} - a.s.$ and $\mathbb{P}[V_t^{\varphi'} \ge 1 - e^{-\lambda t}] \ge 1 - e^{-bt}$.

To complete the proof, for any positive constant $c < \lambda$ and for a large enough time $t > 0$, define the trading strategy $\varphi_s := \varphi_s' e^{(\lambda - c)t}$. Then clearly $\varphi_s \in \mathscr{H}$ and we have (at large time t) that $V_t^{\varphi} = e^{(\lambda - c)t}V_t^{\varphi'}$, which implies that $V_t^{\varphi} \ge -e^{-ct}$, $\mathbb{P} - a.s.$ and that

$$
\begin{aligned}
\mathbb{P}\left[V_t^{\varphi} \ge e^{ct}\right] &= \mathbb{P}\left[V_t^{\varphi'} \ge e^{-\lambda t}\right] \\
&\ge \mathbb{P}\left[V_t^{\varphi'} \ge 1 - e^{-\lambda t}\right] (\text{since } t \text{ is large enough}) \\
&\ge 1 - e^{-bt},
\end{aligned} \tag{31}
$$

which shows existence of a trading strategy φ_t that generates *AEA* with *GDP-F* in the Schwartz commodity futures model, as required. □

Remark 17. Note that we have proved this result after proving Lemma 16 under the condition that the mean-reversion level ν of the log spot prices process X_t is zero. This assumption was made to simplify computations. Indeed, if we assume that $\nu := \mu - \sigma^2/2\kappa \ne 0$, then Theorem 15 is still stated and proved similarly. We only modify the proof of Lemma 16 by replacing the term $(\gamma^2/2\sigma^2)t$ in (25) with the term $(\kappa\nu\gamma/\sigma^2 - \kappa^2\nu/\sigma -$

$\gamma^2/2\sigma^2)t$ and the constant b in the inequality (30) would be obtained as $I(\kappa\nu/\sigma + \gamma^2/2\kappa\sigma^2 - \nu\gamma/\sigma^2 - (1/\kappa)\lambda)$, where I is the same rate function in (28).

4. Discussion and Conclusion

The novelty of our works and the impact of our main theorem in this paper are highlighted in the following comparative discussion. Using our modeling notations, Föllmer and Schachermayer considered a financial security prices process S_t evolving as simple case geometric Ornstein-Uhlenbeck process $S_t := e^{X_t}$, where

$$dX_t = -\kappa X_t dt + \sigma Z_t. \tag{32}$$

And they stated and proved in Theorem 4.2 of [5] the following:

If $\lambda \in (0, \sigma^2/8 + \kappa/4)$ and any $c_1 < \lambda$, then there is a contingent claim $X_T \in \mathscr{K}_T$ such that, for any $c_2 < \lambda - c_1$, we have

(i) $X_T \geq -e^{-c_2 T}$ for large T,

(ii) $\lim_{T\longrightarrow\infty} \ln \mathbb{P}[X_T < e^{c_1 T}] = -(\sigma^2/8 + \kappa/4 - \lambda)/(\sigma^2/8 + \kappa/2 - \lambda) =: b'$.

(1) Note first that the Schwartz one-factor model for a storable commodity which we considered in (3)-(4) corresponds to their security modeling setting under the case $\nu := \mu - \sigma^2/2\kappa = 0$ which we assumed in Lemma 16. And although stated under similar conditions, the proof of our result above (Theorem 15) is established in part using our own method.

(2) Moreover, we assumed no short-selling of the underlying physical commodity in our work, a condition which was absent in the security modeling setting of [5]. Furthermore, the authors of [5] considered the security asset in trading is paying no dividend while in our storable commodity setting we considered the futures contract in trading with a constant positive convenience yields δ.

(3) Next, statement (ii) of their result above implies $\mathbb{P}[X_T \geq e^{c_1 T}] \geq 1 - e^{-b'T}$ for large time horizon $T > 0$, which is similar to statement (ii) of our main theorem. But one sees the difference with our corresponding exponential decaying probability rate $b = (\gamma^2/2\sigma^2 + \kappa/4 - \lambda)^2/(\gamma^2/2\sigma^2 + \kappa/2 - \lambda)$, where the term $\sigma^2/8$ in the expression of b' is replaced by $\gamma^2/2\sigma^2$ with $\gamma := \kappa\mu + \delta = \sigma^2/2 + \delta$.

We clearly notice here that if $\delta = 0$, then $b = b'$ and our result will correspond to Föllmer and Schachermayer's Theorem 4.2.

But we assumed $\delta > 0$; hence, we clearly observe the effect of the constant convenience yield δ in this corresponding storable commodity setting, which, in addition, highlights the difference between the two results regarding the market price of risk expression. Indeed, in the modeling setting of Föllmer and Schachermayer, the market price of risk from (8) of [5] for the security asset is the process $-(\kappa/\sigma)X_t + \sigma/2$, and our market price of risk process for the futures prices which we derived in (6) is $-(\kappa/\sigma)X_t + \gamma/\sigma = -(\kappa/\sigma)X_t + \sigma/2 + \delta/\sigma$. This is greater (by the constant δ/σ) almost surely than the former market price of risk, showing that carrying a physical commodity is riskier in trading the Schwartz' commodity futures than trading a financial security in the corresponding geometric Ornstein-Uhlenbeck setting of Föllmer and Schachermayer.

We conclude at this point that our work in this paper could be viewed as a (storable) commodity version of the works of Föllmer and Schachermayer done in the financial security setting of geometric Ornstein-Uhlenbeck. And Remark 17 highlights under a general level of mean-reversion condition that our result is stronger than this corresponding version (Theorem 4.2) of Fölmer and Schachermayer (2007).

To conclude, we mention a common weak point of the two results, ours and Föllmer and Schachermayer's. Unlike in the works of Mbele Bidima and Rásonyi in [6, 7] done in discrete-time settings, and like other similar works such as [8, 9], the trading opportunity producing asymptotic exponential arbitrage is not explicitly constructed. Nevertheless, our result gives an insight to potential investors that even if arbitrage opportunities are ruled out for finite delivery date $T > 0$ as guaranteed in Proposition 9, trading in log-term (i.e., when T is large enough) in the Schwartz commodity futures exchange may generate risk-less profit for some trading strategy to be still found.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] F. Delbaen and W. Schachermayer, "The fundamental theorem of asset pricing for unbounded stochastic processes," *Mathematische Annalen*, vol. 312, no. 2, pp. 215–250, 1998.

[2] J. M. Harrison and S. R. Pliska, "Martingales and stochastic integrals in the theory of continuous trading," *Stochastic Processes and Their Applications*, vol. 11, no. 3, pp. 215–260, 1981.

[3] N. Dokuchaev, "Mean-reverting market model: speculative opportunities and non-arbitrage," *Applied Mathematical Finance*, vol. 14, no. 4, pp. 319–337, 2007.

[4] Y. M. Kabanov and D. O. Kramkov, "Asymptotic arbitrage in large financial markets," *Finance and Stochastics*, vol. 2, no. 2, pp. 143–172, 1998.

[5] H. Föllmer and W. Schachermayer, "Asymptotic arbitrage and large deviations," *Mathematics and Financial Economics*, vol. 1, pp. 213–249, 2008.

[6] M. L. Mbele Bidima and M. Rasonyi, "On long-term arbitrage opportunities in Markovian models of financial markets," *Annals of Operations Research*, vol. 200, pp. 131–146, 2012.

[7] M. L. D. Mbele Bidima and M. Rásonyi, "Asymptotic exponential arbitrage and utility-based asymptotic arbitrage in Markovian models of financial markets," in *Acta Applicandae Mathematicae*, vol. 138, pp. 1–15, Springer, 2015.

[8] K. Du and A. D. Neufeld, "A note on asymptotic exponential arbitrage with exponentially decaying failure probability," *Journal of Applied Probability*, vol. 50, no. 3, pp. 801–809, 2013.

[9] F. Haba and A. Jacquier, "Asymptotic arbitrage in the Heston model," *International Journal of Theoretical and Applied Finance*, vol. 18, no. 8, 2015.

[10] J. C. Cox, J. E. Ingersoll Jr., and S. A. Ross, "The relation between forward prices and futures prices," *Journal of Financial Economics*, vol. 9, no. 4, pp. 321–346, 1981.

[11] S. E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer Science & Business Media, 2004.

[12] R. Aïd, L. Campi, and D. Lauthier, "A note on the spot-forward no-arbitrage relations in a trading-production model for commodities," 2015, https://arxiv.org/abs/1501.00273.

[13] H. Geman, *Commodities and Commodity Derivatives*, Wiley Finance, 2005.

[14] P. P. Liu and K. Tang, "No-arbitrage conditions for storable commodities and the modeling of futures term structures," in *International Journal of Mathematics and Mathematical Sciences*, vol. 34, pp. 1675–1687, Elsevier, 2010.

[15] E. S. Schwartz, "The stochastic behavior of commodity prices: Implications for valuation and hedging," *Journal of Finance*, vol. 52, no. 3, pp. 923–973, 1997.

[16] F. Delbaen and W. Schachermayer, *The Mathematics of Arbitrage*, Springer Science & Business Media, 2006.

[17] A. Etheridge and Baxter M, *A Course in Financial Calculus*, Cambridge University Press, 2002.

[18] D. Florens-Landais and H. Pham, "Large deviations in estimation of an Ornstein-Uhlenbeck model," *Journal of Applied Probability*, vol. 36, no. 1, pp. 60–77, 1999.

[19] A. Dembo and O. Zeitouni, *Large Deviations Techniques and Applications*, vol. 38 of *Applications of Mathematics (New York)*, Springer, New York, NY, USA, 2nd edition, 1998.