Research Article

On Generalized Arakawa–Kaneko Zeta Functions with Parameters a, b, c

Nestor G. Acala¹ and Edward Rowe M. Aleluya²

¹Mathematics Department, College of Natural Sciences and Mathematics, Mindanao State University–Main Campus, Marawi, Lanao Del Sur, Philippines

²Department of Physical Sciences and Mathematics, College of Science and Environment, Mindanao State University–Naawan, Naawan, Misamis Oriental, Philippines

Correspondence should be addressed to Nestor G. Acala; nestor.acala@gmail.com

Received 29 February 2020; Accepted 22 June 2020; Published 27 July 2020

Academic Editor: Teodor Bulboacă

Copyright © 2020 Nestor G. Acala and Edward Rowe M. Aleluya. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

For $k \in \mathbb{Z}$, the generalized Arakawa–Kaneko zeta functions with a, b, c parameters are given by the Laplace–Mellin integral $\xi_k(s, x; a, b, c) = (1/\Gamma(s)) \int_0^\infty (Li_k(1 - (ab)^{-t})/b^t - a^{-t})c^{-xt}t^{s-1}dt$, where $\Re(s) > 0$ and $x > 0$ if $k \geq 1$, and $\Re(s) > 0$ and $x > |k| + 1$ if $k \leq 0$. In this paper, an interpolation formula between these generalized zeta functions and the poly-Bernoulli polynomials with a, b, c parameters is obtained. Moreover, explicit, difference, and Raabe’s formulas for $\xi_k(s, x; a, b, c)$ are derived.

1. Introduction

In the second half of the 19th century, it was well known that the Riemann zeta function may be represented by the normalized Mellin transform,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt, \quad \Re(s) > 1. \quad (1)$$

Arakawa and Kaneko [1], following expression (1), defined the Arakawa–Kaneko zeta function $\xi_k(s)$ for any integers $k \geq 1$ by

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-t})}{e^t - 1} t^{s-1} dt, \quad (2)$$

where Li_k denotes the k -th polylogarithm $Li_k(z) = \sum_{n=1}^\infty (z^n/n^k)$. The integral converges for $\Re(s) > 0$ and the function ξ_k continues analytically to an entire function of the whole s -plane. Moreover, Arakawa and Kaneko were able to express values of this function at negative values with the aid of the generalized Bernoulli numbers $B_n^{(k)}$ called poly-Bernoulli numbers. These numbers were introduced by Kaneko [2] using the generating function:

$$\frac{Li_k(1 - e^{-z})}{1 - e^{-z}} = \sum_{n=0}^\infty B_n^{(k)} \frac{z^n}{n!}, \quad (3)$$

whose values, when $k = 1$, are just the classical Bernoulli numbers. Arakawa and Kaneko [1] showed that for $m = 0, 1, 2, \dots$,

$$\xi_k(-m) = \sum_{l=0}^m (-1)^l \binom{m}{l} B_l^{(k)}. \quad (4)$$

In [3], Coppo and Candelpergher introduced the more general Arakawa–Kaneko zeta function defined for $\Re(s) > 0$ and $x > 0$ by

$$\xi_k(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-t})}{1 - e^{-t}} e^{-xt} t^{s-1} dt, \quad (5)$$

which is a very natural extension of the classical Arakawa–Kaneko zeta function. In particular, for $x = 1$, $\xi_k(s, 1) = \xi_k(s)$ and for $k = 1$, $\xi_1(s, x) = s\xi(s + 1, x)$.

Note that, $\xi_k(s, x)$ satisfies the explicit formula,

$$\xi_k(s, x) = \sum_{m=0}^\infty \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{1}{(x+j)^s}. \quad (6)$$

Moreover, they also expressed the special values of this function at negative integral points by means of poly-Bernoulli polynomials, that is, for all nonnegative integers n and $x > 0$,

$$\xi_k(-n, x) = (-1)^n B_n^{(k)}(-x), \quad (7)$$

where poly-Bernoulli polynomials $B_n^{(k)}(x)$ are defined by the generating function:

$$e^{-xz} \frac{Li_k(1 - e^{-z})}{1 - e^{-z}} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{z^n}{n!}, \quad (8)$$

whose values at $x = 0$ are precisely the poly-Bernoulli numbers. The polynomials $B_n^{(k)}(x)$ which were introduced by Bayad and Hamahata in [4] satisfy the recurrence relation:

$$B_n^{(k)}(x) = \sum_{m=0}^n (-1)^m \binom{n}{m} B_{n-m}^{(k-1)} \sum_{l=0}^m \frac{(-1)^l}{n-l+1} \binom{m}{l} B_l(x), \quad (9)$$

for all $k \geq 1$ and $n \geq 0$.

2. Generalization of Arakawa–Kaneko Zeta Functions and Poly-Bernoulli with a, b Parameters

The generalization of Arakawa–Kaneko zeta functions using a, b parameters was introduced by Jolany and Corcino [5] which are defined via Laplace-Mellin integral:

$$\xi_k(s, x; a, b) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Li_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{-xt} t^{s-1} dt. \quad (10)$$

It is defined for $\Re(s) > 0$ and $x > 0$ if $k \geq 1$, and for $\Re(s) > 0$ and $x > |k| + 1$ if $k \leq 0$. Its explicit formula is given by

$$\begin{aligned} \xi_k(s, x; a, b) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \sum_{j=0}^n \binom{n}{j} (-1)^j (x + j \ln a + (j+1) \ln b)^{-s} \\ &\quad \cdot (x \ln c - j \ln a - (j+1) \ln b)^n. \end{aligned} \quad (11)$$

Moreover, Jolany et al. [6] introduced generalizations of poly-Bernoulli numbers and poly-Bernoulli polynomials

with a, b parameters, $a \neq b$, using the following generating functions, respectively:

$$\begin{aligned} \frac{Li_k(1 - (ab)^{-t})}{b^t - a^{-t}} &= \sum_{n=0}^{\infty} B_n^{(k)}(a, b) \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \\ \frac{Li_k(1 - (ab)^{-t})}{b^t - a^{-t}} e^{xt} &= \sum_{n=0}^{\infty} B_n^{(k)}(x; a, b) \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|}. \end{aligned} \quad (12)$$

Using these new generalizations, Jolany and Corcino [5] obtained the interpolation formula between the generalized Arakawa–Kaneko zeta functions and generalized poly-Bernoulli polynomials with a, b parameters given by

$$\xi_k(-n, x; a, b) = (-1)^n B_n^{(k)}(-x; a, b). \quad (13)$$

3. Generalization of Poly-Bernoulli Polynomials with a, b, c Parameters

For $a, b, c > 0$ and $a \neq b$, Jolany et al. [6] defined the generalized polynomials $B_n^{(k)}(x; a, b, c)$ with a, b, c parameters as follows:

$$\frac{Li_k(1 - (ab)^{-t})}{b^t - a^{-t}} c^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x; a, b, c) \frac{t^n}{n!}, \quad |t| < \frac{2\pi}{|\ln a + \ln b|}. \quad (14)$$

In this section, some fundamental properties of the generalized poly-Bernoulli polynomials with a, b, c parameters are presented without proofs. These properties are parallel to the results in [5] for the poly-Bernoulli polynomials with a, b parameters. These results are given in the following theorems.

Theorem 1 (explicit formula). For $k \in \mathbb{Z}$ and $n \geq 0$,

$$\begin{aligned} B_n^{(k)}(x; a, b, c) &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^m (-1)^j \binom{m}{j} \\ &\quad \cdot (x \ln c - j \ln a - (j+1) \ln b)^n. \end{aligned} \quad (15)$$

Theorem 2 (recurrence formula). For all $k \geq 1$ and $n \geq 0$,

$$B_n^{(k)}(x; a, b, c) = (\ln a + \ln b) \sum_{m=0}^n (-\ln a)^m \binom{n}{m} B_{n-m}^{(k-1)}(a, b) \sum_{l=0}^m \frac{(-\ln a)^{-l}}{n-l+1} \binom{m}{l} B_l(x; a^{-1}, b, c). \quad (16)$$

As a direct result, by setting $a = e$, $b = 1$, and $c = e$ in Theorem 2, the recurrence formula of the classical poly-Bernoulli polynomials (1) is obtained.

The next theorem shows the relationship of the generalized poly-Bernoulli polynomials with a, b, c parameters and the classical poly-Bernoulli polynomials.

Theorem 3 For $a, b, c > 0$ and $n \geq 0$,

$$B_n^{(k)}(x; a, b, c) = (\ln a + \ln b)^n B_n^{(k)} \left(\frac{x \ln c - \ln b}{\ln a + \ln b} \right). \quad (17)$$

The next theorem presents the second recurrence formula of the generalized poly-Bernoulli polynomials with a, b, c parameters.

Theorem 4 (recurrence formula (2)). For $k \in \mathbb{Z}$ and $n \geq 2$,

$$\begin{aligned} B_0^{(k)}(x; a, b, c) &= 1; \\ B_1^{(k)}(x; a, b, c) &= \frac{1}{2} [B_1^{(k-1)}(x; a, b, c) + (x \ln c - \ln b) B_0^{(k)}(x; a, b, c)]; \\ B_n^{(k)}(x; a, b, c) &= \frac{1}{n+1} \left\{ B_n^{(k-1)}(x; a, b, c) + (x \ln c - \ln b) (\ln a + \ln b)^{n-1} B_0^{(k)}(x; a, b, c) \right. \\ &\quad + (x \ln c - \ln b) \sum_{m=1}^{n-1} (\ln a + \ln b)^{n-m-1} \binom{n}{m} B_m^{(k)}(x; a, b, c) \\ &\quad \left. - \sum_{m=1}^{n-1} (\ln a + \ln b)^{n-m} \binom{n}{m-1} B_m^{(k)}(x; a, b, c) \right\}. \end{aligned} \quad (18)$$

As a direct result, by setting $a = e$, $b = 1$, and $c = e$ in Theorem 4, the second recurrence formula of the classical poly-Bernoulli polynomials given in (1) is obtained.

4. Generalization of Arakawa–Kaneko Zeta Functions with a, b, c Parameters

In this section, we give the definition of the generalized Arakawa–Kaneko zeta function with a, b, c parameters and obtain an interpolation formula between these generalized zeta functions and the poly-Bernoulli polynomials with a, b, c parameters. Moreover, explicit, difference, and Raabe's formulas for $\xi_k(s, x; a, b, c)$ will be derived.

Definition 1. For $k \in \mathbb{Z}$, the generalized Arakawa–Kaneko zeta functions with a, b, c parameters are given by the Laplace-Mellin integral,

$$\xi_k(s, x; a, b, c) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - (ab)^{-t})}{b^t - a^{-t}} c^{-xt} t^{s-1} dt, \quad (19)$$

where $\Re(s) > 0$ and $x > 0$ if $k \geq 1$, and for $\Re(s) > 0$ and $x > |k| + 1$ if $k \leq 0$.

It can be seen that $\xi_k(s, x; e, 1, e)$ is just the classical Arakawa–Kaneko zeta functions $\xi_k(x, s)$.

The following lemma gives a relation between the generalized Arakawa–Kaneko zeta functions with a, b, c parameters and the classical Arakawa–Kaneko zeta functions.

Lemma 1. For $k \in \mathbb{Z}$,

$$\xi_k(s, x; a, b, c) = (\ln a + \ln b)^{-s} \xi_k \left(s, \frac{x \ln c + \ln b}{\ln a + \ln b} \right). \quad (20)$$

Proof. By applying Definition 1, we get

$$\xi_k(s, x; a, b, c) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-t \ln ab})}{1 - e^{-t \ln ab}} e^{-t(x \ln c + \ln b)} t^{s-1} dt. \quad (21)$$

By changing variables $z = (\ln a + \ln b)t$, we obtain

$$\begin{aligned} \xi_k(s, x; a, b, c) &= \frac{1}{(\ln a + \ln b)^s} \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-z})}{1 - e^{-z}} \\ &\quad e^{-z((x \ln c + \ln b)/(\ln a + \ln b))} z^{s-1} dz \\ &= \frac{1}{(\ln a + \ln b)^s} \xi_k \left(s, \frac{x \ln c + \ln b}{\ln a + \ln b} \right). \end{aligned} \quad (22)$$

□

Theorem 5 (interpolation formula). The function $s \rightarrow \xi_k(s, x; a, b, c)$ has analytic continuation to an entire function on the whole complex s -plane and for any positive integer n ,

$$\xi_k(-n, x; a, b, c) = (-1)^n B_n^{(k)}(-x; a, b, c). \quad (23)$$

Proof. To prove that $s \rightarrow \xi_k(s, x; a, b, c)$ has analytic continuation to an entire function on the whole complex s -plane, it is sufficient to show that $s \rightarrow \xi_k(s, x)$ has such a property. The details of this fact come from (7), and the interpolation formula which shows the relationship of the classical Arakawa–Kaneko zeta functions and classical poly-Bernoulli polynomials established by [3]. Hence, by using Lemma 1, Theorem 3, and equation (7), we obtain

$$\begin{aligned}
\xi_k(-n, x; a, b, c) &= (\ln a + \ln b)^{-(-n)} \xi_k\left(-n, \frac{x \ln c + \ln b}{\ln a + \ln b}\right) \\
&= (\ln a + \ln b)^n (-1)^n B_n^{(k)}\left(\frac{-x \ln c - \ln b}{\ln a + \ln b}\right) \\
&= (-1)^n B_n^{(k)}(-x; a, b, c).
\end{aligned} \tag{24}$$

As an immediate consequence of the previous theorems in this section, the explicit formula for the generalized Arakawa–Kaneko zeta functions with a, b, c parameters is obtained. $\square$

Corollary 1 (explicit formula). For $k \in \mathbb{Z}$,

$$\begin{aligned}
\xi_k(s, x; a, b, c) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \sum_{j=0}^n \binom{n}{j} (-1)^j \\
&\quad \cdot (x \ln c + j \ln a + (j+1) \ln b)^{-s}.
\end{aligned} \tag{25}$$

$$\begin{aligned}
\xi_k(s, x; a, b, c) &= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \int_0^{\infty} (1 - e^{-t \ln ab})^n e^{-t(x \ln c + \ln b)} t^{s-1} dt \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \int_0^{\infty} \sum_{j=0}^n \binom{n}{j} (-1)^j e^{-t(x \ln c + j \ln a + (j+1) \ln b)} t^{s-1} dt \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \sum_{j=0}^n \binom{n}{j} (-1)^j \cdot \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-t(x \ln c + j \ln a + (j+1) \ln b)} t^{s-1} dt \\
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)^k} \sum_{j=0}^n \binom{n}{j} (-1)^j \frac{1}{(x \ln c + j \ln a + (j+1) \ln b)^s}.
\end{aligned} \tag{27}$$

As a direct result, setting $a = e$, $b = 1$, and $c = e$, the explicit formula of the generalized Arakawa–Kaneko zeta functions $\xi_k(s, x; a, b, c)$ reduces to the explicit formula of the classical Arakawa–Kaneko zeta functions $\xi_k(s, x)$ which is given in equation (6).

Raabe's formula is a fundamental and universal property in the theory of zeta functions and plays an important role in special functions. This formula holds for several types of zeta

Proof. Definition 1 and definition of polylogarithmic function give

$$\begin{aligned}
\xi_k(s, x; a, b, c) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Li_k(1 - (ab)^{-t})}{b^t - a^{-t}} c^{-xt} t^{s-1} dt \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{Li_k(1 - e^{-t \ln ab})}{1 - e^{-t \ln ab}} e^{-t(x \ln c + \ln b)} t^{s-1} dt \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{(1 - e^{-t \ln ab})^n}{(n+1)^k} e^{-t(x \ln c + \ln b)} t^{s-1} dt.
\end{aligned} \tag{26}$$

Consequently,

functions and provides a powerful link between zeta integrals and Dirichlet series. In the next theorem, an interesting link between integral of the generalized Arakawa–Kaneko zeta functions with a, b, c parameters and Dirichlet series is presented. $\square$

Lemma 2 (difference formula)

$$\begin{aligned}
&\xi_k\left(s, x + \frac{\ln ab}{\ln c}; a, b, c\right) - \xi_k(s, x; a, b, c) \\
&= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{j+1} (x \ln c + j \ln a + (j+1) \ln b)^{-s}.
\end{aligned} \tag{28}$$

Proof

$$\begin{aligned}
 & \xi_k\left(s, x + \frac{\ln ab}{\ln c}; a, b, c\right) - \xi_k(s, x; a, b, c) \\
 &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-t \ln ab})}{1 - e^{-t \ln ab}} \left[e^{-t(x \ln c + \ln ab + \ln b)} - e^{-t(x \ln c + \ln b)} \right] t^{s-1} dt \\
 &= -\frac{1}{\Gamma(s)} \int_0^\infty Li_k(1 - e^{-t \ln ab}) e^{-t(x \ln c + \ln b)} t^{s-1} dt \\
 &= -\frac{1}{\Gamma(s)} \sum_{m=0}^\infty \frac{1}{(m+1)^k} \int_0^\infty \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j e^{-t(x \ln c + j \ln a + (j+1) \ln b)} t^{s-1} dt \\
 &= \sum_{m=0}^\infty \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{j+1} \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(x \ln c + j \ln a + (j+1) \ln b)} t^{s-1} dt \\
 &= \sum_{m=0}^\infty \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{j+1} \frac{1}{(x \ln c + j \ln a + (j+1) \ln b)^s}.
 \end{aligned} \tag{29}$$

The Raabe's formula for generalization of Arakawa-Kaneko zeta functions with a, b, c parameters is presented by the following theorem. $\square$

Theorem 6 (Raabe's formula)

$$\begin{aligned}
 & \int_0^{(\ln ab / \ln c)} \xi_k(s, x + w; a, b, c) dw \\
 &= \frac{1}{(s-1) \ln c} \sum_{m=0}^\infty \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j \\
 & \quad \cdot (x \ln c + j \ln a + (j+1) \ln b)^{1-s}.
 \end{aligned} \tag{30}$$

Proof

$$\begin{aligned}
 & \int_0^{(\ln ab / \ln c)} \xi_k(s, x + w; a, b, c) dw \\
 &= \int_0^{(\ln ab / \ln c)} \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-t \ln ab})}{1 - e^{-t \ln ab}} e^{-t((x+w) \ln c + \ln b)} t^{s-1} dt dw \\
 &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1 - e^{-t \ln ab})}{1 - e^{-t \ln ab}} e^{-t(x \ln c + \ln b)} t^{s-1} \int_0^{(\ln ab / \ln c)} e^{-wt \ln c} dw dt \\
 &= \frac{1}{(\ln c) \Gamma(s)} \int_0^\infty Li_k(1 - e^{-t \ln ab}) e^{-t(x \ln c + \ln b)} t^{s-2} dt.
 \end{aligned} \tag{31}$$

Hence, utilizing Lemma 2 results in

$$\begin{aligned}
 & \int_0^{(\ln ab / \ln c)} \xi_k(s, x + w; a, b, c) dw \\
 &= \frac{1}{\ln c} \cdot \frac{\Gamma(s-1)}{\Gamma(s)} \left[\frac{1}{\Gamma(s-1)} \int_0^\infty Li_k(1 - e^{-t \ln ab}) e^{-t(x \ln c + \ln b)} t^{s-2} dt \right] \\
 &= \frac{1}{(s-1) \ln c} \left(\xi_k(s-1, x; a, b, c) - \xi_k\left(s-1, x + \frac{\ln ab}{\ln c}; a, b, c\right) \right) \\
 &= \frac{1}{(s-1) \ln c} \sum_{m=0}^\infty \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j (x \ln c + j \ln a + (j+1) \ln b)^{1-s}.
 \end{aligned} \tag{32}$$

As a direct result of Raabe's formula and interpolation formula, the following corollary that presents the Raabe's formula in terms of the generalized poly-Bernoulli polynomials with a, b, c parameters is obtained and stated as follows. $\square$

Corollary 2. *Raabe's formula in terms of generalization of poly-Bernoulli polynomials with parameters a, b, c is given as follows:*

$$\begin{aligned} & \int_0^{(\ln ab / \ln c)} B_n^{(k)}(-x-w; a, b, c) dw \\ &= \frac{(-1)^{n+1}}{(n+1)\ln c} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j \\ & \quad \cdot (x \ln c + j \ln a + (j+1) \ln b)^{n+1}. \end{aligned} \quad (33)$$

Proof. By Theorems 5 and 6,

$$\begin{aligned} & \int_0^{(\ln ab / \ln c)} B_n^{(k)}(-x-w; a, b, c) dw \\ &= \int_0^{(\ln ab / \ln c)} (-1)^n \xi_k(-n, x+w; a, b, c) dw \\ &= (-1)^n \int_0^{(\ln ab / \ln c)} \xi_k(-n, x+w; a, b, c) dw \\ &= \frac{(-1)^{n+1}}{(n+1)\ln c} \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^j \\ & \quad \cdot (x \ln c + j \ln a + (j+1) \ln b)^{n+1}. \end{aligned} \quad (34)$$

Hence, the desired result is obtained. $\square$

5. Conclusion and Recommendation

A significant result of the paper is the derivation of an interpolation formula between the generalized poly-Bernoulli polynomials with parameters a, b, c and the generalized Arakawa-Kaneko zeta functions with parameters a, b, c . By applying this interpolation formula, several properties of the generalized Arakawa-Kaneko zeta functions with parameters a, b, c are established. It seems interesting to define another generalization of Arakawa-Kaneko zeta functions with parameters a, b, c through multiple polylogarithms of index set $(k_1, k_2, \dots, k_r)$, where $(k_1, k_2, \dots, k_r)$ are any r -tuples of positive integers, and relate these to the multi-poly-Bernoulli polynomials of parameters a, b, c introduced by Corcino et al. in [7].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

- [1] T. Arakawa and M. Kaneko, "Multiple zeta values, poly-Bernoulli numbers, and related zeta functions," *Nagoya Mathematical Journal*, vol. 153, pp. 189–209, 1999.
- [2] M. Kaneko, "Poly-Bernoulli numbers," *Journal de Théorie des Nombres de Bordeaux*, vol. 9, pp. 199–206, 1997.
- [3] M.-A. Coppo and B. Candelpergher, "The Arakawa-Kaneko zeta function," *The Ramanujan Journal*, vol. 22, no. 2, pp. 153–162, 2010.
- [4] A. Bayad and Y. Hamahata, "Polylogarithms and poly-Bernoulli polynomials," *Kyushu Journal of Mathematics*, vol. 65, no. 1, pp. 15–24, 2011.
- [5] H. Jolany and R. B. Corcino, "Explicit formula for generalization of poly-Bernoulli numbers and polynomials with a, b, c parameters," *Journal of Classical Analysis*, vol. 6, no. 2, pp. 119–135, 2015.
- [6] H. Jolany, M. R. Darafsheh, and R. E. Alikelaye, "Generalizations on poly-Bernoulli numbers and polynomials," *International Journal of Mathematical Combinatorics*, vol. 2, pp. 7–14, 2010.
- [7] C. Corcino, R. Corcino, T. Komatsu, and H. Jolany, "On multi-poly Bernoulli polynomials," *Journal of Inequalities and Special Functions*, vol. 10, no. 2, pp. 21–34, 2019.