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Research Article

On a Conjecture of Yamaguchi and Yokura

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Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of simply connected elliptic spaces. Our paper investigates the conjecture proposed by T. Yamaguchi and S. Yokura, states that $\dim \operatorname{Ker} \pi_*(p)_{\mathbb{Q}} \leq \dim \operatorname{Ker} H_*(p;\mathbb{Q}) + 1$. Our goal is to prove this conjecture when F and B satisfy the condition $\pi_{\text{even}}(F)_{\mathbb{Q}} = \pi_{\text{even}}(B)_{\mathbb{Q}} = 0$. We go also on to establish a well-known conjecture of Hilali for a class of spaces which puts it into the context of fibration.

1. Introduction

We begin with a description of the conjecture referred to in the title. In this paper, all spaces are simply connected CW-complexes and are of finite type, i.e., have finite dimensional rational cohomology.

A space X is said to be elliptic if the dimensions of cohomology and homotopy are both finite [1]. For these spaces, Hilali [2] conjectured in 1990.

Conjecture 1 (Hilali). *Let X be a simply connected rationally elliptic space; then,*

$$(\mathrm{H}) \dim \pi_*(X)_{\mathbb{Q}} \leq \dim H_*(X;\mathbb{Q}). \tag{1}$$

Generally, speaking about cohomology is delicate, invariant, and difficult to compute. Recently, Yamaguchi and Yokura proposed another version to the conjecture (H) of a map [3].

Conjecture 2 (Yamaguchi–Yokura). *Let $p\colon E \longrightarrow B$ be a continuous map between two elliptic spaces, then*

$$(YY) \dim \operatorname{Ker} \pi_*(p)_{\mathbb{Q}} \leq \dim \operatorname{Ker} H_*(p;\mathbb{Q}) + 1, \tag{2}$$

where

$$\begin{aligned} \operatorname{Ker} \pi_*(p)_{\mathbb{Q}} &:= \oplus_{i\geq 1} \operatorname{Ker}\left(\pi_i(p)_{\mathbb{Q}}\colon \pi_i(E)_{\mathbb{Q}} \longrightarrow \pi_i(B)_{\mathbb{Q}}\right),\\ \operatorname{Ker} H_*(p;\mathbb{Q}) &:= \oplus_{i\geq 0} \operatorname{Ker}\left(H_i(p;\mathbb{Q})\colon H_i(E;\mathbb{Q}) \longrightarrow H_i(B;\mathbb{Q})\right). \end{aligned} \tag{3}$$

In particular, if $B \simeq \{*\}$, we obtain the conjecture (H). Then, a positive answer to the conjecture (YY) would give a positive answer to the conjecture (H).

Although we will recall some basic facts about Sullivan minimal models, our proofs assume a working familiarity with them. Our reference for rational homotopy theory is [1]. The rational homotopy type of X is encoded in a differential graded algebra (A, d) called the Sullivan minimal model of X. This is a free-graded algebra $A = \Lambda V$ generated by a graded vector space $V = \oplus_{i\geq 2} V^i$ and with decomposable differential, i.e.,

$$d\colon V^i \longrightarrow \left(\Lambda^{\geq 2} V\right)^{(i+1)}. \tag{4}$$

Notice that $(\Lambda V, d)$ determines the rational homotopy type of X. Especially, there are isomorphisms:

$$\begin{aligned} &H^*(\Lambda V, d) \cong H^*(X;\mathbb{Q}) \text{ as graded commutative algebras},\\ &V \cong \pi_*(X)_{\mathbb{Q}} \text{ as graded vector spaces}. \end{aligned} \tag{5}$$

Although our results are stated and proved in purely algebraic terms, they do admit topological interpretations via this correspondence. Therefore, we can also characterize an elliptic space in terms of its Sullivan minimal model. A space X with Sullivan minimal model $(\Lambda V, d)$ is elliptic if V and $H^*(\Lambda V, d)$ are both finite dimensional. Also, we can reformulate the conjecture (H) algebraically as follows.

Conjecture 3. *If $(\Lambda V, d)$ is a simply connected elliptic Sullivan minimal model, then*

$$\dim V \leq \dim H^*(\Lambda V, d)\,(\mathrm{H}). \tag{6}$$

This conjecture is open in general, but has been proved in some interesting cases (see [2, 4–9]).

Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration. The KS-model for p is a short exact sequence

$$(\Lambda W, d) \longrightarrow (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, d_F), \tag{7}$$

of DGA, with $(\Lambda W, d)$ and $(\Lambda V, d_F)$ as the Sullivan minimal models for B and F, respectively (see [1], Proposition 15.5). The differential D satisfies $D(w) = d(w)$ for $w \in W$ and $D(v) - d_F(v) \in \Lambda^+ W \otimes \Lambda V$ for $v \in V$. The DGA $(\Lambda W \otimes \Lambda V, D)$ is a Sullivan model for the total space E but is not, in general, minimal.

In view of the notation above, the algebraic version of the conjecture (YY) is given.

Conjecture 4. *If $(\Lambda W, d) \xrightarrow{J} (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, d_F)$ is the KS-model for a rational fibration of elliptic spaces, then*

$$(YY)\ \dim \operatorname{Coker} H^*(J, D_1) \leq \dim \operatorname{Coker} H^*(J) + 1, \tag{8}$$

where D_1 is the linear part of D.

This conjecture is affirmed for spherical fibration and TNCZ fibration whose fibre satisfies the conjecture (H) (see [3]). In a previous joined work, the authors with Hilali have shown this conjecture for fibrations whose fibre has at most two oddly generators and also in the case of $H \longrightarrow G \longrightarrow G/H$, where G is a compact connected Lie group and H is a closed subgroup of G (see [10]).

Recall that a fibration $F \xrightarrow{i} E \longrightarrow B$ is totally non-cohomologous to zero (abbreviated TNCZ), if the induced homomorphism $H^*(i)$: $H^*(E; \mathbb{Q}) \longrightarrow H^*(F; \mathbb{Q})$ is surjective. It is equivalent to requiring that the Serre spectral sequence collapses at E_2-term. In this case, there is an isomorphism: $H^*(E; \mathbb{Q}) \cong H^*(F; \mathbb{Q}) \otimes H^*(B; \mathbb{Q})$ of $H^*(B; \mathbb{Q})$-modules.

We end this section with some notations and conventions. In general, we use V or W to denote a positively graded rational vector space of the finite type. The cohomology of a DGA (A, d) is denoted $H^*(A, d)$ or just $H^*(A)$, and let $[x] \in H^*(A, d)$ stand for the cohomology class of the cocycle $x \in A$.

As an overriding hypothesis, we assume that all spaces appearing in this paper are rational simply connected elliptic spaces.

2. The Conjecture of Yamaguchi and Yokura

The topological aspect in this section is centered around the following question.

Question 1. Let $(\Lambda W, d) \xrightarrow{J} (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, d_F)$ be the KS-model of a fibration with $W^{\text{even}} = V^{\text{even}} = 0$, is it true that $\dim \operatorname{Coker} H^*(J, D_1) \leq \dim \operatorname{Coker} H^*(J) + 1$?

Our most general results here are as follows.

Theorem 1. *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration. Suppose*

(1) F has the rational homotopy type of a product of odd-dimensional spheres

(2) p admits a section

Then, the conjecture (YY) is true.

Recall that a fibration $F \longrightarrow E \xrightarrow{p} B$ admits a section if there is a map s: $B \longrightarrow E$ such that $pos \simeq Id_B$. However, in [11], Lemma 3, Thomas showed that a fibration admits a section if and only if there exists a KS-model:

$$(\Lambda W, d) \longrightarrow (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, d_F), \tag{9}$$

such that $Dv - d_F(v) \in \Lambda^+ W \otimes \Lambda^+ V$ for $v \in V$.

Proof of Theorem 1. In the following, we make the identification $W^n \cong \pi_n(B)_{\mathbb{Q}}$, where $(\Lambda W, d)$ is the Sullivan minimal model of B. Hypothesis (11) implies that the Sullivan minimal model $(\Lambda V, d_F)$ for F has trivial differential, $d_F = 0$, with $V^{\text{even}} = 0$. Write $V = \langle v_1, v_2, \ldots, v_n \rangle$ with $|v_i| \leq |v_j|$ whenever $i < j$ and each $|v_i|$ is odd. So, the KS-model for $F \longrightarrow E \xrightarrow{p} B$ is of the form

$$(\Lambda W, d) \xrightarrow{J} (\Lambda W \otimes \Lambda(v_1, v_2, \ldots, v_n), D) \longrightarrow (\Lambda(v_1, v_2, \ldots, v_n), 0). \tag{10}$$

The second assumption implies that $Dv_1 = 0$ and $D(v_i) \in \Lambda^+ W \otimes \Lambda^+(v_1, v_2, \ldots, v_{i-1})$ for $2 \leq i \leq n$. This allows us to deduce that D is decomposable, and then

$$\dim \operatorname{Coker} H^*(J, D_1) = \dim V = n. \tag{11}$$

On the contrary, we put

$$\Gamma = \{v_1, v_1 v_2, v_1 v_2 v_3, \ldots, v_1 v_2 v_3 \ldots v_i \text{ for } 4 \leq i \leq n\}. \tag{12}$$

Furthermore, a direct argument shows that every element in Γ is a nonexact D-cycle. Since the elements of Γ all have different degrees, then they are linearly independent. Therefore, from (11) we have

$$\begin{aligned} \dim \operatorname{Coker} H^*(J) + 1 &\geq n + 1 \\ &\geq \dim \operatorname{Coker} H^*(J, D_1). \end{aligned} \tag{13}$$

□

Example 1. Let us consider the fibration

$$\mathbb{S}^3 \times \mathbb{S}^5 \longrightarrow X \xrightarrow{p} \mathbb{CP}^3, \tag{14}$$

given by the KS-model

$$(\Lambda(x, y), d) \xrightarrow{J} (\Lambda(x, y) \otimes \Lambda(u, v), D) \longrightarrow (\Lambda(u, v), d_F), \tag{15}$$

where $|x| = 2$, $|y| = 7$, $|u| = 3$, and $|v| = 5$, and the nonzero differentials are given by: $Dy = x^4$ and $Dv = x^3$. Hence, we have dim Coker $H^*(J, D_1) = 2$ and Coker $H^*(J) = \mathbb{Q}\{[u], [ux], [ux^2], [y - xv]\}$. Then,

$$1 + \dim \operatorname{Coker} H^*(J) = 5 \geq \dim \operatorname{Coker} H^*(J, D_1). \tag{16}$$

We can see that p does not admit a section, though it satisfies the conjecture (YY).

Theorem 2. *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration. Suppose*

(1) $\pi_{even}(F)_{\mathbb{Q}} = 0$

(2) B has the rational homotopy type of product of at least $\dim \pi_*(F)_{\mathbb{Q}} - 2$ odd-dimensional spheres.

Then, the conjecture (YY) holds.

Proof. According to the dimension of $\pi_*(F)_{\mathbb{Q}}$, we distinguish two cases:

Case I: $\dim \pi_*(F)_{\mathbb{Q}} \leq 2$, so B has the rational homotopy type of a point. This implies that $\operatorname{Ker} \pi_*(p)_{\mathbb{Q}} = \pi_*(F)_{\mathbb{Q}}$ and $\operatorname{Ker} H_*(p; \mathbb{Q}) = H_*(F; \mathbb{Q})$. From hypothesis (11) and [9], Theorem 1.2, we deduce that p satisfies the conjecture (YY).

Case II: $\dim \pi_*(F)_{\mathbb{Q}} \geq 3$, let $n = \dim \pi_*(F)_{\mathbb{Q}}$ and $m = \dim \pi_*(B)_{\mathbb{Q}}$.

The first hypothesis implies that the Sullivan minimal model $(\Lambda V, d_F)$ for F is oddly generated, i.e., $V^{\text{even}} = 0$, and then we write $V = \langle v_1, v_2, \ldots, v_n \rangle$ with $|v_i| \leq |v_j|$ whenever $i < j$ and each $|v_i|$ are odd. From the second hypothesis, the Sullivan minimal model $(\Lambda W, d)$ for B has trivial differential, $d = 0$, with $W^{\text{even}} = 0$; more precisely, we denote $(\Lambda(u_1, u_2, \ldots, u_m), 0)$ with $|u_j|$ is odd and $m \geq n - 2$. Therefore, the KS-model of p is given by

$$(\Lambda(u_1, \ldots, u_m), 0) \xrightarrow{J} (\Lambda(u_1, \ldots, u_m) \otimes \Lambda(v_1, \ldots, v_n), D) \longrightarrow (\Lambda(v_1, \ldots, v_n), d_F). \tag{17}$$

Then, for degree reasons, D is decomposable. Hence, we clearly have

$$\dim \operatorname{Coker} H^*(J, D_1) = n. \tag{18}$$

In order to prove

$$\dim \operatorname{Coker} H^*(J) + 1 \geq \dim \operatorname{Coker} H^*(J, D_1), \tag{19}$$

it suffices to find at least n elements in $H^*(\Lambda(u_1, \ldots, u_m) \otimes \Lambda(v_1, \ldots, v_n), D)/\operatorname{Im} H^*(J)$. For this, we put

$$\Gamma = \{\mu_E, u_1 u_2, \ldots, \widehat{u}_i, \ldots, u_m \mu_F \text{ for } 1 \leq i \leq m\}. \tag{20}$$

Here, the notation $\widehat{u}_i$ means that the element u_i is removed, and μ_E and μ_F denote the fundamental class of E and F, respectively. Now, for each element α in Γ, we have $D\alpha = 0$, and it is easy to see that α cannot be a D-coboundary. Thus, $[\alpha]$ is in Coker $H^*(J)$ for each α in Γ. Consequently, we have

$$\dim \operatorname{Coker} H^*(J) + 1 \geq m + 2 \geq n. \tag{21}$$

□

Example 2. Note that condition (30) above is sufficient but not necessary. Indeed, consider the nontrivial fibration $F \longrightarrow E \longrightarrow \mathbb{S}^3 \times \mathbb{S}^5$ given by the following KS-model:

$$(\Lambda(x, y), 0) \xrightarrow{J} (\Lambda(x, y) \otimes \Lambda(z, t, u, v, w), D) \longrightarrow (\Lambda(z, t, u, v, w), d_F), \tag{22}$$

with $|x| = |y| = |z| = |t| = 3$, $|u| = |v| = 5$, $|w| = 7$, $Dx = Dy = Dz = Dt = 0$, $Du = xy + zt$, $Dv = zt$, and $Dw = zv$. A careful check reveals that D defines a differential. Since $\operatorname{Im} H^*(J) = \mathbb{Q}\{1, [x], [y], [xy]\}$ and $\mathbb{Q}\{[z], [t], [tv], [zw]\} \subset H^*(\Lambda(x, y, z, t, u, v, w), D)$, we deduce that

$$\dim \operatorname{Coker} H^*(J) + 1 \geq 5. \tag{23}$$

Thus, the conjecture (YY) is true though $\dim \pi_*(B)_{\mathbb{Q}} < \dim \pi_*(F)_{\mathbb{Q}} - 2$.

In the remainder of this section, we show the conjecture (YY) for certain fibrations whose total space has a two-stage Sullivan minimal model $(\Lambda U, D)$, i.e., U decomposes as $U \cong W \oplus V$ with $dW = 0$ and $dV \subset \Lambda W$. Furthermore, if $(\Lambda U, D)$ is elliptic, then W may have generators of odd or even degree but V must have generators of odd degree only.

Proposition 1. *Let $(\Lambda W \otimes \Lambda V, D)$ be a two-stage Sullivan minimal model with W^{even} and V have the same dimensional, then the following KS-extension*

$$\left(\Lambda W^{\text{odd}}, 0\right) \xrightarrow{J} (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda W^{\text{even}} \otimes \Lambda V, d_F), \tag{24}$$

satisfies the conjecture (YY).

Proof. The condition $\dim W^{\text{even}} = \dim V$ implies that $(\Lambda W^{\text{even}} \otimes \Lambda V, d_F)$ is the Sullivan minimal model of an F_0-space ([1], Section 32). Moreover, since $DW^{\text{even}} = 0$ and $DV^{\text{odd}} \subset \Lambda W^{\text{odd}} \otimes \Lambda W^{\text{even}}$, because $(\Lambda W \otimes \Lambda V, D)$ is supposed two-stage Sullivan minimal model, then J is a pure fibration and by Theorem 2 of [11], it is TNCZ. Furthermore, from [3], we deduce the result. □

Proposition 2. *Let $(\Lambda W \otimes \Lambda V, D)$ be a two-stage Sullivan minimal model with odd degree only and assume that $D\colon V \longrightarrow \Lambda^2 W$ is an isomorphism, then the following KS-extension*

$$(\Lambda W, 0) \xrightarrow{J} (\Lambda W \otimes \Lambda V, D) \longrightarrow (\Lambda V, d_F), \tag{25}$$

satisfies the conjecture (YY).

Proof. Suppose that $\dim W = n$, since $D\colon V \longrightarrow \Lambda^2 W$ is an isomorphism, and ΛW is an exterior algebra, we have $\dim V = ((n^2 - n)/2)$. On the contrary, we know from Proposition 2.1 of [12]

$$\dim H^*(\Lambda W \otimes \Lambda V, D) \geq 2^{((n^2-n)/2)}. \tag{26}$$

As $DW = 0$, then every element in ΛW is a D-cycle and since $DV \cong \Lambda^2 W$; thus, the cohomology represented by elements of word-length at least two is bounded. This proves that

$$\begin{aligned} \dim \operatorname{Im} H^*(J) &= \dim W + 1 \\ &= n + 1, \end{aligned} \tag{27}$$

taking into account zeroth cohomology, and then we obtain

$$\begin{aligned} \dim \operatorname{Coker} H^*(J) + 1 &\geq 2^{((n^2-n)/2)} - (n+1) + 1 \\ &\geq \frac{n^2 - n}{2}. \end{aligned} \tag{28}$$

The abovementioned computation works for $n \neq 2$. If $n = 2$, referring to the Sullivan minimal model in this case, we have $(\Lambda W \otimes \Lambda V, D) = (\Lambda(x, y, z), D)$ with the degrees of all elements are odd and nonzero differential $Dz = xy$. It is easy to check that

$$\dim \operatorname{Coker} H^*(J) + 1 = 4 \geq 1 = \dim \operatorname{Coker} H^*(J, D_1). \tag{29}$$

□

3. The Hilali Conjecture

In this section, we consider a particular question suggested by Hilali and Mamouni in [6].

Question 2. If $F \longrightarrow E \longrightarrow B$ is a fibration where F and B are elliptic spaces and both satisfy the conjecture (H), is it true that E will too?

As a first result concerning this question, we have the following.

Theorem 3. *Let $\mathbb{S}^{(2n+1)} \longrightarrow E \xrightarrow{p} B$ be a fibration in which B satisfies the conjecture (H). If p admits a section, then E satisfies the conjecture (H).*

A key ingredient to the proof of this theorem is as follows.

Proposition 3. *Let $\mathbb{S}^{(2n+1)} \longrightarrow E \xrightarrow{p} B$ be a fibration such that p admits a section. Then, we have $H^*(E;\mathbb{Q}) \cong H^*(\mathbb{S}^{(2n+1)};\mathbb{Q}) \otimes H^*(B;\mathbb{Q})$.*

Proof. Since B is simply connected, then this fibration is oriented. Hence, we may formulate the Gysin sequence for p as follows ([13], p.375):

$$\cdots \longrightarrow H^{(i-2n-2)}(B;\mathbb{Q}) \xrightarrow{\delta} H^i(B;\mathbb{Q}) \xrightarrow{p^*} H^i(E;\mathbb{Q}) \xrightarrow{\partial} H^{(i-2n-1)}(B;\mathbb{Q}) \xrightarrow{\delta} H^{(i+1)}(B;\mathbb{Q}) \longrightarrow \cdots, \tag{30}$$

where $\delta(b) = b \smile e$ for some $e \in H^{(2n+2)}(B;\mathbb{Q})$. Since $H^i(B;\mathbb{Q}) = 0$ for $i < 0$, then the long exact sequence (30) restricts to isomorphisms:

$$0 \longrightarrow H^i(B;\mathbb{Q}) \xrightarrow{p^*} H^i(E;\mathbb{Q}) \longrightarrow 0, \quad \text{for } i \leq 2n. \tag{31}$$

For $i \geq 2n + 1$, sequence (30) will be rewritten as follows:

$$0 \longrightarrow H^{(2n+1)}(B;\mathbb{Q}) \xrightarrow{p^*} H^{(2n+1)}(E;\mathbb{Q}) \xrightarrow{\partial} H^0(B;\mathbb{Q}) \xrightarrow{\delta} H^{(2n+2)}(B;\mathbb{Q}) \longrightarrow \cdots. \tag{32}$$

Our hypothesis implies that p^* is injective. So, the sequence (32) breaks up into split short exact sequences:

$$0 \longrightarrow H^i(B;\mathbb{Q}) \xrightarrow{p^*} H^i(E;\mathbb{Q}) \xrightarrow{\partial} H^{(i-2n-1)}(B;\mathbb{Q}) \longrightarrow 0, \quad \text{for } i \geq 2n + 1. \tag{33}$$

From (31) and (33), we obtain

$$H^i(E;\mathbb{Q}) \cong H^i(B;\mathbb{Q}) \oplus H^{(i-2n-1)}(B;\mathbb{Q}). \tag{34}$$

On the contrary, we clearly have

$$\oplus_{(j+k)=i} H^j\left(\mathbb{S}^{(2n+1)};\mathbb{Q}\right) \otimes H^k(B;\mathbb{Q}) \cong H^i(B;\mathbb{Q}) \oplus H^{(i-2n-1)}(B;\mathbb{Q}), \tag{35}$$

and then by the Künneth Formula, we deduce that

$$H^*(E;\mathbb{Q}) \cong H^*\left(\mathbb{S}^{2n+1};\mathbb{Q}\right) \otimes H^*(B;\mathbb{Q}). \tag{36}$$

□

Remark 1. This result is also true if $e = 0$.

Proof of Theorem 3. We have proved in Proposition 3 that

$$H^*(E;\mathbb{Q}) \cong H^*\left(\mathbb{S}^{(2n+1)};\mathbb{Q}\right) \otimes H^*(B;\mathbb{Q}). \tag{37}$$

So, by taking their dimension, we obtain

$$\begin{aligned} \dim H^*(E;\mathbb{Q}) &\geq \dim H^*\left(\mathbb{S}^{(2n+1)};\mathbb{Q}\right) \cdot \dim H^*(B;\mathbb{Q}) \\ &\geq 2\dim \pi_*(B)_{\mathbb{Q}}\ (B \text{ satisfies the conjecture (H)}) \\ &\geq \dim \pi_*(E)_{\mathbb{Q}}, \end{aligned} \tag{38}$$

as required. □

Proposition 4. *Let $F \longrightarrow E \longrightarrow B$ be a TNCZ fibration in which F and B satisfy the conjecture (H), and then E will too.*

Proof. By assumption, we have $H^*(E;\mathbb{Q}) \cong H^*(F;\mathbb{Q}) \otimes H^*(B;\mathbb{Q})$. Next, we argue exactly as in the Proof of Theorem 3 to show that $\dim H^*(E;\mathbb{Q}) \geq \dim \pi_*(E)_{\mathbb{Q}}$.

A much stronger consequence follows if we restrict the fibre. □

Corollary 1. *Let $F \longrightarrow E \longrightarrow B$ be a fibration, in which F is an F_0-space with rank $\pi_{odd}(F)_{\mathbb{Q}} \leq 3$. Then, E satisfies the conjecture (H) once B satisfies it.*

Proof. It is an immediate consequence from [14]. □

Proposition 5. *Let $F \longrightarrow E \longrightarrow B$ be a fibration such that $\pi_{even}(F)_{\mathbb{Q}} = \pi_{even}(B)_{\mathbb{Q}} = 0$, and then E satisfies the conjecture (H).*

The proof of this proposition is omitted. It can be proved using the result of the paper [9].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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