Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2020, Article ID 1527973, 12 pages
https://doi.org/10.1155/2020/1527973

Research Article

Cycle Intersection for SO(p, q)-Flag Domains

Faten Abu Shoga

Department of Mathematics, Islamic University of Gaza, P.O. Box 108, Gaza, Gaza Strip, State of Palestine

Correspondence should be addressed to Faten Abu Shoga; fshoga@iugaza.edu.ps

Received 4 August 2019; Revised 31 October 2019; Accepted 19 November 2019; Published 2 March 2020

Academic Editor: Chin-Chia Wu

A real form G_0 of a complex semisimple Lie group G has only finitely many orbits in any given compact G-homogeneous projective algebraic manifold $Z = G/Q$. A maximal compact subgroup K_0 of G_0 has special orbits C which are complex submanifolds in the open orbits of G_0. These special orbits C are characterized as the closed orbits in Z of the complexification K of K_0. These are referred to as *cycles*. The cycles intersect Schubert varieties S transversely at finitely many points. Describing these points and their multiplicities was carried out for all real forms of SL$(n, \mathbb{C})$ by Brecan (Brecan, 2014) and (Brecan, 2017) and for the other real forms by Abu-Shoga (Abu-Shoga, 2017) and Huckleberry (Abu-Shoga and Huckleberry). In the present paper, we deal with the real form SO(p, q) acting on the SO $(2n$, C)-manifold of maximal isotropic full flags. We give a precise description of the relevant Schubert varieties in terms of certain subsets of the Weyl group and compute their total number. Furthermore, we give an explicit description of the points of intersection in terms of flags and their number. The results in the case of G/Q for all real forms will be given by Abu-Shoga and Huckleberry.

1. Background

Let G be a complex semisimple Lie group and B a Borel subgroup. The compact algebraic homogeneous space $Z = G/B$ is called a complex flag manifold. Let G_0 be a real form of G in the sense that there is an antiholomorphic involution $\tau : G \longrightarrow G$ such that G_0 is the fixed point set of τ. It is well known [1] that there are only finitely many G_0-orbits on Z. Therefore, at least one of them must be open. Such open orbits are called *flag domains*.

Fix a Cartan involution $\theta : G_0 \longrightarrow G_0$. The fixed point set of θ in G_0 is a maximal compact subgroup of G_0 denoted by K_0, and K is its complexification.

If D is a flag domain on Z, then K_0 has exactly one open orbit in D that is a complex submanifold and denoted by C_0. The complex submanifold C_0 is both the unique K-orbit in D that is compact and the unique K_0-orbit in D that is complex. This is the origin of what is known as "Matsuki duality."

Consider an Iwasawa decomposition $G_0 = K_0A_0N_0$ of the real form G_0. We refer to a Borel subgroup $B \subset G$ as an Iwasawa–Borel subgroup of G if it contains an Iwasawa factor A_0N_0. Of course, these are just the Borel subgroups which occur as the isotropy groups at points of the closed G_0-orbit in G/B.

In general, given any Borel subgroup B in G and a point $z \in G/B$, the closure $S = cl(\mathcal{O})$ of the orbit $\mathcal{O} = B.z$ is called the Schubert variety or Schubert cycle. An Iwasawa–Schubert variety $S = cl(\mathcal{O})$ is transversal (relative to an open G_0-orbit D) if $S \cap D$ is nonempty and contained in the open B-orbit $\mathcal{O}$, and $\operatorname{codim} S = q - r$ where $r = \dim C_0$ and the intersection $S \cap C_0$ is transversal at each of its points.

Thus, although the determination of the r-codimensional Iwasawa–Schubert varieties which have nonempty intersection with D, along with their points of intersection with C_0, is apparently a problem of a combinatorial nature, such information has complex analytic significance.

Since $S \cap D \neq \varnothing$ implies that $S \cap C_0 \neq \varnothing$, the first goal of this paper is to determine which Schubert varieties S have nonempty intersection with C_0. After doing so, we then describe this intersection in the case where S is of complementary dimension to C_0. The Schubert varieties are determined by the elements of the Weyl group W_I of a distinguished maximal torus T_I in the Borel subgroup B_I which fixes a certain base point in $\gamma_{c\ell}$. These Schubert varieties are denoted by S_w where $w \in W_I$. Consequently, our results are formulated in terms of conditions on elements w in the Weyl group W_I. In this paper, we discussed both even and odd cases for p and q. (In some steps, we discuss only the even case because the odd case is included in the even case.)

2. Flag Intersection for SO(p, q)

2.1. Preliminaries. Consider the semisimple Lie group $G = \mathrm{SO}(m, \mathbb{C})$ with the real form $\mathrm{SO}(p, q)$. In this case, it is convenient to choose the bilinear form b on C^{2n} depending on p and q. If p or q is even, then we choose it in the usual way:

$$b(v, w) = -\sum_{i=1}^{q} v_i w_i + \sum_{i=q+1}^{m} v_i w_i. \tag{1}$$

Let σ be the complex conjugation $\sigma(v) = \overline{v}$, then the Hermitian form $h(v, w)$ of signature (p, q) which defines the real form is defined by

$$h(v, w) = b(v, \sigma(w)) = -\sum_{i=1}^{q} v_i \sigma(w_i) + \sum_{i=q+1}^{m} v_i \sigma(w_i). \tag{2}$$

If both p and q are odd, then the complex bilinear form is

$$b(v, w) = -\sum_{i=1}^{q-1} v_i w_i + \left(v_q w_{q+1} + v_{q+1} w_q\right) + \sum_{i=q+2}^{m} v_i w_i, \tag{3}$$

and the Hermitian form $h(v, w)$ is defined by

$$h(v, w) = -\sum_{i=1}^{q} v_i \overline{w}_i + \sum_{i=q+1}^{m} v_i \overline{w}_i. \tag{4}$$

The real form is $G_0 = \mathrm{SO}(p, q) = \mathrm{SO}(2n, \mathbb{C}) \cap \mathrm{SU}(p, q)$.

A Cartan involution $\theta = \mathfrak{so}(p, q) \longrightarrow \mathfrak{so}(p, q)$ in the Lie algebra level is given by $\theta(g) = -g^t$, and the maximal compact subgroup associated with it is $K_0 := S(O(p) \times O(q)) \subset S(U(p) \times U(q))$.

2.2. Introduction to the Flags in Z. Recall that the (indefinite metric) orthogonal group $G = \mathrm{SO}(p, q)$ acts on the space of full flags by two orbits: closed orbit and open orbit. The closed orbit is the set of all maximally b-isotropic full flags F with respect to b, where the maximally b-isotropic full flag F is defined as a sequence of $(2n+1)$-vector spaces:

$$F = \left(\{0\} = V_0 \subset V_1 \subset \ldots \subset V_{2n} = \mathbb{C}^{2n}\right), \tag{5}$$

such that $\dim V_i = i$, for all $0 \le i \le 2n$, and $V_{2n-i} = V_i^{\perp}$, for $1 \le i \le n$. In particular, V_i is isotropic for $1 \le i \le n$. The set of all maximal isotropic flags is denoted by Z. The manifold structure of Z arises from the fact that the Lie group G acts transitively on it with $Z = G/B$ where B is the stabilizer of any particular maximally b-isotropic flag in Z.

The signature of a flag $F = (0 \subset V_1 \subset \ldots \subset V_{2n}) \in Z$ with respect to h consists of three sequences:

$$\begin{aligned} a &: 0 \le a_1 \le a_2 \le \ldots \le a_{2n}, \\ b &: 0 \le b_1 \le b_2 \le \ldots \le b_{2n}, \\ d &: 0 \le d_1 \le d_2 \le \ldots \le d_{2n}, \end{aligned} \tag{6}$$

with $\mathrm{sign}(V_i) = (a_i, b_i, d_i)$ where a_i (resp., b_i) denotes the dimension of a maximal negative (positive) subspace and d_i the degeneracy V_i of the restriction of h to V_i and $a_i + b_i + d_i = i$ $\forall 1 \le i \le 2n$. The 3-tuple (a, b, d) is called the signature of the flag F. If $d = 0$, we refer to F as being nondegenerate and write $\mathrm{sign}(F) = (a, b)$.

2.3. Base Points. To parametrize the Schubert cells, it is important to focus on a maximal Torus T_I in an Iwasawa–Borel subgroup B_I of G which fixes the associated flag F_I. On the contrary, we need another maximal torus T_S which defines a basis to determine the base cycles called the standard basis. These two tori are conjugate.

In the following, we define bases depending on p and q as well as the positive and negative spaces which will be essential for understanding the base cycle. Although the goal of the paper is to handle the case where m is even, since in this paragraph the discussion for the odd case is essentially the same, we include it. Also, the fundamental maximal torus T_S is defined in each case by requiring that each basis vector is a T_S-eigenvector.

(i) If $m = 2n$ and q are even, the ordered b-isotropic basis is

$$\begin{aligned} & e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n-1} + ie_{2n}, e_{2n-1} \\ & \quad - ie_{2n}, \ldots, e_1 - ie_2, \end{aligned} \tag{7}$$

where

$$\begin{aligned} E^- &:= \langle e_1 + ie_2, e_3 + ie_4, \ldots, e_{q-1} + ie_q, e_1 - ie_2, e_3 \\ & \quad - ie_4, \ldots, e_{q-1} - ie_q \rangle, \\ E^+ &:= \langle e_{q+1} + ie_{q+2}, \ldots, e_{2n-1} + ie_{2n}, e_{q+1} \\ & \quad - ie_{q+2}, \ldots, e_{2n-1} - ie_{2n} \rangle. \end{aligned} \tag{8}$$

(ii) If $m = 2n$ and q are odd, the ordered b-isotropic basis is

$$e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n-1} + ie_{2n}, e_q, e_{q+1}, e_{2n-1} - ie_{2n}, \ldots, e_1 - ie_2, \tag{9}$$

where

$$\begin{aligned} E^- &= \langle e_1 + ie_2, e_3 + ie_4, \ldots, e_q, e_1 - ie_2, e_3 \\ & \quad - ie_4, \ldots, e_{q-1} - ie_q \rangle, \\ E^+ &= \langle e_{q+1}, e_{q+2} + ie_{q+3}, \ldots, e_{2n-1} + ie_{2n}, e_{q+2} \\ & \quad - ie_{q+3}, \ldots, e_{2n-1} - ie_{2n} \rangle. \end{aligned} \tag{10}$$

Note that the vectors e_q and e_{q+1} are isotropic and $b(e_q, e_{q+1}) = -1$. Moreover, $h(e_3, e_3) = -1$ and

$h(e_4, e_4) = 1$ which means that $e_q \in E^-$ and $e_{q+1} \in E^+$.

(iii) If $m = 2n+1$ and q are even, the ordered b-isotropic basis is

$$e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n} + ie_{2n+1}, e_{q+1}, e_{2n} - ie_{2n+1}, \ldots, e_1 - ie_2, \tag{11}$$

where

$$\begin{aligned} E^- &:= \langle e_1 + ie_2, e_3 + ie_4, \ldots, e_{q-1} + ie_q, e_1 - ie_2, e_3 - ie_4, \ldots, e_{q-1} - ie_q \rangle, \\ E^+ &= \langle e_{q+1}, e_{q+2} + ie_{q+3}, \ldots, e_{2n-1} + ie_{2n}, e_{q+2} - ie_{q+3}, \ldots, e_{2n-1} - ie_{2n} \rangle. \end{aligned} \tag{12}$$

(iv) If $m = 2n+1$ and q are odd, the ordered b-isotropic basis is

$$e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n} + ie_{2n+1}, e_q, e_{2n} - ie_{2n+1}, \ldots, e_1 - ie_2, \tag{13}$$

where

$$\begin{aligned} E^- &= \langle e_1 + ie_2, e_3 + ie_4, \ldots, e_q, e_1 - ie_2, e_3 - ie_4, \ldots, e_{q-2} - ie_{q-1} \rangle, \\ E^+ &= \langle e_{q+1} + ie_{q+2}, \ldots, e_{2n} + ie_{2n+1}, e_{q+1} - ie_{q+2}, \ldots, e_{2n} - ie_{2n+1} \rangle. \end{aligned} \tag{14}$$

Remark 1. If $m = 2n+1$, we have the vector e_q or e_{q+1} in the ordered b-isotropic basis. In this case, the vector sits in a fixed position in the middle of the basis and the terms at the beginning skip over e_q or e_{q+1}.

The above bases define the standard maximal torus T_S in each case as the subgroup of diagonal matrices, i.e.,

$$\begin{aligned} T_S &= \{g \in G_0 : g = \operatorname{diag}(t_1, t_2, \ldots, t_n, -t_n, \ldots, -t_2, -t_1), \\ &\quad t_i \in \mathbb{C}\} \quad \text{if } m = 2n, \\ \text{or } T_S &= \{g \in G_0 : g = \operatorname{diag}(t_1, t_2, \ldots, t_n, 0, -t_n, \ldots, -t_2, -t_1), \\ &\quad t_i \in \mathbb{C}\} \quad \text{if } m = 2n+1. \end{aligned} \tag{15}$$

For any of the ordered bases above, we denote by F_S the associated flag in Z, where the reordering on these bases will determine the flag domains, the base cycles, and the intersection points.

Just as in the case of T_S, the maximal torus T_I is defined to have a certain basis of eigenvectors which depends on m being odd or even.

(i) If m is even, then the basis is

$$\begin{aligned} &e_1 + e_{2q}, \ldots, e_q + e_{q+1}, e_{2q+1} + ie_{2q+2}, e_{2q+3} + ie_{2q+4}, \ldots, e_{2n-1} + ie_{2n}, \\ &e_{2n-1} - ie_{2n}, \ldots, e_{2q+3} - ie_{2q+4}, e_{2q+1} - ie_{2q+2}, e_q - e_{q+1}, \ldots, e_1 - e_{2q}. \end{aligned} \tag{16}$$

(ii) If m is odd, then the basis is

$$\begin{aligned} &e_1 + e_{2q}, e_2 + e_{2q-1}, \ldots, e_q + e_{q+1}, e_{2q+1}, e_{2q+2} + ie_{2q+3}, e_{2q+4} + ie_{2q+5}, \ldots, e_{2n} + ie_{2n+1}, \\ &e_{2q+1}, e_{2n} - ie_{2n+1}, \ldots, e_{2q+4} - ie_{2q+5}, e_{2q+2} - ie_{2q+3}, e_q - e_{q+1}, \ldots, e_2 - e_{2q-1}, e_1 - e_{2q}. \end{aligned} \tag{17}$$

We denote by B_I the Iwasawa–Borel subgroup of G which fixes the associated flag F_I.

2.4. Weyl Groups, Flag Domains, and Base Cycles. Let us use shorthand notation for the bases used above. By (r, s) – form, we mean a basis $(r_1, \ldots, r_n, s_n, \ldots, s_1)$ where $b(r_i, r_j) = b(s_i, s_j) = 0$ and $b(r_i, s_j) = \delta_{ij}$ for all i and j. This occurs in the even-dimensional case. By (r, t, s)-form, which occurs in the odd-dimensional case, r_i and s_i satisfy the same conditions and t is a single vector with $b(t, t) = \pm 1$ and $b(t, r_i) = b(t, s_i) = 0$ for all i.

Here, we discuss the Weyl groups $W(T_S)$ and $W(T_I)$ by their actions on these bases. If T is either of these tori, then it stabilizes the spaces spanned by r_i and s_i for $i = 1, \ldots, n$ for both kinds of bases. In this, we regard T as a product $T = T_1 \cdot \ldots \cdot T_n$, in the second case acting trivially on the space spanned by t. In both cases, an arbitrary permutation in S_n acting diagonally on these spaces by $(r_i, s_i) \longmapsto (r_{\pi(i)}, s_{\pi(i)})$ normalizes the T-action and is in the given orthogonal group G.

A simple reflection $(r_i, s_i) \longmapsto (s_i, r_i)$ for one such i, and being the identity on the other 2-dimensional spaces, is denoted by a sign change. This normalizes T but has negative determinant. In the first case, this means that only an even number of sign changes are allowed. In the second case, we may couple the sign change with the map $t \longmapsto -t$ so that this slightly modified sign change has positive determinant. Thus, in both cases, we have the action of $S_n \ltimes \mathbb{Z}_2^n$ normalizing the T-representation on the basis at hand. This is in fact the action of the full Weyl group.

As in the cases of $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{SO}^*(2n)$, to describe the flag domains and their K_0-base cycles, we use the Weyl group $W(T_S)$ of the standard torus T_S. The base point for α is the flag F_S defined by the standard basis. Then, all base points are $F_\alpha := w(F_S)$ where $w \in \mathbb{Z}_2^{n-1}$ if $m = 2n$ and $w \in \mathbb{Z}_2^n$ if $m = 2n+1$ is associated with the sign-change vector α. As before, the base cycles and flag domains are the orbits $K.F_\alpha$ and $G_0.F_\alpha$, respectively, and α defines a non-degenerate sign with $\operatorname{sign}(F_\alpha) = (a, b)$ and then $D_\alpha = D_{a,b}$.

In our particular case, there are two maximal tori, the maximal torus T_I and the maximal torus T_S, which were defined earlier. Recall that these two maximal tori T_I and T_S are conjugates, and a conjugation induces an isomorphism ψ of the associated Weyl groups. In this case, the bijective map ψ is described, depending on the case, as follows:

(i) If $m = 2n$ and q are even, define W_{T_S} to be the Weyl group with respect to the basis (7) and let W_I be the Weyl group with respect to the basis (16). In this case, the bijective map between W_I and W_{T_S} is $\psi(\mp(2i-1)) = \pm i, \psi(\mp 2i) = \pm(q-i+1)$ if $1 \le i \le q/2$ and $\psi(\pm i) = \pm i$ if $i > q$.

(ii) If $m = 2n$ and q are odd, define W_{T_S} to be the Weyl group with respect to the basis (9) and let W_I be the Weyl group with respect to the basis (16); then, the bijective map in this case is $\psi : W_I \longrightarrow W_{T_o}$ defined by $\psi(\mp(2i-1)) = \pm i, \psi(\mp 2i) = \pm(q-i)$ if $1 \le i \le (q-1)/2$ and $\psi(\pm i) = \pm(i-1)$ if $i > q$ and $\psi(-q) = -n$.

(iii) If $m = 2n+1$ and q are even, define W_{T_e} to be the Weyl group with respect to the basis (11) and let W_I be the Weyl group with respect to the basis (17); then, we can define the bijective map between them to be $\psi(\mp(2i-1)) = \pm i, \psi(\mp 2i) = \pm(q-i+1)$ if $1 \le i \le q/2$ and $\psi(\pm i) = \pm i$ if $i > q$.

(iv) If $m = 2n+1$ and q are odd, define W_{T_o} to be the Weyl group with respect to the basis (13). Let W_I be the Weyl group with respect to the basis (17). The bijective map ψ between W_I and W_{T_o} can be defined as $\psi(\mp(2i-1)) = \pm i, \psi(\mp 2i) = \pm(q-i)$ if $1 \le i \le (q-1)/2$ and $\psi(\pm i) = \pm(i)$ if $i > q$.

Moreover, through an argument quite similar to Proposition 1, the base cycle C_α is the set

$$C_\alpha = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \le i \le m\}, \quad (18)$$

where E^- and E^+ defined in Section 2.3 and the intersection dimensions are determined by α (resp., (a,b)). The flag domains are parametrized by the signature of their base cycles.

2.5. The Length of the Elements of $S_n \ltimes \mathbb{Z}_2^n$. Recall that every B-Schubert cell S contains exactly one T-fixed point. Hence, at least in $Z = G/B$ where the Weyl group acts freely and transitively on Fix(T), we should be able to compute $dim_{\mathbb{C}} S$ in terms of a corresponding Weyl element. In fact, if z_0 is the B-fixed point in $Z = G/B$ and $w \in W$, then the complex dimension of the Schubert cell $S_w = B.w(z_0)$ is the length $\ell(w)$.

As indicated above, we used a special way of writing the Weyl elements which is defined in the paper [2]. For dimension computations, let us state how we can compute the length of elements $w \in S_n \ltimes \mathbb{Z}_2^n$ relative to the notation for the Weyl group elements used in [2].

Lemma 1 (see [2]). *Fix $w \in S_n \ltimes \mathbb{Z}_2^n$ and construct $\tilde{w} \in S_n \ltimes \mathbb{Z}_2^n$ by the following algorithm:*

(1) Starting from left to right in w, using simple reflections, place all positive numbers in w step by step in a sequence of n–empty boxes beginning from the first one in $\tilde{w}$, in the same order as they appeared in w.

(2) From left to right in w, replace a negative number with its absolute value in the n–empty boxes starting from right to left.

If $\tilde{w} = (\tilde{w}(1), \tilde{w}(2), \ldots, \tilde{w}(n))$, then define $L(\tilde{w}) = (n^2 - n)/2 -$ number of $\{\tilde{w}(i) : i < k$ and $\tilde{w}(i) < \tilde{w}(k)\}$, and if we have m negative signs in w in the following positions $j_1, j_2, \ldots, j_m$, then define $f(w) = \sum_{i=1}^{m}[(n-j_i)]$. It follows that the length of w is

$$l(w) = L(\tilde{w}) + f(w) + m. \quad (19)$$

Example 1. Let $w = 5(-34)(-12) \in S_5 \ltimes \mathbb{Z}_2^5$, then by following the above remark, we have

$$5(-34)(-12) \Longrightarrow 542(-3-1) \Longrightarrow 542(13), \quad (20)$$

so $w = 5(-34)(-12)$ and $\tilde{w} = 542(13)$, and then $L(\tilde{w}) = 8$ and $f(w) = 4$.

Hence, $l(w) = 8 + 4 + 2 = 14$.

2.6. Description of the B_I-Orbits. In the previous section, we have chosen a flag F_I defined by the basis

$$e_1 - e_{2n}, e_2 - e_{2n-1}, \ldots, e_n - e_{n+1}, e_n + e_{n+1}, \ldots, e_2 + e_{2n-1}, e_1 + e_{2n}, \quad (21)$$

which belongs to the G_0-closed orbit γ^{cl}. Given a Weyl element $w \in W$, we consider the Schubert cell $B_I.F_w$ and an element $F = b(F_w)$ in it which is defined by an ordered basis $(b.v_{w_1}, \ldots, b.v_{w_{2n}})$, $b \in B_I$, where $v_i = e_i - e_{2n-i+1}, 1 \le i \le n$, and $v_i = e_{2n-i+1} + e_i, n+1 \le i \le 2n$. If $F \in C_\alpha$, then it can be defined by an ordered basis $(\varepsilon_1, \ldots, \varepsilon_{2n})$ where for all k the vector ε_k is either in V^- or in V^+ and is a linear combination

$$\varepsilon_k = \sum_{j \le k} c_j b.v_{w_j}. \quad (22)$$

In this case, we call the basis a *split basis*. According to the above discussion for $\mathrm{Sp}(2n, \mathbb{C})$, we can write the formulas for $b.v_{w_j}$ for any element $b \in B_I$ as follows:

$$\begin{aligned} b.v_{w_j} &= \eta_j\left(e_{-w_j} + e_{2n+w_j+1}\right) + \zeta_j\left(e_{-w_j} - e_{2n+w_j+1}\right) + B_j, \\ &\quad -n \le w_j \le -1, \\ b.v_{w_j} &= \zeta_j\left(e_{w_j} - e_{2n-w_j+1}\right) + B_j, \quad 1 \le w_j \le n, \end{aligned} \quad (23)$$

where B_j does not involve the basis vectors e_{w_j} and e_{2n-w_j+1}.

Remark 2. When discussing the B_I-orbit of the base point F_I, it is important to explicitly understand the orbits $B_I.(e_i - e_{2n-i+1})$ and $B_I.(e_i + e_{2n-i+1})$. As we explained above, we have the following facts.

If F_I and B_I are as above, then the orbits of interest are $B_I.(e_i \pm e_{2n-i+1})$, $i = 1, \ldots, n$. In this case, the orbits $B_I.(e_i + e_{2n-i+1})$ and $B_I.(e_i - e_{2n-i+1})$ have points of the forms

$$b(e_i + e_{2n-i+1}) = \lambda(e_i + e_{2n-i+1}) + \cdots + b_n(e_n + e_{n+1}) + a_n(e_n - e_{n+1}) + \cdots + a_1(e_1 - e_{2n}),$$
$$b(e_i - e_{2n-i+1}) = \lambda(e_i - e_{2n-i+1}) + a_{i-1}(e_{i-1} - e_{2n-i}) + \cdots + a_1(e_1 - e_{2n}), \tag{24}$$

respectively, with $\lambda \neq 0$. Note that, in the above orbits, if b is chosen appropriately, then we can arrange $b(e_i + e_{2n+i-1}) = e_i$ or $b(e_i + e_{2n+i-1}) = e_{2n-i+1}$ for all i. Does this play a role in the proof of Theorem 1.

2.7. Flag Domains and Base Cycles. In order to parametrize the flag domains, it is enough to parametrize the closed K-orbits (see [3], Sections 4.2 and 4.3). For this, we choose a base point $F_S \in \mathrm{Fix}(T_S)$. It follows that the set $\mathscr{C}$ of closed K-orbits can be identified with the Weyl group orbit $W_G(T_S).F_S$ (see [3], Corollary 4.2.4). As we have observed above, $W_G(T_S) \cong S_n \ltimes \mathbb{Z}_2^n$ where the S_n factor can be identified with $W_K(T_S)$. In fact, it is exactly the stabilizer of $K.F_S$ in $\mathscr{C}$. Consequently, $\mathscr{C}$ can be identified with $\mathbb{Z}_2^n.F_S$. We regard an element of $\mathbb{Z}_2^n$ as a vector α of length n consisting of plus and minus signs, and thus, $\mathscr{C}$ can be identified with the set of such vectors. In concrete terms, the K-orbits (resp., G_0-orbits) of base points $(\pm e_1, \ldots, \pm e_n)$ are exactly the base cycles (resp., flag domains) in Z. The respective flag domains are denoted by D_α.

Observe that α defines a nondegenerate signature (a, b) and that $D_\alpha \subset D_{a,b} := \{F : \mathrm{sign}(F) = (a, b)\}$. Conversely, given F with $\mathrm{sign}(F) = (a, b)$, we may choose a basis $(v, w) := (v_1, \ldots, v_n, w_n, \ldots, w_1)$ which defines F and which has the following properties:

(1) b is in canonical form

(2) (v, w) is h-orthogonal

(3) $\|v_j\|_h^2 = \pm 1$ with $\mathrm{sgn}(\|v_j\|_h) = \alpha_j$

(4) $\|w_j\|_h^2 = -\|v_j\|_h^2$

If $\mathrm{sign}(F) = \mathrm{sign}(\widehat{F})$, then we choose bases (v, w) and $(\widehat{v}, \widehat{w})$ for F and $\widehat{F}$, respectively, and note that the transformation which takes the one basis to the other is in G_0. This argument shows that $D_{a,b} = D_\alpha$.

Denote $E_+ = \langle e_1, \ldots, e_n \rangle$ and $E_- = \langle e_{n+1}, \ldots, e_{2n} \rangle$, where $\mathbb{C}^{2n} = E_+ \oplus E_-$.

Proposition 1. *For a fixed open orbit D_α, the base cycle C_α is the set*

$$C_\alpha = \{F \in Z : V_i = (V_i \cap E^-) \oplus (V_i \cap E^+), 1 \le i \le 2n\}. \tag{25}$$

Proof. Firstly, it was shown above that there is a base point F_α with the splitting condition. Since K_0 is the product $K_0 = K_0^+ \times K_0^-$, it acts transitively on all such flags. Let $F \in C_\alpha$ where $V_i^- = V_i \cap E^-$ and $V_i^+ = V_i \cap E^+$. Define F^+ and F^- to be the sets of maximally isotopic flags in E^+ and E^-, respectively. Recall that $K_0 = K_0^+ \times K_0^-$ where K_0^+ and K_0^- act transitively on the sets F^+ and F^-, respectively, which implies that K_0 acts transitively on the set of maximal isotropic flags obtained by put flags from F^+ and F^- in a way such that the new flag has signature α. Hence, C_α is a complex manifold. But K_0 has a unique orbit in D_α which is a complex manifold. Therefore, C_α is the base cycle. □

3. Conditions for $S_w \cap C_\alpha \neq \emptyset$

Here, we deal with the homogeneous space Z of maximally isotropic full flags of the complex orthogonal symmetric group $G = \mathrm{SO}(2n, \mathbb{C})$ equipped with the action of the real form $G_0 = \mathrm{SO}(p, q)$. The general results here are stated in terms of algorithms (See Definitions 1 and 2); in fact, it seems impossible to avoid this. In Corollary 1, we give concrete formulas for the intersection points in $S_w \cap C_\alpha$ if the intersection is nonempty and S_w is of complementary dimension. Also, the number of intersection points with C_0 is explicitly computed in Theorem 2.

Since the Schubert variety Sw is determined by the Weyl element w, we will describe in this section the conditions for an element w of the Weyl group which parametrize the Schubert variety Sw that satisfies $S_w \cap D_\alpha \neq \emptyset$ for some flag domain D_α. As would be expected, a special type of permutation plays a fundamental role.

Definition 1. An element $w \in W$ is called *a harmonic permutation* if it satisfies the following conditions:

If q is even, the number $-(2i-1), 1 \le i \le q/2$, sits in any place to the left of the number $(2i)$ or $(-2i), 1 \le i \le q/2$, in the one line notation of the permutation and the order of the numbers or $-(q+i)$, where $1 \le i \le p-q$ is arbitrary.

If q is odd, the number $-(2i-1), 1 \le i \le (q-1)/2$, sits in any place to the left of the number $(2i), 1 \le i \le (q-1)/2$, and the number $-q$ sits in the last position in the one line notation of the permutation and the order of the numbers $q+i$ or $-(q+i)$, where $1 \le i \le (p-q)/2$ is arbitrary.

Example 2. In $\mathrm{SO}(4,2)$, the relevant pairs are (-12) and $(-1-2)$. As a result, we have 6 harmonic permutations. They are $(-12-3), (-1-32), (-3-12), (-1-23), (-13-2)$, *an d* $(3-1-2)$.

Recall that the fixed point in the closed orbit γ^{cl} is the flag associated with the following ordered basis:

(i) If m is even, then the basis is

$$\begin{aligned} &e_1 + e_{2q}, \ldots, e_q + e_{q+1}, e_{2q+1} + ie_{2q+2}, e_{2q+3} \\ &\quad + ie_{2q+4}, \ldots, e_{2n-1} + ie_{2n}, \\ &e_{2n-1} - ie_{2n}, \ldots, e_{2q+3} - ie_{2q+4}, e_{2q+1} \\ &\quad - ie_{2q+2}, e_q - e_{q+1}, \ldots, e_1 - e_{2q}. \end{aligned} \tag{26}$$

(ii) If m is odd, then the basis is

$$\begin{aligned} & e_1+e_{2q}, e_2+e_{2q-1}, \ldots, e_q+e_{q+1}, e_{2q+1}, e_{2q+2}+ie_{2q+3}, e_{2q+4} \\ & \quad +ie_{2q+5}, \ldots, e_{2n}+ie_{2n+1}, \\ & e_{2q+1}, e_{2n}-ie_{2n+1}, \ldots, e_{2q+4}-ie_{2q+5}, e_{2q+2}-ie_{2q+3}, e_q \\ & \quad -e_{q+1}, \ldots, e_2-e_{2q-1}, e_1-e_{2q}. \end{aligned} \tag{27}$$

Remark 3. For v_i, any such basis vector, and $b \in B_I$, the form of $b.v_i$ is given as follows:

(1) $b.(e_i - e_{2q-i+1}) = \lambda(e_i - e_{2q-i+1}) + \cdots + b_n(e_q - e_{q+1}) + a_n(e_q + e_{q+1}) + \cdots + a_1(e_1 + e_{2q})$

(2) $b.(e_i + e_{2q-i+1}) = \lambda(e_i + e_{2q-i+1}) + a_{i-1}(e_{i-1} - e_{2q-i}) + \cdots + a_1(e_1 - e_{2q})$

(3) $b.(e_{2q+2j-1} - ie_{2q+2j}) = \lambda(e_{2q+2j-1} - ie_{2q+2j}) + \cdots + b_m(e_{m-1} - e_m) + a_m(e_{m-1} + e_m) + \cdots + a_1(e_1 + e_{2q})$ where $m = 2n$ or $2n+1$

(4) $b.(e_{2q+2j-1} + ie_{2q+2j}) = \lambda(e_{2q+2j-1} + ie_{2q+2j}) + a_{i-1}(e_{2q+2j+1} + ie_{2q+2j+2}) + \cdots + a_1(e_1 + e_{2q})$

(5) $b.(e_{2q+1}) = \lambda(e_{2q+1}) + a_n(e_{2n} + ie_{2n+1}) + \cdots + a_1(e_1 + e_{2q})$ if $m = 2n+1$

$\lambda \neq 0$ in all cases above. Note that, in the above orbits, if b is chosen appropriately, then we can arrange all linear combination for every one of the above vectors to be in the standard basis of T_S-eigenvectors. See Theorem 1.

In the following result, $\psi : W_I \longrightarrow W_S$ denotes the bijective map between Weyl groups which was introduced in Section 2.

Proposition 2. *If w is a harmonic permutation, then the flag $F_{\psi(w)}$ belongs to the orbit $B_I(F_w)$.*

Proof. We handle the case where $m = 2n$. The proof for $m = 2n+1$ goes analogously. Let us first prove the theorem for the case that q is even. For this, let $w \in W_I$ be a harmonic permutation and define $F_w = w.(F_I)$ to be the isotropic full flag associated with w. Denote by

$$\{0\} \subset \langle u_{w_1} \rangle \subset \langle u_{w_1}, u_{w_2} \rangle \subset \ldots \subset \langle u_{w_1}, \ldots, u_{w_n} \rangle, \tag{28}$$

the first n subspaces of $F_w = w(F_I)$, where u_{w_i} is a vector from the basis above. Let $\widetilde{w} \in W_S$ be the image of w under the bijective map ψ, and let $Y_{\widetilde{w}}$ be the isotropic flag associated with $\widetilde{w}$ such that the first half of $Y_{\widetilde{w}}$ is

$$\{0\} \subset \langle \varepsilon_{\widetilde{w}_1} \rangle \subset \langle \varepsilon_{\widetilde{w}_1}, \varepsilon_{\widetilde{w}_2} \rangle \subset \ldots \subset \langle \varepsilon_{\widetilde{w}_1}, \ldots, \varepsilon_{\widetilde{w}_n} \rangle. \tag{29}$$

Our claim here is that this flag is an intersection point in $B.F_w \cap C_\alpha$. To prove this, we will construct b with $b(F_w) = Y_{\widetilde{w}}$. From the definition of harmonic permutation, there are two possibilities for w_1: $|w_1| = 2i - 1 \leq q$ or $|w_1| > q$.

Case 1: if $|w_1| > q$, then $u_{w_1} = v_{\widetilde{w}_1}$, so the orbit $B_I.\langle u_{w_1} \rangle$ contains the point $\langle \varepsilon_{\widetilde{w}_1} \rangle$.

Case 2: if $|w_1| = 2i - 1 \leq q$, then we must consider the orbit $B_I.\langle e_{2\widetilde{w}_1-1} - e_{2q+w_1+1} \rangle$. By Remark 3 above, the orbit $B_I.\langle e_{2\widetilde{w}_1-1} - e_{2q+w_1+1} \rangle$ contains points of the form

$$\begin{aligned} y = \Big\langle & \alpha_1\left(e_{2\widetilde{w}_1-1} + e_{2q+w_1+1}\right) + \alpha_2\left(e_{2\widetilde{w}_1} + e_{2q+w_1}\right) \\ & + \alpha_3\left(e_{2\widetilde{w}_1} - e_{2q+w_1}\right) + \alpha_4\left(e_{2\widetilde{w}_1-1} - e_{2q+w_1+1}\right)\Big\rangle, \end{aligned} \tag{30}$$

where $\alpha_1 = \pm\alpha_4$ and $\alpha_2 = \pm\alpha_3$. By taking $\alpha_1 = \alpha_4 = 1/2$ and $\alpha_2 = \alpha_3 = (1/2)i$, it follows that $y = \langle e_{2\widetilde{w}_1-1} + ie_{2\widetilde{w}_1} \rangle = \langle v_{\widetilde{w}_1} \rangle$.

To construct the j-vector of $b(v_{w_j})$ to obtain the subspace $V_{\widetilde{w}_j}$, we must consider three cases:

Case 1: if $|w_j| > q$, then $w_j = \widetilde{w}_j$ and $\varepsilon_{\widetilde{w}_j} = u_{w_j}$, so the orbit $B_I.\langle u_{w_j} \rangle$ contains the point $\varepsilon_{\widetilde{w}_j}$.

Case 2: if $|w_j| = 2i - 1 \leq q$, then our job goes through the orbit $B_I.\langle e_{2\widetilde{w}_j-1} - e_{2q+w_j+1} \rangle$. By using Remark 3, we see that the orbit $B_I.\langle e_{2\widetilde{w}_j-1} - e_{2q+w_j+1} \rangle$ contains points of the form

$$\begin{aligned} y = \Big\langle & \alpha_1\left(e_{2\widetilde{w}_j-1} + e_{2q+w_j+1}\right) + \alpha_2\left(e_{2\widetilde{w}_j} + e_{2q+w_j}\right) \\ & + \alpha_3\left(e_{2\widetilde{w}_j} - e_{2q+w_j}\right) + \alpha_4\left(e_{2\widetilde{w}_j-1} - e_{2q+w_j+1}\right)\Big\rangle, \end{aligned} \tag{31}$$

where $\alpha_1 = \pm\alpha_4$ and $\alpha_2 = \pm\alpha_3$. By taking $\alpha_1 = \alpha_4 = 1/2$ and $\alpha_2 = \alpha_3 = (1/2)i$, it follows that $y = \langle e_{2\widetilde{w}_j-1} + ie_{2\widetilde{w}_j} \rangle = \langle \varepsilon_{\widetilde{w}_j} \rangle$. Therefore, b is constructed to obtain the flag

$$\{0\} \subset \langle \varepsilon_{\widetilde{w}_1} \rangle \subset \langle \varepsilon_{\widetilde{w}_1}, \varepsilon_{\widetilde{w}_2} \rangle \subset \ldots \subset \langle \varepsilon_{\widetilde{w}_1}, \ldots, \varepsilon_{\widetilde{w}_j} \rangle. \tag{32}$$

Case 3: if $|w_j| = 2i$, then the orbit is $B_I.\langle e_{|w_j|} + e_{|2\widetilde{w}_j|-1} \rangle$ which is relevant. In this case, the points

$$\begin{aligned} y = \Big\langle & \alpha_1\left(e_{|w_j|-1} + e_{|2\widetilde{w}_j|}\right) + \alpha_2\left(e_{|w_j|} + e_{|2\widetilde{w}_j|-1}\right) \\ & + \alpha_3\left(e_{|w_j|-1} + ie_{|w_j|}\right)\Big\rangle \end{aligned} \tag{33}$$

belong to the orbit $B_I.\langle e_{|w_j|} + e_{|2\widetilde{w}_j|-1} \rangle$. For $\alpha_1 = -i$, $\alpha_2 = 1$, and $\alpha_3 = i$, we have $y = \langle e_{|2\widetilde{w}_j|-1} + ie_{|2\widetilde{w}_j|} \rangle$. Therefore, the j-vector of b is constructed in this case as well.

Thus, by induction, we observe that $b \in B_I$ can be constructed with $b(F_w) = Y_{\widetilde{w}}$.

We complete the proof by handling the case where q is odd. For this point, we can repeat the steps of the proof above for $w_j > q$ and $|w_j| = 2i - 1$. For the case where $|w_j| = 2i$, we apply the same method as above, only changing $2|\widetilde{w}_j|$ by $2(|\widehat{w}_j| + 1)$. If $w_n = -q$, then $\widehat{w}_j = -n$. In this case, the orbit of relevance is $B_I.\langle e_q - e_{q+1} \rangle$. As we see from Remark 3, $y = \alpha_1(e_q + e_{q+1}) + \alpha_2(e_q - e_{q+1})$ belongs to this orbit. The desired result is then obtained by taking $\alpha_1 = 1, \alpha_2 = -1$, and the point $y = e_{q+1}$ belongs to the orbit $B_I.\langle e_q - e_{q+1} \rangle$. Therefore, the element of B_I is also constructed in the case where q is odd. □

Theorem 1 (harmonic permutation theorem). *The following are equivalent:*

(i) w *is harmonic*

(ii) $B_I(F_w) \cap D_\alpha \neq \emptyset$ *for some* α

Under either of these conditions for every α*, the nonempty intersection* $B_I(F_w) \cap C_\alpha$ *contains a* T_S*-fixed point.*

Proof. (i) $\Longrightarrow$ (ii) is exactly the statement of Proposition 2.

(ii) $\Longrightarrow$ (i) Assuming that w is not harmonic permutation, we will show that $S_w \cap D_\alpha = \emptyset$, for all α, i.e., S_w has no T_S-fixed points. Assume on the contrary that there exists $b \in B_I$ such that $b.(F_w) \in S_w \cap C_\alpha$. For $b(F_w)$ to be fixed, b has to have a certain shape, and at each stage where the condition of harmonicity is violated, we should prove that there is no such b. For this, recall that the complex bilinear form b has been defined to satisfy the following orthogonality condition:

$$\begin{aligned} b\left(e_j - e_{2q-j+1}, e_j + e_{2q-j+1}\right) &= 1, \quad 1 \leq j \leq q, \\ b\left(e_j \pm e_{2q-j+1}, e_k \pm e_{2q-k+1}\right) &= 0, \quad \text{for } k \neq j. \end{aligned} \tag{34}$$

Since w is not a harmonic permutation, there exists a pair of the form $(-(2i-1), 2i)$ or of the form $(-(2i-1), -2i)$, where $1 \leq i \leq q/2$, such that $\mp 2i$ sits to the left of $-(2i-1)$. Assume that $w_j = 2i$ is the first even number which appears in w such that $2i$ sits to the left of $-(2i-1)$. Then, b does in fact yield a T_S-fixed partial flag, i.e.,

$$\{0\} \subset \langle \varepsilon_{\widetilde{w}_1} \rangle \subset \langle \varepsilon_{\widetilde{w}_1}, \varepsilon_{\widetilde{w}_2} \rangle \subset \ldots \subset \langle \varepsilon_{\widetilde{w}_1}, \ldots, v_{\widetilde{w}_{j-1}} \rangle. \tag{35}$$

Now, we check that there is no $b \in B_I$ so that $b.(v_{w_j})$ defines an extended partial flag which is T_S-fixed. For this, we consider the orbit $B_I.\langle e_{|w_j|} + e_{|2\widetilde{w}_j|-1} \rangle$ which contains points of the form

$$\alpha_1\left(e_1 + e_{2q}\right) + \alpha_2\left(e_2 + e_{2q-1}\right) + \cdots + \alpha_j\left(e_{\left|w_j\right|} + e_{\left|2\widetilde{w}_j\right|-1}\right). \tag{36}$$

This is a linear combination of h-isotropic vectors for all α_i. Recall that the flag of intersection has non-h-isotropic vectors, and to have one of these vectors in this step which is linearly independent of all vectors in the flag (35) above, we should add the point $\langle (e_{|w_j|-1} + ie_{|w_j|}) \rangle$ from the flag (35) to the linear combination in (36), but this vector is not in the flag (35). Thus, as was claimed, no $b \in B_I$ has the property that $b(F_w)$ is T_S-fixed.

If $w_j = -2i$ is the first even number such that $-2i$ sits to the left of $-(2i-1)$, then the relevant orbit is $B_I.\langle e_{|w_j|} + e_{|2\widetilde{w}_j|-1} \rangle$ which contains points of the form

$$\begin{aligned} &\alpha_1\left(e_1 + e_{2q}\right) + \cdots + \alpha_q\left(e_q + e_{q+1}\right) + \alpha_{q+1}\left(e_q - e_{q+1}\right) \\ &\quad + \cdots + \alpha_s\left(e_{\left|w_j\right|} - e_{\left|2\widetilde{w}_j\right|-1}\right). \end{aligned} \tag{37}$$

To have the nonisotropic point which is linearly independent of all points in the flag (35), we should add any of the points $(e_{|w_j|-1} + ie_{|w_j|})>$, $(e_{|w_j|-1} - ie_{|w_j|})>$, $(e_{2q-|w_j|+1} + ie_{2q-|w_j|+2})>$, or $(e_{2q-|w_j|+1} - ie_{2q-|w_j|+2})>$ from the flag (35) to the linear combination in (36). But the flag (35) does not contain any of these points. So again, for all $b \in B_I$, the flag $b(F_w)$ is not T_S-fixed, and consequently, $S_\alpha \cap D_\alpha = \emptyset$. ☐

4. Introduction to the Combinatorics

For the remainder of this paper, we only discuss the intersection properties of the Iwasawa–Schubert cells which are of complementary dimension to C_0. Recall that the maximal compact subgroup of $\mathrm{SO}(p,q)$ is $K_0 := S(O(p) \times O(q)) \subset S(U(p) \times U(q))$. For E^+ and E^- as in Section 2.4, the base cycle in the flag domain $D_{a,b}$ is given by

$$C_0 = \left\{ F \in Z : \dim V_i \cap E^- = \sum_{j=1}^{i} a_j \text{ and } \dim V_i \cap E^+ = \sum_{j=1}^{i} b_j, 1 \leq i \leq m \right\}. \tag{38}$$

If $m = 2n$, we have two cases: If q is even, the dimension of C_0 is $(p(p-2)/4) + (q(q-2)/4)$. If q is odd, then C_0 has the dimension $((p-1)^2/4) + ((q-1)^2/4)$. If $m = 2n+1$, then the dimension of the base cycle is $((p-1)^2/4) + (q(q-2)/4)$ if q is even and is $(p(p-2)/4) + ((q-1)^2/4)$ if q is odd. Since we have restricted to the case where $Z = G/B$ is the manifold of complete flags, it follows that $\dim S_w = pq/2$ if q and n are even and $\dim S_w = (pq-1)/2$ if q or n is odd.

Definition 2. A harmonic permutation $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ is called *a perfect harmonic permutation* if it is constructed by the following algorithm:

(A)Start with a sequence of n empty boxes which are to be filled in order to construct w.

(B)Consider the pairs $(-(2j-1), 2j)$ and $(-(2j-1), -2j)$ for all $1 \leq j \leq q/2$.

(C)If q is even,

(1) The pairs in step B are $(-1, 2)$, $(-1, -2)$, $(-3, 4)$, $\ldots$, $(-(q-1), q)$, $(-(q-1), -q)$.

(2) Step by step, starting from $(-1, 2)$ until $(-(q-1), -q)$, for each $1 \leq j \leq q/2$, we have two pairs of the forms $(-(2j-1), 2j)$ and $(-(2j-1), -2j)$, so choose only one pair of them for each step and omit the other from the above list.

(3) If we choose the pair of the form $(-(2j-1), 2j)$, place this pair in any box in w such that the components $-(2j-1)$ and $2j$ of this pair sit as close as possible to each other.

(4) If we choose the pair of the form $(-(2j-1), -2j)$, place this pair in any box in w such that the components $-(2j-1)$ and $-2j$ of this pair sit as close as possible to each other and all pairs $(-(2i-1), 2i)$ or $(-(2i-1), -2i)$ with $i > j$ are sitting to the left of this pair and the pairs of the form $(-(2i-1), -2i)$ with $i < j$ are sitting in a decreasing order with respect to i to the right of $(-(2j-1), -2j)$.

(5) Once a pair is placed, its position can be ignored so that the places at the immediate left and right of this pair become adjacent.

(6) After all pairs are placed, the remaining numbers $\pm(q+i)$ for $1 \le i \le (p-q)/2$ are placed in the remaining spots in the strictly increasing order with respect to $|w(i)|$ such that all numbers $\pm(q+i)$ for $1 \le i \le (p-q)/2$ are sitting to the left of the pairs $(-(2i-1), -2i)$ $1 \le i \le q/2$. If the number of negative signs in all pairs from steps 2 and 3 is even, then all the remaining numbers are positive, and if the number of negative signs is odd, then all the remaining numbers are positive except the number n is negative.

(D)If q is odd,

(1) The pairs in step B are $(-1,2), (-3,4), (-5,6), \ldots, (-(q-2), q-1), -q$.

(2) In the last box of w, put the number $-q$.

(3) Step by step, starting from $(-1,2)$ until $(-(q-2), q-1)$, choose a pair and place this pair in any box of the first $n-1$ boxes in w such that the components $-(2j-1)$ and $2j$ of this pair sit as close as possible to each other. This means that once a pair is placed, it can be ignored so that the places at the immediate left and right of this pair become adjacent.

(4) After all pairs are placed, the numbers $\pm(q+i)$ for $1 \le i \le p-q/2$ are placed in the remaining spots in the strictly increasing order with respect to $|w(i)|$. If the number of negative signs in all pairs from steps 1 and 2 is even, then all the remaining numbers are positive, and if the number of negative signs is odd, then all the remaining numbers are positive except the number n is negative.

Remark 4. If q is odd, then the signature of the flag domain D_α is + in the last position.

Remark 5. If $W = S_n \ltimes \mathbb{Z}_2^n$, then a perfect harmonic permutation consists only of pairs of the form $(-(2j-1), 2j)$ for all $1 \le j \le q/2$. It is constructed as above, in particular such that the sign of n is +.

Example 3. If $p = 6, q = 4$, then the perfect harmonic permutations are

$$\begin{aligned}
&(-12)(-34)5,\ (-12)5(-34),\ 5(-12)(-34),\ (-3\,4)(-1\,2)5,\\
&\cdot(-3\,4)5(-1\,2), 5(-3\,4)(-1\,2),\\
&(-12)-5(-3-4), -5(-12)(-3-4), -5(-3-4)(-12),\\
&\cdot(-34)-5(-1-2), -5(-34)(-1-2),\\
&5(-3-4)(-1-2),\ (-3-1\,2\,4)5, 5(-3-1\,2\,4),\\
&-5(-3-12-4).
\end{aligned} \tag{39}$$

Example 4. If $p = 10, q = 6$, then the element $78(-5\,6)(-3-4)(-1\,2)$ is a perfect harmonic permutation, while the element $78(-3-4)(-1\,2)(-5\,6)$ is not a perfect harmonic permutation.

Recall that the dimension of the cells corresponds to the length of the word w in the Weyl group, i.e., if $F_w = w.F_I$, then $\dim(B.F_w) = l(w)$. So if we want to discuss the dimension of the Schubert cell, it is enough to discuss the length of Weyl elements.

Proposition 3. *Every perfect harmonic permutation $w \in S_n \ltimes \mathbb{Z}_2^{n-1}$ has dimension $pq/2$ if q is even and $(pq-1)/2$ if q is odd.*

Proof. Given a perfect harmonic permutation w, we consider three cases. These depend on which of the pairs $(-1\,2)$, $(-3\,4)$, $(-1-2)$, or $(-3-4)$ is contained in w. In each of these cases, our proof goes by induction on the dimension of the flag manifold. Without loss of generality, let $p > q \ge 6$ because if $q \le 6$, then the permutation has only one or two pairs and the proof becomes trivial. □

Case 1. The permutation w contains the pairs $(-1\,2)$ and $(-3\,4)$.

First, we remove the pairs $(-1\,2)$ and $(-3\,4)$ from w to have a new permutation v consisting of the numbers $\{5, 6, \ldots, n\}$. We define a function $f : \{5, \ldots, n\} \longrightarrow \{1, \ldots, n-4\}$ by $f(i) = i-4$ for all $1 \le i \le n-4$. This is a bijective map which sends v to $\widehat{w} \in S_{n-4} \ltimes \mathbb{Z}_2^{n-5}$. Note that $\widehat{w}$ is a perfect harmonic permutation in $S_{n-4} \ltimes \mathbb{Z}_2^{n-5}$. Thus by the induction assumption, $l(\widehat{w}) = (p-4)(q-4)/2$. Since $v = f^{-1}(\widehat{w})$, it follows that v has the same length as $\widehat{w}$. So we put the numbers 1234 to the left of v to have an element $\widetilde{w} \in S_n \ltimes \mathbb{Z}_2^{n-1}$ with length $(p-4)(q-4)/2$. To split the sign of 1 and 3 (i.e., to change the positive sign to the negative sign), we add $n-4+1$ to $(p-4)(q-4)/2$ to send 3 to the last position and add $n-4+2$ to send 1 to the position before the last position. Consequently, we have the element $(24)v(13)$ with length $((p-4)(q-4)/2) + n - 4 + 1 + n - 4 + 2 = ((p-4)(q-4)/\ 2) + 2n - 5$, and it follows that the element $(24)v(-3-1)$ has length $((p-4)(q-4)/2) + 2n - 4$. We then return to the original w and remove only the pair $(-1\ \ 2)$. Then, we define g to be the number of positions to the left of (-34). As a result, we have $n-4-g$ positions to the left of $(-3\,4)$, and -3 should cross $n-4-g+1$ positions to end up in the last position and 4 should cross g boxes to end up in the first position.

Finally, we return to the original w and define h to be the number of positions to the left of $(-1\,2)$ and $n-2-f$ to the right. In this situation, -1 must cross $n-2-h+1$ positions to end up in the last position and 2 must cross g positions to end up in the first position. It follows that the length of w is $((p-4)(q-4)/2) + 2n - 4 + n - 4 - g + 1 + g + n - 2 - h + 1 + h = pq/2$.

If q is odd, then the length of $\widehat{w} \in S_{n-4} \ltimes \mathbb{Z}_2^{n-5}$ is $l(\widehat{w}) = (p-4)(q-4) - 1/2$. After applying the same steps as above, it follows that $l(w) = (pq-1)/2$.

Case 2. The permutation w contains the pair $(-1\ -2)$.

Note that this case only occurs if q is even. Since w is a perfect harmonic permutation, the pair $(-1-2)$ sits in the last 2 positions of w. We remove the pair $(-1\ -2)$ from w to obtain $\widehat{v}$. Since q is even, a similar argument to that shown above shows that $l(\widehat{v}) = (p-2)(q-2)/2$ because q is even. We put the pair $(1\,2)$ to the left of $\widehat{v}$. It follows that $\widetilde{v} = (1\,2)\widehat{v}$ has the same length as $\widehat{v}$. To split the sign of 1 and 2, each of them must cross $n-2$ positions. Having made this move, we then apply 2 reflections to obtain the pair $(-1\ -2)$ in the last two positions. Hence, the length of w becomes $l(w) = ((p-2)(q-2)/2) + 2(n-2) + 2 = pq/2$.

Case 3. The permutation w contains the pairs $(-1\,2)$ and $(-3\ -4)$.

Note that this case appears only if q is even. In this case, these two pairs appear in w in the following form: $(-3-4-1\,2)$ or $(-3-1\,2-4)$, or the pair $(-3-4)$ sits in the last two positions of w. If the pair $(-3-4)$ sits in the last two positions of w, then the argument goes as in Case 2 above.

If we have the form $(-3-1\,2-4)$, then we must add $n-4$ to $(p-2)(q-2)/2$ to put 3 in the last position and add $n-2$ to $(p-2)(q-2)/2$ to put 4 in the last position. Then, we must add 1 to split the signs and 3 to send -3 to its position in the original$w$; then, the length of w becomes $((p-2)(q-2)/2) + (n-4) + (n-2) + 1 + 3 = pq/2$.

If we have the form $(-3-4-1\,2)$, then we must add $n-4$ to $(p-2)(q-2)/2$ to put 3 in the last position and add $n-1$ to $(p-2)(q-2)/2$ to put 4 in the position before the last position. Then, we must add 1 to split the signs, 3 to send -3 to its position in the original w, and 1 to send -4 to its position in the original w. It follows that the length of w is $((p-2)(q-2)/2) + (n-4) + (n-3) + 1 + 3 + 1 = pq/2$.

Proposition 4. *Every perfect harmonic permutation $w \in S_n \ltimes \mathbb{Z}_2^n$ has dimension $pq/2$.*

Proof. The argument goes exactly along the lines as that of the above proposition. Here, it is in fact simpler because only the pairs $(-1\,2)$ and $(-3\,4)$ appear. □

5. Intersection Points of Schubert Duality

Let $w \in W_I$ be a perfect harmonic permutation, in particular so that S_w is of complementary dimension to the cycles. Recall that, in this case, either $S_w \cap C_\alpha = \varnothing$ or is pointwise T_S-fixed (see Section 2.3). The main goal in this section is to compute all such intersection points. The argument in the case of $\mathrm{SO}(p,q)$ is carried out by means of algorithms. Nevertheless, we are able to provide formulas for the cardinality of $S_w \cap C_\alpha$ when it is nonzero and the total number of cycles C_α for which this intersection is nonempty (see Theorem 2).

Proposition 5. *If w is a perfect harmonic permutation so that $B_I.F_w$ intersects a cycle C_α at a point given by the ε-basis $(\varepsilon_1,\dots,\varepsilon_m)$ of T_S-eigenvectors, then for any such eigenvector ε_k, it follows that the ε-basis is given by $\varepsilon_k = e_{2j-1} + ie_{2j}$ or $e_{2j-1} - ie_{2j}$ if q and p are even and $\varepsilon_k = e_{2j-1} + ie_{2j}$, $e_{2j-1} - ie_{2j}$, e_q, e_{q+1}, or e_{2q+1} if q or p is odd, depending on the dimension m and the signature α.*

Proof. This is a consequence of the following:

(1) w is a perfect harmonic permutation and the flag basis is that of $w(F_I)$.

(2) We have the following cases:

(i) $b.v_j = \eta_j(e_r + e_{2q-r+1}) + \zeta_j(e_r - e_{2q-r+1}) + \widetilde{\eta}_j(e_{r+1} + e_{2q-r}) + \widetilde{\zeta}_j(e_{r+1} - e_{2q-r}) + B_j = K_j + B_j$

(ii) $b.v_j = \eta_j(e_{2q+r} + ie_{2q+r+1}) + B_j = K_j + B_j$
Here, $\widetilde{\eta}_j = \pm i\eta_j$ and $\widetilde{\zeta} = \pm i\zeta_j$ and B_j does not involve the basis vectors e_r, e_{r+1}, e_{2q-r+1}, and e_{2q-r}.

(iii) $b.v_j = \eta_j(e_{2q+1}) + B_j = K_j + B_j$, if $m = 2n+1$

(3) $\eta_j \neq 0$ and $\widetilde{\eta}_j \neq 0$.

(4) The intersection $S_w \cap C_\alpha$ is a flag defined by T_S-eigenvectors.

From the expression for v_j, it is obvious that all of the possibilities in the statement occur. Furthermore, since $\eta_j \neq 0$ and $\widetilde{\eta}_j \neq 0$, for every j, a nonzero contribution from K_j occurs in the sum

$$v_j = \sum_{k\le j} c_{kj} b.v_{w_k}. \tag{40}$$

Since $e_{2j-1} + ie_{2j}$ and $e_{2j-1} - ie_{2j}$ and e_q or e_{q+1} do not occur in $b.v_{w_k}$ for $k<j$, it follows that $\varepsilon_j = K_j + E_j$ in the standard basis. Finally, since ε_j is a T_S-eigenvector, it follows that $\varepsilon_j = K_j$ where K_j of the ε_j forms indicated in the proposition.

As a result of the above proposition, the following corollary gives us all intersection points of $S_w \cap C_\alpha$. □

Corollary 1. *Let D_α be a flag domain parametrized by a sequence α and $w \in W_I$ be a perfect harmonic permutation such that $S_w \cap D_\alpha \neq \phi$. Then, the following algorithm produces us all intersection points of $S_w \cap C_\alpha$:*

If q is even,

(i) Consider a copy of α denoted by β.

(ii) For any pair $(-(2j-1), 2j)$, $1 \le j \le q/2$, in w, if the corresponding signature of it in β is $+-$, then replace this $+-$ in β by

$$\begin{aligned}&\langle e_{2q-2j+1} + ie_{2q-2j+2}, e_{2j-1} - ie_{2j}\rangle\\ &\text{or } \langle e_{2q-2j+1} - ie_{2q-2j+2}, e_{2j-1} + ie_{2j}\rangle,\end{aligned} \tag{41}$$

and if the corresponding signature of it in β is $-+$, then replace this $-+$ in β by

$$\begin{aligned}&\langle e_{2j-1} + ie_{2j}, e_{2q-2j+1} - ie_{2q-2j+2}\rangle\\ &\text{or } \langle e_{2j-1} - ie_{2j}, e_{2q-2j+1} + ie_{2q-2j+2}\rangle.\end{aligned} \tag{42}$$

(iii) For any pair $(-(2j-1), -2j)$, $1 \le j \le q/2$, in w, if the corresponding signature of it in β is $+-$, then replace this $+-$ in β by

$$\langle e_{2q-2j+1} + ie_{2q-2j+2}, e_{2j-1} + ie_{2j}\rangle \text{ or } \langle e_{2q-2j+1} - ie_{2q-2j+2}, e_{2j-1} - ie_{2j}\rangle, \quad (43)$$

and if the corresponding signature of it in β is $-+$, then replace this $-+$ in β by

$$\langle e_{2j-1} + ie_{2j}, e_{2q-2j+1} + ie_{2q-2j+2}\rangle \text{ or } \langle e_{2j-1} - ie_{2j}, e_{2q-2j+1} - ie_{2q-2j+2}\rangle. \quad (44)$$

(iv) For the remaining numbers, for each $q+1 \le j \le n-1$, replace the corresponding $+$ in β by $(e_{2j-1} + ie_{2j})$, and for $\pm n$, replace the corresponding $+$ by $(e_{2n-1} \pm ie_{2n})$.

If q is odd,

(i) For any pair $(-(2j-1), 2j), 1 \le j \le (q-1)/2$, in w, if the corresponding signature of it in β is $+-$, then replace this $+-$ in β by

$$\langle e_{2q-2j+1} + ie_{2q-2j+2}, e_{2j-1} - ie_{2j}\rangle \text{ or } \langle e_{2q-2j+1} - ie_{2q-2j+2}, e_{2j-1} + ie_{2j}\rangle, \quad (45)$$

and if the corresponding signature of it in β is $-+$, then replace this $-+$ in β by

$$\langle e_{2j-1} + ie_{2j}, e_{2q-2j+1} - ie_{2q-2j+2}\rangle \text{ or } \langle e_{2j-1} - ie_{2j}, e_{2q-2j+1} + ie_{2q-2j+2}\rangle. \quad (46)$$

(ii) For the number $-q$, replace the corresponding $+$ in β by e_{q+1}.

(iii) For the remaining numbers, for each $q+1 \le j \le n-1$, replace the corresponding $+$ in β by $(e_{2j-1} + ie_{2j})$, and for $\pm n$, replace the corresponding $+$ by $(e_{2n-1} \pm ie_{2n})$.

Each point obtained from this algorithm is a point of the intersection of $S_w \cap D_\alpha$.

Theorem 2. *A Schubert variety S_w which is parametrized by a perfect harmonic permutation w intersects $2^{q/2}$ flag domains if q is even and intersects the base cycles of these flag domains in 2^q points. If q is odd, it intersects $2^{(q-1)/2}$ flag domains and intersects the base cycles of these flag domains in 2^{q-1} points.*

Proof. Let $w \in W_I$ be a perfect harmonic permutation. We first show that if q is even, then $S_w \cap C_\alpha$ consists of 2^q points and 2^{q-1} points if q is odd. Since w is a perfect harmonic permutation, we have two cases. In the first case, if w contains the pair $(-1\,2)$, then the pair $(-1\,2)$ sits inside consecutive boxes in w. The goal here is to show that there are exactly 4 possibilities for this pair which can be completed to the maximal isotropic flag. We begin by considering the B_I-orbit of $\langle e_1 - e_{2q}\rangle$. By Remark 3, the elements in this orbit have the form

$$\beta_1\left(e_1 + e_{2q}\right) + \cdots + \beta_{2n}\left(e_1 - e_{2q}\right). \quad (47)$$

The question is how many 1-dimensional subspaces (spanned by vectors in Proposition 5) do we have such that these subspaces can be completed to the maximal isotropic flag. To compute this number, we denote by v_1 the vector we have from the first step which spans the 1-dimensional subspace; in the second step, we consider the orbit $B_I.\langle e_2 + e_{2q-1}\rangle$, which has points of the form

$$\beta_1\left(e_1 + e_{2q}\right) + \beta_2\left(e_2 + e_{2q-1}\right). \quad (48)$$

The 2-dimensional subspace in the flag is spanned by linear combinations of the form

$$v_2 = \beta_1\left(e_1 + e_{2q}\right) + \beta_2\left(e_2 + e_{2q-1}\right) + \gamma v_1. \quad (49)$$

Note that v_2 should be in E^+ or in E^- and of the form stated in Proposition 5, and therefore, v_1 should contain the terms e_1 and e_2 or the terms e_{2q-1} and e_{2q}. Thus, we have the following 4 possibilities of v_1 as follows: $e_1 - ie_2$, $e_1 + ie_2, e_{2q-1} + ie_{2q}$, and $e_{2q-1} - ie_{2q}$.

If $v_1 = e_1 \mp ie_2$, then for $\beta_1 = \pm i, \beta_2 = 1$, and $\gamma = \mp i$, the vector v_2 is $v_2 = e_{2q-1} \pm ie_{2q}$, so the 2-dimensional subspace corresponding to the pair $(-1\,2)$ is spanned by

$$\langle e_1 \mp ie_2, e_{2q-1} \pm ie_{2q}\rangle. \quad (50)$$

If $v_1 = e_{2q-1} \mp ie_{2q}$, then for $\beta_1 = 1, \beta_2 = \pm i$, and $\gamma = \mp i$, it follows that v_2 is $v_2 = e_1 \pm ie_2$. Thus, the 2-dimensional subspace corresponding to the pair $(-1\,2)$ is spanned by

$$\langle e_{2q-1} \mp ie_{2q}, e_1 \pm ie_2\rangle. \quad (51)$$

Having constructed the 2-dimensional space, we ignore the pair (-12) in w and repeat the same steps for the next pairs step by step. So if w contains the pair $(-(2j-1), 2j), 1 \le j \le q/2$, then in the same way, the only possible 2-dimensional subspaces which can be completed to maximal isotropic flags are the subspaces spanned by

$$\langle e_{2j-1} \mp ie_{2j}, e_{2q-2j+1} \pm ie_{2q-2j+2}\rangle \text{ or } \langle e_{2q-2j+1} \pm ie_{2q-2j+2}, e_{2j-1} \mp ie_{2j}\rangle. \quad (52)$$

The second case is where w contains the pair $(-1-2)$ which sits inside consecutive boxes in w. Look at the orbit $B_I.\langle e_1 - e_{2q}\rangle$, then this orbit has points of the form

$$\beta_1\left(e_1 + e_{2q}\right) + \cdots + \beta_{2n-1}\left(e_2 - e_{2q-1}\right) + \beta_{2n}\left(e_1 - e_{2q}\right). \quad (53)$$

We also consider the orbit $B_I.\langle e_2 - e_{2q-1}\rangle$ which has points of the form

$$\beta_1\left(e_1 + e_{2q}\right) + \cdots + \beta_{2n-1}\left(e_2 - e_{2q-1}\right). \quad (54)$$

Then, if we have 1-dimensional subspace spanned by v_1 from the first step, the 2-dimensional subspace in our flag spanned by v_1 and a vector v_2 is of the form $v_2 = \beta_1 (e_1 + e_{2q}) + \beta_2 (e_2 + e_{2q-1}) + \beta_3 (e_2 - e_{2q-1}) + \gamma v_1$. So we have 4 possibilities of v_1 which can be extended to the maximal isotropic flag. They are $e_1 - ie_2, e_1 + ie_2$, $e_{2q-1} + ie_{2q}$, *and* $e_{2q-1} - ie_{2q}$.

If $v_1 = e_1 \mp ie_2$, then for $\beta_1 = \mp i, \beta_2 = 0, \beta_3 = -1$, and $\gamma = \pm i$, the vector v_2 is $v_2 = e_{2q-1} \mp ie_{2q}$. Thus, the 2-dimensional subspace corresponding to the pair $(-1-2)$ is spanned by

$$\langle e_1 \mp ie_2, e_{2q-1} \mp ie_{2q} \rangle. \tag{55}$$

If $v_1 = e_{2q-1} \mp ie_{2q}$, then for $\beta_1 = 1, \beta_2 = 0, \beta_3 = \mp i$, and $\gamma = \mp i$, the vector v_2 is $v_2 = e_1 \pm ie_2$. As a result, the 2-dimensional subspace corresponding to the pair $(-1-2)$ is spanned by

$$\langle e_{2q-1} \mp ie_{2q}, e_1 \mp ie_2 \rangle. \tag{56}$$

Having determined the 2-dimensional subspace, we ignore the pair $(-1-2)$ from w and repeat the same steps for the next pairs step by step. More generally, if w contains the pair $(-(2j-1), -2j), 1 \le j \le q/2$, then by the same method, the only possible 2-dimensional subspaces which can be completed to the maximal isotropic flag are

$$\begin{aligned} &\langle e_{2j-1} \mp ie_{2j}, e_{2q-2j+1} \mp ie_{2q-2j+2} \rangle \\ &\text{or } \langle e_{2q-2j+1} \mp ie_{2q-2j+2}, e_{2j-1} \mp ie_{2j} \rangle. \end{aligned} \tag{57}$$

Therefore, for each pair of w, we have 4 possibilities.

For the remaining numbers, recall that w is a perfect harmonic permutation. In particular, all numbers in the remaining boxes sit in an increasing order. Thus, the only possibility for these which can be completed to the maximal isotropic flag is the following point: if n is positive, the point is the flag associated with the ordered basis

$$e_{2q+1} + ie_{2q+2}, \dots, e_{2n-1} + ie_{2n}, \tag{58}$$

and if n is negative, then the point is the flag associated with the ordered basis

$$e_{2q+1} + ie_{2q+2}, \dots, e_{2n-3} + ie_{2n-2}, e_{2n-1} - ie_{2n}. \tag{59}$$

Hence, if q is even, then w contains $q/2$ pairs and each pair has 4 possibilities. Therefore, in this case, the number of possible intersection points is $4^{q/2}.1 = 2^q$. If q is odd, then w contains $(q-1)/2$ pairs and each pair has 4 possibilities; then, the number of possible intersection points is $4^{(q-1)/2}.1 = 2^{q-1}$.

Finally, we show that the points described above belong to exactly $q/2$ flag domains if q is even and to $(q-1)/2$ flag domains if q is odd. For this, recall that, for each pair in w, we have 4 possibilities of 2-dimensional subspaces and note that the signature of these subspaces is $+-$ and $-+$, and the signature of the remainder is $++\dots++$. Thus, the number of flag domains which have nonempty intersection with S_w for the given w is $2^{q/2}$ if q is even and is $2^{(q-1)/2}$ if q is odd.

Also, since each two of these 4 possibilities of 2-dimensional subspaces have the same signature, then for any fixed signature for a flag domain, there are $2^{q/2}$ points belonging to the base cycle of that flag domain if q is even and $2^{(q-1)/2}$ points belonging to the same base cycle of that flag domain if q is odd. □

Remark 6. In the case of the group SL_{2n}^{C} with the real form $SU(p,q)$, Brecan [4, 5] shows that the number of Iwasawa–Schubert varieties which intersect at least one base cycle and have the minimal dimension pq is $(2n-1).(2n-3)\dots(2n-2q+1)$. But for the group $SO(2n,\mathbb{C})$ with the real form $SO(p,q)$, if q is even, the number of Schubert varieties S_w which have the minimal dimension $pq/2$ and intersect at least one base cycle is $n.(n-2)\dots(n-q+2)$, and if q is odd, the number of Schubert varieties S_w which have the minimal dimension $(pq-1)/2$ and intersect at least one base cycle is $(n-2).(n-4)\dots(n-q+1)$.

In the case of $SO(2n+1,\mathbb{C})$ with the real form $SO(p,q)$, if q is even, the number of Schubert varieties S_w which have the minimal dimension $pq/2$ and intersect at least one base cycle is $(n-1).(n-3)\dots(n-q+1)$, and if q is odd, the number of Schubert varieties S_w which have the minimal dimension $pq/2$ and intersect at least one base cycle is $(n-2).(n-4)\dots(n-q+1)$.

The following remark, which is a consequence of the proof of Theorem 2, describes all intersection points of S_w with the base cycles C_α.

Remark 7. To determine all intersection points between the base cycles and the Iwasawa–Schubert variety S_w of complementary dimension, we will define a set for each case of q:

If q is even, let $w \in W_I$ be a perfect harmonic permutation and define the following:

$\text{Swite}_w := \{\psi(w_r) : w_r$ is obtained from w by switching none, some, or all pairs $(-(2i-1), 2i)$ by $(2i, -(2i-1))$ or $(-2i, 2i-1)$ or $((2i-1), -2i)$ and switching none, some, or all pairs $(-(2i-1), -2i)$ by $(-2i, -(2i-1))$ or $(2i, (2i-1))$ or $((2i-1), 2i), 1 \le i \le q/2\}$. Define $\mathbb{F}_e(\text{Fix}\, T_S)$ to be the set of all maximally b-isotropic flags associated with the basis

$$\begin{aligned} &e_1 + ie_2, e_3 + ie_4, \dots, e_{2n-1} + ie_{2n}, e_{2n-1} \\ &\quad - ie_{2n}, \dots, e_1 - ie_2. \end{aligned} \tag{60}$$

Let $M_w \subset \mathbb{F}_e(\text{Fix}\, T_S)$ be the set of all maximally b-isotropic flags associated with all elements in Swite_w. Note that we have $q/2$ of the pairs $(-(2i-1), 2i)$ and $(-(2i-1), -2i), 1 \le i \le q/2$, in any $w \in W_I$, and for each pair, we have 4 possibilities to switch it, so the cardinality of Swite_w is $4^{q/2} = 2^q$. The set Swite_w gives us all intersection points of S_w, and each $2^{q/2}$ of these points belongs to only one flag domain where these points of intersection sit in the base cycle of that flag domain.

If q is odd, let $w \in W_I$ be a perfect harmonic permutation and define the following:

$\text{Swito}_w := \{\psi(w_r) : w_r$ is obtained from w by switching none, some, or all pairs

$(-(2i-1), 2i)$ by $(2i, -(2i-1))$ or $(-2i, 2i-1)$ or $((2i-1), -2i), 1 \le i \le q/2\}$. Define $\mathbb{F}_o(\text{Fix}\, T_S)$ to be the set of all maximal b-isotropic flags associated with the basis

$$e_1 + ie_2, e_3 + ie_4, \ldots, e_{q-2} + ie_{q-1}, e_{q+2} + ie_{q+3}, \ldots, e_{2n-1} + ie_{2n}, e_q,$$
$$e_{q+1}, e_{2n-1} - ie_{2n}, \ldots, e_{q+2} - ie_{q+3}, e_{q-2} - ie_{q-1}, \ldots, e_1 - ie_2. \quad (61)$$

Let $M_w \subset \mathbb{F}_o(\mathrm{Fix}\, T_S)$ be the set of all maximally b-isotropic flags associated with all elements in Swito_w. Note that we have $(q-1)/2$ of the pairs $(-(2i-1), 2i), 1 \le i \le q/2$, in any $w \in W_I$, and for each pair, we have 4 possibilities for switching it. Hence, the cardinality of Swito_w is $4^{(q-1)/2} = 2^{q-1}$. The set Swito_w gives us all intersection points of S_w, and each $2^{(q-1)/2}$ of these points belongs to only one flag domain where these points of intersection sit in the base cycle of that flag domain.

Example 5. In $G_0 = \mathrm{SO}(6,4)$, fix $w = (-35-142)$ as a perfect harmonic permutation, then

$$\mathrm{Swite}_w = \left\{ \begin{array}{c} (251-3-4), (25-1-34), (-2513-4), (-25-134), (-3512-4), \\ (351-2-4), (-35-124), (35-1-24), (25-4-31), (-25-431), (254-3-1), (-2543-1), \\ (-35-421), (35-4-21), (-3542-1), (354-2-1) \end{array} \right\}. \quad (62)$$

Example 6. In $G_0 = \mathrm{SO}(5,3)$, fix $w = (-124-3)$ as a perfect harmonic permutation, then

$$\mathrm{Swito}_w = \{(1-23-4), (-213-4), (2-13-4), (-123-4)\}. \quad (63)$$

Recall that, in the cases of $\mathrm{SP}(2n,\mathbb{R})$ and $\mathrm{SO}^*(2n)$, every flag domain intersects all Schubert varieties of complementary dimension. But in the case of $\mathrm{SO}(p,q)$, we do not have this property except in a very special case. We explain this case in the following example.

Example 7. If $n = q+1$, then the flag domain parametrized by the sequence

$$\alpha = + - + - \ldots + - + -+ \quad (64)$$

intersects all Schubert varieties of dimension $pq/2$ if q is even. And the flag domain parametrized by the sequence

$$\beta = + - + - \ldots + - + - + + \quad (65)$$

intersects all Schubert varieties of dimension $(pq-1)/2$ if q is odd.

Data Availability

The research data used to support this study are included within the article.

Disclosure

Parts of this work are from the author's PhD thesis at Ruhr-Universität, Bochum, Germany [6].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was partially supported by L'Oreal UNESCO "For Women in Science 2018". The author would like to thank Prof. Dr. Dr. h.c. mult. Alan T. Huckleberry and Prof. Dr. Peter Hienzner for initially introducing her this research area and this topic in particular.

References

[1] J. A. Wolf, "Exhaustion functions and cohomology vanishing theorems for open orbits on complex flag manifolds," *Mathematical Research Letters*, vol. 2, no. 2, 1995.
[2] F. S. Abu-Shoga, "On the Weyl group of SO (2n, C)," *IOSR Journal of Mathematics (IOSR-JM)*, vol. 15, pp. 55–58, 2019.
[3] G. Fels, A. Huckleberry, and J. A. Wolf, "Cycle spaces of flag domains," *Progress in Mathematics*, Birkhuser-Verlag, vol. 245Boston, MA, USA, 2006.
[4] A. Brecan, "Schubert duality for SL (n, R)-flag domains," 2014, https://arxiv.org/abs/1409.5868.
[5] A.-M. Brecan, "On the intersection pairing between cycles in SU (p, q)-flag domains and maximally real Schubert varieties," *Journal of Algebra*, vol. 492, pp. 324–347, 2017.
[6] F. S. Abu-Shoga, *Combinatorial Geometry of Flag Domains in G/B*, Ph.D. thesis, Ruhr University Bochum, Bochum, Germany, 2017.