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Research Article

Blow-Up Rate Estimates for a System of Reaction-Diffusion Equations with Gradient Terms

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This paper is concerned with the blow-up properties of Cauchy and Dirichlet problems of a coupled system of Reaction-Diffusion equations with gradient terms. The main goal is to study the influence of the gradient terms on the blow-up profile. Namely, under some conditions on this system, we consider the upper blow-up rate estimates for its blow-up solutions and for the gradients.

1. Introduction

In this paper, we consider the following problem:

$$\begin{aligned} u_t &= \Delta u - |\nabla u|^{q_1} + v^{p_1}, \quad (x,t) \in \Omega \times (0,T) \\ v_t &= \Delta v - |\nabla v|^{q_2} + u^{p_2} \quad (x,t) \in \Omega \times (0,T) \\ u(x,0) &= u_0(x) \geq 0, \quad x \in \Omega \\ v(x,0) &= v_0(x) \geq 0, \quad x \in \Omega, \end{aligned} \tag{1}$$

where $p_1, p_2 \in (1,\infty)$; $q_1, q_2 \in (1,2)$;

$\Omega = R^n$ or B_R (a ball in R^n with radius R).

Moreover, for $\Omega = B_R, u$ and v satisfy the zero Dirichlet boundary conditions:

$$u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega, \ t \in (0,T); \tag{2}$$

$u_0, v_0 \in C^2(\overline{\Omega})$ are both nonzero, satisfying the monotonicity conditions:

$$\Delta u_0 - |\nabla u_0|^{q_1} + {v_0}^{p_1} \geq 0, \quad x \in \Omega \tag{3}$$

$$\Delta v_0 - |\nabla v_0|^{q_2} + {u_0}^{p_2} \geq 0, \quad x \in \Omega \tag{4}$$

Moreover, in case of $\Omega = B_R$, (u_0, v_0) should satisfy compatibility conditions:

$$u_0(x) = v_0(x) = 0, \quad x \in \partial\Omega \tag{5}$$

As an application to system (1), a single equation of this system can be considered a simple model in population dynamics, [1, 2]:

$$u_t = \Delta u - |\nabla u|^q + u^p, \quad \text{in } \Omega \times \{t \geq 0\} \tag{6}$$

where $p, q > 1$.

Let the domain Ω represent a territory where a biological species live on. $u(x,t)$ refers to the spatial density of individuals located near a point $x \in \Omega$ at time $t \geq 0$.

In fact, the evolution of this density is the result of three types of mechanisms: displacement, birth, and death. For more details of deriving the evolution equation satisfied by u, see [1].

Basically, under different assumptions on the mechanisms of accidental death, the corresponding term should more generally be a nondecreasing function of the density u and its gradient ∇u.

Moreover, homogeneous Dirichlet's conditions can be added to this model which, for instance, correspond to a nonviable environment in the boundary zone.

It is expected that, with a large size initial function (initial distribution of population u_0), the density u becomes unbounded in a finite time $T > 0$. Therefore, Chipot and Weissler [3] studied the effect of the damping term in this equation on global existence or nonexistence.

The blow-up phenomena in Reaction-Diffusion equations have been intensively studied; see, for instance, [4–7]. One of the studied cases is the Cauchy problem of the semilinear heat equation:

$$\begin{aligned} u_t &= \Delta u + u^p, \quad (x,t) \in \mathrm{R}^n \times (0,T), \\ u(x,t) &= u_0(x) \geq 0, \quad x \in \mathrm{R}^n \end{aligned} \tag{7}$$

The second studied case is zero Dirichlet problem of the semilinear heat equation:

$$\begin{aligned} u_t &= \Delta u + u^p, \quad (x,t) \in B_R \times (0,T), \\ u(x,t) &= 0, \quad x \in \partial B_R \\ u(x,t) &= u_0(x) \geq 0, \quad x \in B_R \end{aligned} \tag{8}$$

where $p > 1$.

For both cases (7) and (8), it has been proved in [8, 9] that if the initial function is nonnegative and suitably large, then blow-up occurs in a finite time. In [5, 10], it has been shown that the upper blow-up rate estimate for this equation is as follows:

$$u(x,t) \leq \frac{C}{(T-t)^{1/(p-1)}}, \quad (x,t) \in \Omega \times (0,T). \tag{9}$$

The blow-up properties of semiliear heat equations with negative sign gradient terms (damping terms) have been studied by some authors as in [3, 7, 11].

One of these equations is population model (6). For $\Omega \subseteq R^n$, it is well known that blow-up can only occur if $p > q$; see [3, 11, 12].

Moreover, if $\Omega = B_R$, then blow-up occurs at the center of B_R, and this follows from the upper point-wise estimate:

$$u(x,t) \leq \frac{c}{|x|^\alpha}, \quad x \in B_R \setminus \{0\}, \ t \in [0,T), \tag{10}$$

where $\alpha > 2/(p-1)$, for $1 < q < 2p/(p+1)$,

while $\alpha > q/(p-q)$, for $2p/(p+1) \leq q < p$.

It is clear that $q/(p-q) > 2/(p-1)$, where $q > 2p/(p+1)$

Therefore, the profile of blow-up solutions of (6) is similar to that of problem (8), where $q < 2p/(p+1)$ (see [12]), while if $q > 2p/(p+1)$, the gradient term causes more effect on the plow-up profile and it becomes more singular.

Moreover, it has been proved in [4, 13, 14], that there are positive constants a and b, such that the upper and lower blow-up rate estimates for this equation, where $< 2p/(p+1)$, take the following form:

$$\frac{a}{(T-t)^{1/(p-1)}} \leq u(x,t) \leq \frac{b}{(T-t)^{1/(p-1)}} \tag{11}$$

In [15–18], the coupled system of Reaction-Diffusion equations was considered:

$$\begin{aligned} u_t &= \Delta u + v^{p_1}, \\ v_t &= \Delta v + u^{p_2}, \end{aligned} \tag{12}$$

$$(x,t) \in \Omega \times (0,T), \tag{13}$$

where $p_1, p_2 > 1$; $\Omega = B_R$ or R^n.

It was shown that if the initial functions satisfy $u_0, v_0 \geq 0$, both being nonzero and large enough, then blow-up occurs in a finite time.

For the Cauchy problem associated with (12), it was proved in [16] that if

$$\max\{\alpha, \beta\} \geq \frac{n}{2}, \tag{14}$$

then blow-up occurs in a finite time, where

$$\begin{aligned} \alpha &= \frac{p_1+1}{p_1 p_2 - 1}, \\ \beta &= \frac{p_2+1}{p_1 p_2 - 1} \end{aligned} \tag{15}$$

Later, in [4, 15], it was proved that the upper blow-up rate estimates of this system are as follows:

$$\begin{aligned} u(x,t) &\leq \frac{C_1}{(T-t)^\alpha}, \quad (x,t) \in \Omega \times (0,T), \\ v(x,t) &\leq \frac{C_2}{(T-t)^\beta}, \quad (x,t) \in \Omega \times (0,T). \end{aligned} \tag{16}$$

for some $C_1, C_2 > 0$.

The system (1) has been studied in [19], where $\Omega \in R^3$ and is a bounded convex domain and

$$\begin{aligned} &p = p_1 = p_2 \quad \left(\alpha = \beta = \frac{1}{1-p}\right); \\ &q = q_1 = q_2; \quad p > q > 1. \end{aligned} \tag{17}$$

It has been shown that if a classical solution of this system blows ups (becomes unbounded) in the W-norm, where

$$W(t) = \int_\Omega \left(u^{2p} + v^{2p}\right) dx \tag{18}$$

then blow-up time for this problem can be estimated from below as follows:

$$T \geq \frac{1}{2AW_0^2} \tag{19}$$

where $W(0) = W_0 = \int_\Omega (u_0^{2p} + v_0^{2p}) dx$

and A is a constant which depends on the data.

For the blow-up times and applications of other parabolic systems with damping terms (such as Keller-Segel system with Neumann and Robin boundary conditions), we refer to [20, 21].

In this paper, with some restricted conditions on system (1), we show that the upper blow-up rate estimates for this solution and its gradients terms take the following forms:

$$\begin{aligned} u(x,t) &\le C_1 (T-t)^{-\alpha}, \\ |\nabla u(x,t)| &\le C_1 (T-t)^{-(1+2\alpha)/2}, \\ v(x,t) &\le C_2 (T-t)^{-\beta}, \\ |\nabla v(x,t)| &\le C_2 (T-t)^{-(1+2\beta)/2}, \end{aligned} \tag{20}$$

where $(x,t) \in \Omega \times (0,T)$ and $C_1, C_2 > 0, \alpha, \beta$ are given in (15).

2. Local Existence and Blow-up

Set

$$\begin{aligned} F_1(v, \nabla u) &= v^{p_1} - |\nabla u|^{q_1}, \\ F_2(u, \nabla v) &= u^{p_1} - |\nabla v|^{q_1} \end{aligned} \tag{21}$$

Since the system (1) is uniformly parabolic and its equations have the same principle parts and $F_1, F_2 \in C^1(R \times R^n)$, also the growths of the nonlinearities in F_1 and F_2 with respect to the gradient terms are subquadratic; $u_0, v_0 \in C^2(\overline{\Omega})$, and satisfying (5), it follows that the local existence and uniqueness of classical solution to the for system (1), where $\Omega = B_R$, with zero Dirichlet boundary conditions, are guaranteed by standard parabolic theory (see Theorem 7.1, [22, 23]).

i.e., there exists $T > 0$, such that

$$u \in C^{2,1}(\Omega \times (0,T)) \cap C\left(\overline{\Omega} \times [0,T)\right). \tag{22}$$

Also, the gradient terms are bounded as long as the components of the solution are bounded; see [23].

In case of $\Omega = R^n$, these results can also be extended to the Cauchy problem associated with system (1) (see Theorem 8.1, [22, 24]).

Moreover, from the monotonicity assumptions (3) and (4) and since u_0, v_0 are nonnegative, it follows by the maximum principle [6] that in the interval of existence the solutions of system (1) are nondecreasing in time and nonnegative.

i.e., $u(x,t) \ge 0,\ v(x,t) \ge 0,\ in\ \Omega \times (0,T)$.

On the other hand, since the existence and uniqueness of system (1) can only be locally guaranteed and according to known blow-up results to the single equation (6), blow-up may occur in this problem in a finite time. Therefore, some authors were interested in studying the blow-up properties and numerical solutions of system (1); see for instance [19, 25].

3. Upper Blow-Up Rate Estimates

In the next theorem, we derive the upper blow-up rate estimates for any blow-up solution of system (1) and its gradients.

Theorem 1. *Assume that p_1, p_2, q_1, and q_2 satisfy the following two conditions:*

(i) $\max\{\alpha, \beta\} \ge n/2$,

(ii) $1 < q_1 < (2\alpha+2)/(2\alpha+1),\ 1 < q_2 < (2\beta+2)/(2\beta+1)$,

where α, β are given in (15).

Let (u,v) be a blow-up solution of the Cauchy (Dirichlet) problem of system (1), with the above conditions, which blows up at $T < \infty$. There exist two positive constants C_1 and C_2 such that upper blow-up rate estimates for (u,v) and $(\nabla u, \nabla v)$ are as follows:

$$\begin{aligned} u(x,t) &\le C_1 (T-t)^{-\alpha}, \\ |\nabla u(x,t)| &\le C_1 (T-t)^{-(1+2\alpha)/2}, \\ v(x,t) &\le C_2 (T-t)^{-\beta}, \\ |\nabla v(x,t)| &\le C_2 (T-t)^{-(1+2\beta)/2}, \end{aligned} \tag{23}$$

in $\Omega \times (0,T)$.

Proof. For $t \in (0,T)$, set

$$\begin{aligned} M_u(t) &= \sup_{\Omega \times (0,t]} \left[u(x,t) + |\nabla u(x,t)|^{2\alpha/(1+2\alpha)}\right], \\ M_v(t) &= \sup_{\Omega \times (0,t]} \left[v(x,t) + |\nabla v(x,t)|^{2\beta/(1+2\beta)}\right]. \end{aligned} \tag{24}$$

Clearly, each of M_u, M_v is continuous, nondecreasing, and nonnegative function on $(0,T)$. Moreover, $M_u \longrightarrow \infty$ or $M_v \longrightarrow \infty$ as $t \longrightarrow T$ and that follows from (u,v) blowing up at T.

It will be shown later that we can find $\delta \in (0,1)$ such that

$$\delta \le M_u^{-1/2\alpha}(t) M_v^{1/2\beta}(t) \le \frac{1}{\delta} \tag{25}$$

for $T/2 < t < T$

So that consequently both M_u and $M_v$diverge as $t \longrightarrow T$.

In order to prove this theorem, we will use a rescaling method as in [4] and the proof will have five steps.

Step 1 (rescaling). If M_u diverges as $t \longrightarrow T$, then we can apply the following procedure.

Letting $t_0 \in (0,T)$, we can choose $(x_1, t_1) \in \Omega \times (0, t_0]$ such that

$$u(x_1,t_1) + \left|\nabla u(x_1,t_1)\right|^{2\alpha/(1+2\alpha)} \ge \frac{1}{2} M_u(t_0). \tag{26}$$

Define the new rescaled functions as follows:

$$\varphi_1^\gamma(y,s) = \gamma^{2\alpha} u\left(\gamma y + x_1, \gamma^2 s + t_1\right), \tag{27}$$

$$\varphi_2^\gamma(y,s) = \gamma^{2\beta} v\left(\gamma y + x_1, \gamma^2 s + t_1\right), \tag{28}$$

$$(y,s) \in \Omega_\gamma \times \left(\frac{-t_1}{\gamma^2}, \frac{T-t_1}{\gamma^2}\right), \tag{29}$$

where $\gamma = \gamma(t_0) = M_u^{-1/2\alpha}(t_0)$ is a scaling factor and

$$\Omega_\gamma = \{y \in R^n : \gamma y + x_1 \in \Omega\}. \tag{30}$$

It is clear that

$$\Omega_\gamma := \begin{cases} R^n & if\ \Omega = R^n, \\ B_{R/\gamma} & if\ \Omega = B_R. \end{cases} \tag{31}$$

Next, we aim to show that $(\varphi_1^\gamma, \varphi_2^\gamma)$ is a solution of the system:

$$\begin{aligned} \varphi_{1s}^\gamma - \Delta\varphi_1^\gamma &= -\gamma^{\mu_1} |\nabla\varphi_1^\gamma|^{q_1} + (\varphi_2^\gamma)^{p_1}, \\ \varphi_{2s}^\gamma - \Delta\varphi_2^\gamma &= -\gamma^{\mu_2} |\nabla\varphi_2^\gamma|^{q_2} + (\varphi_1^\gamma)^{p_2}, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \mu_1 &= 2\alpha + 2 - (2\alpha + 1) q_1, \\ \mu_2 &= 2\beta + 2 - (2\beta + 1) q_2. \end{aligned} \tag{33}$$

From assumption (ii), we get $\mu_1, \mu_2 > 0$.

Clearly,

$$\begin{aligned} \varphi_{1s}^\gamma &= \gamma^{2\alpha+2} u, \\ \nabla\varphi_1^\gamma &= \gamma^{2\alpha+1} \nabla u, \\ \Delta\varphi_1^\gamma &= \gamma^{2\alpha+2} \Delta u. \end{aligned} \tag{34}$$

From (1) and (34), we get

$$\begin{aligned} \frac{1}{\gamma^{(2\alpha+2)}} \varphi_{1s}^\gamma = \frac{1}{\gamma^{(2\alpha+2)}} \Delta\varphi_1^\gamma + \frac{1}{\gamma^{q_1(2\alpha+1)}} |\nabla\varphi_1^\gamma|^{q_1} \\ + \frac{1}{\gamma^{2p_1\beta}} (\varphi_2^\gamma)^{p_1}. \end{aligned} \tag{35}$$

Hence, the first equation of the system (32) can be obtained by multiplying the last equation by $\gamma^{(2\alpha+2)}$. The same way can be used to show that φ_2^γ satisfies the second equation of the system (32).

Now, we restrict s to $s \in (-\gamma^{-2} t_1, 0]$ to show that

$$\varphi_1^\gamma (y, s) + |\nabla\varphi_1^\gamma (y, s)|^{2\alpha/(1+2\alpha)} \le 1, \tag{36}$$

for $(y, s) \in \Omega_\gamma \times (-\gamma^{-2} t_1, 0]$.

From (34), we obtain

$$|\nabla\varphi_1^\gamma (y, s)|^{2\alpha/(1+2\alpha)} = \gamma^{2\alpha} |\nabla u|^{2\alpha/(1+2\alpha)}. \tag{37}$$

Clearly,

$$\begin{aligned} u(x, t) + |\nabla u(x, t)|^{2\alpha/(1+2\alpha)} \le M_u(t_0), \\ (x, t) \in \Omega \times (0, t_1]. \end{aligned} \tag{38}$$

From (32), (37), and (38), we get (36).

Moreover,

$$\varphi_2^\gamma + |\nabla\varphi_2^\gamma|^{2\beta/(1+2\beta)} \le M_u^{-\beta/\alpha}(t_0) M_v(t_0), \tag{39}$$

for $(y, s) \in \Omega_\gamma \times (-\gamma^{-2} t_1, 0]$.

On the other hand, from (26), we obtain

$$\varphi_1^\gamma (0, 0) + |\nabla\varphi_1^\gamma (0, 0)|^{2\alpha/(1+2\alpha)} \ge \frac{1}{2} \tag{40}$$

If $M_v \longrightarrow \infty$ as $t \longrightarrow T$, the same procedure can be repeated by changing the roles of u and.

Step 2 (Schauder's estimates). In this step, we find the interior Schauder's estimates of the functions φ_1, φ_2 on the sets

$$S_K = \{y \in \Omega_\gamma, |y| \le k\} \times [-k, kN], \tag{41}$$

where $k > 0$, $N = 0, 1$. Assuming that φ_1 and φ_2 satisfy in S_{2k} the condition

$$\begin{aligned} 0 \le \varphi_1^\gamma + |\nabla\varphi_1^\gamma|^{2\alpha/(1+2\alpha)} \le A, \\ 0 \le \varphi_2^\gamma + |\nabla\varphi_2^\gamma|^{2\beta/(1+2\beta)} \le A \end{aligned} \tag{42}$$

Our claim is as follows: for any positive and small enough values of k, A, and σ, there exists a constant $L = L(k, A, \sigma)$ such that

$$\begin{aligned} \|\varphi_1^\gamma\|_{C^{2+\sigma,1+\sigma/2}(S_k)} \le L, \\ \|\varphi_2^\gamma\|_{C^{2+\sigma,1+\sigma/2}(S_k)} \le L \end{aligned} \tag{43}$$

From (42), we deduce that $\varphi_1^\gamma, \varphi_2^\gamma, \nabla\varphi_1^\gamma$, and $\nabla\varphi_2^\gamma$ are uniformly bounded functions in S_{2k}. So, the functions $(\varphi_1^\gamma)^{p_1}, (\varphi_2^\gamma)^{p_2}, |\nabla\varphi_1^\gamma|^{q_1}$, and $|\nabla\varphi_2^\gamma|^{q_2}$ are uniformly bounded in S_{2k}. Therefore, the right hand side of each equation in (32) is uniformly bounded function in S_{2k}. By applying the interior regularity theory (see [23]), we get locally uniform estimates in $C^{1+\sigma,(1+\sigma)/2}$-norms. Consequently, on the right hand side of each equation in (32), we can obtain locally uniform estimates in Hölder norms $C^{\sigma,\sigma/2}$. Therefore, the parabolic interior Schauder's estimates (43) are held; see [23].

Step 3 (the proof of (25)). Suppose that the lower bound of (25) is not held. So, there is a sequence t_j, such that $t_j \longrightarrow T$ as $j \longrightarrow \infty$, and

$$M_u^{-1/2\alpha}(t_j) M_v^{1/2\beta}(t_j) \longrightarrow 0 \quad as\ j \longrightarrow \infty \tag{44}$$

Thus, $M_u \longrightarrow \infty$ as $t_j \longrightarrow T$.

Now, for each t_j which plays the same role of t_0, as in Step 1, we can scale about the corresponding point (x_j^*, t_j^*) for each, where $t_j^* \le t_j$. We get the corresponding rescaled solution $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$:

$$\varphi_1^{\gamma_j}(y,s) = \gamma_j^{2\alpha} u\left(\gamma_j y + x_j^*, \gamma_j^2 s + t_j^*\right), \tag{45}$$

$$\varphi_2^{\gamma_j}(y,s) = \gamma_j^{2\beta} v\left(\gamma_j y + x_j^*, \gamma_j^2 s + t_j^*\right). \tag{46}$$

where $\gamma_j = \gamma(t_j) = M_u^{-1/2\alpha}(t_j)$ is the scaling factor.

It is clear that $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$ satisfies, as in Step 1, the following problem:

$$\begin{aligned} \varphi_{1s}^{\gamma_j} - \Delta\varphi_1^{\gamma_j} &= -\gamma_j^{\mu_1}\left|\nabla\varphi_1^{\gamma_j}\right|^{q_1} + \left(\varphi_2^{\gamma_j}\right)^{p_1}, \\ \varphi_{2s}^{\gamma_j} - \Delta\varphi_2^{\gamma_j} &= -\gamma_j^{\mu_2}\left|\nabla\varphi_2^{\gamma_j}\right|^{q_2} + \left(\varphi_1^{\gamma_j}\right)^{p_2}, \end{aligned} \tag{47}$$

with

$$\begin{aligned} &\varphi_1^{\gamma_j}(0,0) + \left|\nabla\varphi_1^{\gamma_j}(0,0)\right|^{2\alpha/(1+2\alpha)} \geq \frac{1}{2}, \\ &0 \leq \varphi_1^{\gamma_j} + \left|\nabla\varphi_1^{\gamma_j}\right|^{2\alpha/(1+2\alpha)} \leq 1, \\ &\varphi_2^{\gamma_j} + \left|\nabla\varphi_2^{\gamma_j}\right|^{2\beta/(1+2\beta)} \leq M_u^{-\beta/\alpha}\left(t_j\right) M_v\left(t_j\right), \end{aligned} \tag{48}$$

for $(y,s) \in \Omega_{\gamma_j} \times (-\gamma_j^{-2} t_j^*, 0]$, where

$$\Omega_{\gamma_j} := \begin{cases} R^n & \text{if } \Omega = R^n, \\ B_{R/\gamma_j} & \text{if } \Omega = B_R. \end{cases} \tag{49}$$

Clearly,

$$\Omega_{\gamma_j} \longrightarrow R^n \quad \text{as } j \longrightarrow \infty. \tag{50}$$

From (44) and (48), we see that

$$\varphi_2^{\gamma_j} + \left|\nabla\varphi_2^{\gamma_j}\right|^{2\beta/(1+2\beta)} \longrightarrow 0, \quad \text{as } j \longrightarrow \infty. \tag{51}$$

Thus $\varphi_2^{\gamma_j}$ and $\nabla\varphi_2^{\gamma_j}$ are bounded in $\Omega_{\gamma_j} \times (-\gamma_j^{-2} t_j^*, 0]$ for all j.

By applying Step 2, there is L_k independent of j, such that the uniform Schauder's estimates of $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$ are as follows:

$$\begin{aligned} &\left\|\varphi_1^{\gamma_j}\right\|_{C^{2+\sigma,1+\sigma/2}(\{y\in\Omega_{\gamma_j}, |y|\leq k\}\times[-k,0])} \leq L_k, \\ &\left\|\varphi_2^{\gamma_j}\right\|_{C^{2+\sigma,1+\sigma/2}(\{y\in\Omega_{\gamma_j}, |y|\leq k\}\times[-k,0])} \leq L_k \end{aligned} \tag{52}$$

Since $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$ is defined on a compact set, by the Arzela-Ascoli theorem, there exists a convergent subsequence, and it is denoted by $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$.

Since $\mu_1, \mu_2 > 0$ and $\nabla\varphi_1^{\gamma_j}, \nabla\varphi_2^{\gamma_j}$ are bounded, the limit point (φ_1, φ_2) is a solution of the following system:

$$\begin{aligned} \varphi_{1s} &= \Delta\varphi_1 + \varphi_2^{p_1}, \\ \varphi_{2s} &= \Delta\varphi_2 + \varphi_1^{p_2} \end{aligned} \tag{53}$$

$$\text{in } R^n \times (-\infty, 0] \tag{54}$$

Since $\varphi_2^{\gamma_j} \longrightarrow 0$, where $j \longrightarrow \infty$, it follows that $\varphi_2 \equiv 0$ *in* $R^n \times (-\infty, 0]$.

Consequently, from the second equation of (53), we get

$$\varphi_1 \equiv 0, \quad \text{in } R^n \times (-\infty, 0]. \tag{55}$$

Thus,

$$\varphi_1(0,0) + \left|\nabla\varphi_1(0,0)\right|^{2\alpha/(1+2\alpha)} = 0, \tag{56}$$

which leads to a contradiction with (48), so the lower bound is proved.

If we change the roles of u and v, the upper bound of (25) can be proved similarly as in the last proof.

Step 4 (estimates on doubling M_u). Since $M_u \longrightarrow \infty$ as $t \longrightarrow T$, M_u is a continuous function. For any $t_0 \in (0,T)$, the point t_0^+ can be defined as follows:

$$t_0^+ = \max\left\{t \in \left(t_0, T\right) : M_u(t) = 2M_u\left(t_0\right)\right\}. \tag{57}$$

Clearly,

$$\begin{aligned} &u(x,t) + |\nabla u(x,t)|^{2\alpha/(1+2\alpha)} \leq 2M_u\left(t_0\right), \\ &\qquad (x,t) \in \Omega \times \left(0, t_0^+\right]. \end{aligned} \tag{58}$$

Take $\gamma = \gamma(t_0) = M_u^{-1/2\alpha}(t_0)$.

We claim that there is $0 < A$ which is independent of t_0 such that

$$\frac{\left(t_0^+ - t_0\right)}{\gamma^2\left(t_0\right)} \leq A, \quad t_0 \in \left(\frac{T}{2}, T\right), \tag{59}$$

By supposing that this claim is not true, there is a sequence $t_j \longrightarrow T$, as $j \longrightarrow \infty$ such that

$$\frac{\left(t_j^+ - t_j\right)}{\gamma_j^2\left(t_j\right)} \longrightarrow \infty, \tag{60}$$

where

$$t_j^+ = \max\left\{t \in \left(t_j, T\right) : M_u(t) = 2M_u\left(t_j\right)\right\}. \tag{61}$$

For each t_j, where $T/2 < t_j < t_j^+ < T$, $\forall j$, we can choose $0 < t_j^* \leq t_j$.

As in Step 3, we scale about the corresponding point (x_j^*, t_j^*), and we can get the corresponding rescaled functions $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$ with the scaling factor: $\gamma_j = \gamma(t_j) = M_u^{-1/2\alpha}(t_j)$, which satisfies (47) with the following conditions:

$$\begin{aligned} &\varphi_1^{\gamma_j}(0,0) + \left|\nabla\varphi_1^{\gamma_j}(0,0)\right|^{2\alpha/(1+2\alpha)} \geq \frac{1}{2}, \\ &0 \leq \varphi_1^{\gamma_j} + \left|\nabla\varphi_1^{\gamma_j}\right|^{2\alpha/(1+2\alpha)} \leq 2, \\ &\varphi_2^{\gamma_j} + \left|\nabla\varphi_2^{\gamma_j}\right|^{2\beta/(1+2\beta)} \leq M_u^{-\beta/\alpha}\left(t_j\right) M_v\left(t_j^+\right) \end{aligned} \tag{62}$$

$$(y,s) \in \Omega_{\gamma_j} \times \left(\frac{-t_j^*}{\gamma_j^2}, \frac{t_j^+ - t_j^*}{\gamma_j^2}\right). \tag{63}$$

From (61) and (62), it follows that

$$\varphi_2^{\gamma_j} + \left|\nabla\varphi_2^{\gamma_j}\right|^{2\beta/(1+2\beta)} \le 2^{\beta/\alpha} M_u^{-\beta/\alpha}\left(t_j^+\right) M_v\left(t_j^+\right). \quad (64)$$

From (25), we conclude that

$$M_v(t) \le \delta^{-2\beta} M_u^{\beta/\alpha}(t), \quad t \in \left(\frac{T}{2}, T\right). \quad (65)$$

Therefore, (64) becomes

$$\varphi_2^{\gamma_j} + \left|\nabla\varphi_2^{\gamma_j}\right|^{2\beta/(1+2\beta)} \le \frac{2^{\beta/\alpha}}{\delta^{2\beta}} \quad (66)$$

By applying Step 2, we use the Schauder estimates for $(\varphi_1^{\gamma_j}, \varphi_2^{\gamma_j})$, and we can get a convergent subsequence in $C_{loc}^{2+\sigma,1+\sigma/2}(R^n \times R)$ to the solution of system (53) in $R^n \times R$.

Thus, we get a contradiction because under the assumption (i), all nontrival solutions of system (53) blow up in a finite time; see [17].

So, there is $0 < A$ such that

$$\gamma^{-2}(t_0)\left(t_0^+ - t_0\right) \le A, \quad t_0 \in \left(\frac{T}{2}, T\right). \quad (67)$$

Step 5 (rate estimates).

For any $t_0 \in (T/2, T)$, as in Step 4,

define $M_u(t_1) = 2M_u(t_0)$, where $t_1 = t_0^+ \in (t_0, T)$.

By (67), we have

$$\left(t_1 - t_0\right) \le A M_u^{-1/\alpha}(t_0). \quad (68)$$

We can get $t_2 \in (t, T)$ by using t_1 as a new value of t_0 such that

$$M_u(t_2) = 2M_u(t_1) = 4M_u(t_0). \quad (69)$$

Thus,

$$\left(t_2 - t_1\right) \le A M_u^{-1/\alpha}(t_1) = 2^{-1/\alpha} A M_u^{-1/\alpha}(t_0). \quad (70)$$

So, for any $j \ge 0$, we have

$$\left(t_{j+1} - t_j\right) \le 2^{-j/\alpha} A M_u^{-1/\alpha}(t_0), \quad (71)$$

where the sequence $t_j \longrightarrow T$ as $j \longrightarrow \infty$.

By adding the above inequalities, it follows that

$$(T - t_0) \le \sum_{j\ge 0} 2^{-j/\alpha} A M_u^{-1/\alpha}(t_0). \quad (72)$$

Thus, $(T - t_0) \le (1 - 2^{-1/\alpha})^{-1} A M_u^{-1/\alpha}(t_0)$.

Using (25) results in

$$M_v(t_0) \le \delta^{-2\beta} M_u^{\beta/\alpha}(t_0), \quad t_0 \in \left(\frac{T}{2}, T\right). \quad (73)$$

Thus,

$$M_v(t_0) \le \delta^{-2\beta}\left(1 - 2^{-1/\alpha}\right)^{-\beta} A^{\beta}(T - t_0)^{-\beta}, \quad t_0 \in \left(\frac{T}{2}, T\right). \quad (74)$$

From above, there exist positive two constants a and b such that

$$\begin{aligned} M_u(t_0) &\le a(T - t_0)^{-\alpha}, \quad t_0 \in \left(\frac{T}{2}, T\right), \\ M_v(t_0) &\le b(T - t_0)^{-\beta}, \quad t_0 \in \left(\frac{T}{2}, T\right). \end{aligned} \quad (75)$$

From the last two inequalities and the definitions of M_u and M_v, it follows that there are C_1 and C_2 such that

$$\begin{aligned} u(x,t) + |\nabla u(x,t)|^{2\alpha/(1+2\alpha)} &\le C_1 (T - t)^{-\alpha}, \\ v(x,t) + |\nabla v(x,t)|^{2\beta/(1+2\beta)} &\le C_2 (T - t)^{-\beta}. \end{aligned} \quad (76)$$

Or, we can split the last estimates as follows:

$$\begin{aligned} u(x,t) &\le C_1 (T - t)^{-\alpha}, \\ |\nabla u(x,t)| &\le C_1 (T - t)^{-(1+2\alpha)/2}, \\ v(x,t) &\le C_2 (T - t)^{-\beta}, \\ |\nabla v(x,t)| &\le C_2 (T - t)^{-(1+2\beta)/2} \end{aligned} \quad (77)$$

where $(x,t) \in \Omega \times (0,T)$. □

4. Conclusions and Future Studies

From Theorem 1 and its proof, we can point out the following conclusions:

(i) By (25), it follows that the blow-up in system (1) can only occur simultaneously.

(ii) The gradient terms are bounded for any $t < T$.

(iii) The upper blow-up rate estimates for both of systems (1) and (12) take the same forms. This means that, with the two conditions of Theorem 1, the gradient terms in the system (1) do not effect or make any changes on the profile of blow-up solutions.

Next, we point out some possible future research directions:

(1) One may try to derive the blow-up rate estimates for problem (1), in case one or both assumptions (i) and (ii) of Theorem 1 are not satisfied.

(2) For the semilinear system (12) defined in a ball, and under some restricted assumptions on (u_0, v_0) (nonnegative and radial decreasing functions), it is well known that the blow-up can only occur at the center point (see [18]). However, it is unknown whether and under which condition this result can be extended to the system (1).

Data Availability

https://www.dropbox.com/home?preview=Data+Availability+Statement.pdf.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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