Research Article

A Linear Map Acts as a Leonard Pair with Each of the Generators of $U(sl_2)$

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Let $\mathcal{F}$ denote an algebraically closed field with a characteristic not two. Fix an integer $d \geq 3$; let x , y , and z be the equitable basis of sl_2 over $\mathcal{F}$. Let V denote an irreducible sl_2 -module with dimension $d + 1$; let $A \in \text{End}(V)$. In this paper, we show that if each of the pairs A, x , A, y , and A, z acts on V as a Leonard pair, then these pairs are of Krawtchouk type. Moreover, A is a linear combination of 1 , x , y , and z .

1. Introduction

In this section, we recall some facts concerning Leonard pairs. Leonard pairs were introduced by P. Terwilliger [1] to extend the algebraic approach of Bannai and Ito [2] to a result of D. Leonard concerning the sequences of orthogonal polynomials with finite support, for which the dual sequence of polynomials is also a sequence of orthogonal polynomials [3, 4]. These polynomials arise in connection with the finite-dimensional representations of certain Lie algebras and quantum groups, so one expects Leonard pairs to arise as well. Leonard pairs of Krawtchouk type have been constructed from finite-dimensional irreducible sl_2 -modules [5, 6]. Fix an integer $d \geq 1$. Throughout this paper $\mathcal{F}$ shall denote an algebraically closed field with characteristic not two. Also, V shall denote an $\mathcal{F}$ -vector space of dimension $d + 1$, and $\text{Mat}_{d+1}(\mathcal{F})$ shall denote the $\mathcal{F}$ -algebra of $(d + 1) \times (d + 1)$ matrices with entries in $\mathcal{F}$ having rows and columns indexed by $1, 2, \dots, d + 1$. A square matrix is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. A tridiagonal matrix is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero. A square matrix is said to be *upper* (resp. *lower*) *bidagonal* whenever every nonzero entry appears on or immediately above (resp. below) the main diagonal.

Definition 1. Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. By a Leonard pair on V , we mean an ordered pair A, A^* , where $A: V \rightarrow V$ and $A^*: V \rightarrow V$ are linear transformations that satisfy both (i) and (ii) below:

- (i) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal
- (ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal

We remark that if A, A^* is a Leonard pair on V , then A^*, A is a Leonard pair on V , and for any scalars α, α^* , the pair $A + \alpha I, A^* + \alpha^* I$ is also a Leonard pair on V .

For more details about Leonard pairs, see [7–14].

Definition 2. Let V denote a vector space over $\mathcal{F}$ with dimension $d + 1$. Let $\{v_i\}_{i=0}^d$ denote a basis for V , which satisfies condition (ii) of Definition 1. For $0 \leq i \leq d$, the vector v_i is an eigenvector of A ; let θ_i denote the corresponding eigenvalues. Let $\{v_i^*\}_{i=0}^d$ denote a basis for V , which satisfies condition (i) of Definition 1. For $0 \leq i \leq d$, the vector v_i^* is an eigenvector of A^* ; let θ_i^* denote the corresponding eigenvalues. Let the sequence $\{a_i\}_{i=0}^d$ denote the diagonal of the matrix, which represents A with respect to $\{v_i^*\}_{i=0}^d$. Let the

sequence $\{a_i^*\}_{i=0}^d$ denote the diagonal of the matrix, which represents A^* with respect to $\{v_i\}_{i=0}^d$.

The ordering of $\{\theta_i\}_{i=0}^d$ in Definition 2 is said to be standard. For a standard ordering $\{\theta_i\}_{i=0}^d$ of eigenvalues of, the ordering $\{\theta_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result applies for $\{\theta_i^*\}_{i=0}^d$.

Theorem 1 (see [15]). *Let A, A^* be a Leonard pair on V ; let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be standard ordering of the eigenvalues of A (resp. A^*). Then, there exists a basis of V such that the matrices representing A, A^* with respect to this basis are, respectively,*

$$A = \begin{pmatrix} \theta_0 & & & & \\ & 1 & \theta_1 & & \\ & & 1 & \theta_2 & \\ & & & \ddots & \ddots \\ & & & & 1 & \theta_d \end{pmatrix}, \quad (1)$$

$$A^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & & \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ & & & & \theta_d^* \end{pmatrix},$$

for some sequence of scalars $\varphi_1, \varphi_2, \dots, \varphi_d$ in $\mathcal{F}$, which we refer to as the first split sequence of A, A^* .

Definition 3 (see [16]). Let d denote a nonnegative integer. By a parameter array over $\mathcal{F}$ of diameter d , we mean a sequence of scalars $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ taken from $\mathcal{F}$ that satisfy the following conditions:

$$\theta_i \neq \theta_j, \quad (0 \leq i < j \leq d), \quad (2)$$

$$\theta_i^* \neq \theta_j^*, \quad (0 \leq i < j \leq d), \quad (3)$$

$$\varphi_i \neq 0, \quad (1 \leq i \leq d), \quad (4)$$

$$\phi_i \neq 0, \quad (1 \leq i \leq d), \quad (5)$$

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d), \quad (1 \leq i \leq d), \quad (6)$$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0), \quad (1 \leq i \leq d), \quad (7)$$

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{j-2}^* - \theta_{j+1}^*}{\theta_{j-1}^* - \theta_j^*}, \quad (2 \leq i, j \leq d-1). \quad (8)$$

Theorem 2 (see [15]). *Let d denote a nonnegative integer; let B and B^* denote matrices in $\text{Mat}_{d+1}(\mathcal{F})$. Assume B is lower bidiagonal and B^* is upper bidiagonal. Then, the following are equivalent:*

- (i) *The pair B, B^* is a Leonard pair in $\text{Mat}_{d+1}(\mathcal{F})$.*
- (ii) *There exists a parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ over $\mathcal{F}$ such that*

$$B(i, i) = \theta_i, \quad B^*(i, i) = \theta_i^*, \quad (0 \leq i \leq d),$$

$$B(j, j-1)B^*(j-1, j) = \varphi_j, \quad (1 \leq j \leq d). \quad (9)$$

Suppose (i) and (ii) hold. Then, the parameter array in (ii) is uniquely determined by B, B^* .

Theorem 3 (see [15]). *With reference to Definition 2, let A, A^* be a Leonard pair on V ; let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be standard ordering of the eigenvalues of A (resp. A^*). Via a standard basis, the matrix representing A and A^* are*

$$\begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ 0 & c_2 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots & \ddots \\ & & 0 & c_{d-1} & a_{d-1} & b_{d-1} \\ 0 & & & 0 & c_d & a_d \end{pmatrix} \quad (10)$$

and $\text{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$, where

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*}, \quad (0 \leq i \leq d),$$

$$a_i^* = \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}}, \quad (0 \leq i \leq d),$$

$$b_i = \varphi_{i+1} \frac{\prod_{h=0}^{i-1} (\theta_i^* - \theta_h^*)}{\prod_{h=0}^i (\theta_{i+1}^* - \theta_h^*)}, \quad (0 \leq i \leq d-1),$$

$$c_i = \phi_i \frac{\prod_{h=0}^{d-i-1} (\theta_i^* - \theta_{d-h}^*)}{\prod_{h=0}^{d-i} (\theta_{i-1}^* - \theta_{d-h}^*)}, \quad (1 \leq i \leq d),$$

where $\varphi_0 = 0$ and $\varphi_{d+1} = 0$.

Theorem 4 (see [17]). *Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. Let A, A^* denote a Leonard pair on V . Then, there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta$, and η^* taken from $\mathcal{F}$ such that*

$$A^2 A^* - \beta A A^* A + A^* A^2 - \gamma (A A^* + A^* A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I, \quad (12)$$

$$A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^* (A^* A + A A^*) - \varrho^* A = \gamma A^{*2} + \omega A^* + \eta^* I. \quad (13)$$

The sequence is uniquely determined by the pair A, A^* provided the dimension of V is at least 4.

Relations (12) and (13) are called the Askey–Wilson relations.

2. Leonard Pairs of Classical Type

In [18], the parameters arrays are classified into 13 families, each named for certain associated sequences of orthogonal polynomials. The four families, which arise in this paper, share certain property. Given a parameter array, let β be the common value of (8) minus one.

Definition 4. A parameter array is of classical type whenever $\beta = 2$.

Theorem 5 (see [18]). *A parameter array is of classical type if and only if it is of Racah, Hahn, dual Hahn, or Krawtchouk type.*

Theorem 6 (see [18], Example 5.10). *Fix nonzero $h, h^* \in \mathcal{F}$ and $s, s^*, r_1, r_2, \theta_0, \theta_0^* \in \mathcal{F}$ such that $r_1 + r_2 = s + s^* + d + 1$ and none of $r_1, r_2, s^* - r_1, s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$ and that neither of s, s^* is equal to $-i$ for $2 \leq i \leq 2d$. Let*

$$\begin{aligned}\theta_i &= \theta_0 + hi(i+1+s), & (0 \leq i \leq d), \\ \theta_i^* &= \theta_0^* + h^*i(i+1+s^*), & (0 \leq i \leq d), \\ \varphi_j &= hh^*j(j-d-1)(j+r_1)(j+r_2), & (1 \leq j \leq d), \\ \phi_j &= hh^*j(j-d-1)(j+s^*-r_1)(j+s^*-r_2), & (1 \leq j \leq d).\end{aligned}\tag{14}$$

Then, $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array of Racah type. We refer to the scalars $r_1, r_2, s, s^*, h, h^*, \theta_0$, and θ_0^* as hypergeometric parameters of Φ .

Theorem 7 (see [18], Example 5.11). *Fix nonzero $s, h^* \in \mathcal{F}$ and some $s^*, r, \theta_0, \theta_0^* \in \mathcal{F}$ such that neither of $r, s^* - r$ is equal to $-i$ for $1 \leq i \leq d$ and that s^* is not equal $-i$ for $2 \leq i \leq 2d$. Let*

$$\begin{aligned}\theta_i &= \theta_0 + si, & (0 \leq i \leq d), \\ \theta_i^* &= \theta_0^* + h^*i(i+1+s^*), & (0 \leq i \leq d), \\ \varphi_j &= h^*sj(j-d-1)(j+r), & (1 \leq j \leq d), \\ \phi_j &= -h^*sj(j-d-1)(j+s^*-r), & (1 \leq j \leq d).\end{aligned}\tag{15}$$

Then, $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array of Hahn type. We refer to the scalars r, s, s^*, h^*, θ_0 , and θ_0^* as hypergeometric parameters of Φ .

Theorem 8 (see [18], Example 5.12). *Fix nonzero $h, s^* \in \mathcal{F}$ and $s, r, \theta_0, \theta_0^* \in \mathcal{F}$ such that neither of $r, s - r$ is equal to $-i$ for $1 \leq i \leq d$, and that s is not equal $-i$ for $2 \leq i \leq 2d$. Let*

$$\begin{aligned}\theta_i &= \theta_0 + hi(i+1+s), & (0 \leq i \leq d), \\ \theta_i^* &= \theta_0^* + s^*i, & (0 \leq i \leq d), \\ \varphi_j &= hs^*j(j-d-1)(j+r), & (1 \leq j \leq d), \\ \phi_j &= hs^*j(j-d-1)(j+r-s-d-1), & (1 \leq j \leq d).\end{aligned}\tag{16}$$

Then, $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array of dual Hahn type. We refer to the scalars r, s, s^*, h, θ_0 , and θ_0^* as hypergeometric parameters of Φ .

Theorem 9 (see [18], Example 5.13). *Fix nonzero $r, s, s^* \in \mathcal{F}$, and some $\theta_0, \theta_0^* \in \mathcal{F}$ such that $r \neq ss^*$:*

$$\begin{aligned}\theta_i &= \theta_0 + si, & (0 \leq i \leq d), \\ \theta_i^* &= \theta_0^* + s^*i, & (0 \leq i \leq d), \\ \varphi_j &= rj(j-d-1), & (1 \leq j \leq d), \\ \phi_j &= (r-ss^*)j(j-d-1), & (1 \leq j \leq d).\end{aligned}\tag{17}$$

Then, $\Phi = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ is a parameter array of Krawtchouk type. We refer to the scalars r, s, s^*, θ_0 , and θ_0^* as hypergeometric parameters of Φ .

3. The Lie Algebra sl_2

In this section, we recall some facts concerning the Lie algebra sl_2 .

Definition 5 (see [11]). The Lie algebra sl_2 is the $\mathcal{F}$ -algebra that has a basis e, f, h satisfying the following conditions:

$$\begin{aligned}[h, e] &= 2e, \\ [h, f] &= -2f, \\ [e, f] &= h,\end{aligned}\tag{18}$$

where $[-, -]$ denotes the Lie bracket.

Lemma 1 (see [10]). *With reference to Definition 5, let*

$$\begin{aligned}x &= 2e - h, \\ y &= -2f - h, \\ z &= h.\end{aligned}\tag{19}$$

Then, x, y, z is a basis for sl_2 and

$$\begin{aligned}[x, y] &= 2x + 2y, \\ [y, z] &= 2y + 2z, \\ [z, x] &= 2z + 2x.\end{aligned}\tag{20}$$

We call x, y, z the equitable basis for the Lie algebra sl_2 .

Lemma 2 (see [11]). *For each nonnegative integer d , there is an irreducible finite-dimensional sl_2 -module V_d with basis $v_0, v_1, \dots, v_d$ and action $hv_i = (d-2i)v_i$ ($0 \leq i \leq d$), $fv_i = (i+1)v_{i+1}$ ($0 \leq i \leq d-1$), $fv_d = 0$, $ev_0 = 0$, and $ev_i = (d-i+1)v_{i-1}$ ($1 \leq i \leq d$). Moreover, up to*

isomorphism, V_d is the unique irreducible sl_2 -module of dimension $d + 1$.

Lemma 3 (see [10]). With reference to Lemmas 1 and 2,

$$\begin{aligned}(x + dI)v_0 &= 0, \\ (x + (d - 2i)I)v_i &= 2(d - i + 1)v_{i-1}, \quad (1 \leq i \leq d), \\ (y + (d - 2i)I)v_i &= -2(i + 1)v_{i+1}, \quad (0 \leq i \leq d - 1), \\ (y - dI)v_d &= 0, \\ zv_i &= (d - 2i)v_i, \quad (0 \leq i \leq d).\end{aligned}\quad (21)$$

Definition 6. With reference to Lemmas 1 and 2, fix $\zeta \in \mathbb{C}$. A basis $b = \{b_0, b_1, \dots, b_d\}$ of V_d is said to be a standard $\zeta(z)$ -eigenbasis whenever $\zeta(x)$, $\zeta(y)$, and $\zeta(z)$ act as

$$\begin{aligned}(\zeta(x) + dI)b_0 &= 0, \\ (\zeta(x) + (d - 2i)I)b_i &= 2(d - i + 1)b_{i-1}, \quad (1 \leq i \leq d), \\ (\zeta(y) + (d - 2i)I)b_i &= -2(i + 1)b_{i+1}, \quad (0 \leq i \leq d - 1), \\ (\zeta(y) - dI)b_d &= 0, \\ (\zeta(z) - (d - 2i)I)b_i &= 0, \quad (0 \leq i \leq d).\end{aligned}\quad (22)$$

Lemma 4 (see [19]). With reference to Lemmas 1 and 2,

- (i) For $0 \leq i \leq d$, let $u_i = \sum_{j=0}^{d-i} (-1)^i ((d - j)!/i! (d - j - i)!) v_j$. Then, $u = \{u_0, u_1, \dots, u_d\}$ is a standard x -eigenbasis of V_d .
- (ii) For $0 \leq i \leq d$, let $w_i = \sum_{j=d-i}^d (-1)^i (j!/(d - i)! (j + i - d)!) v_j$. Then, $w = \{w_0, w_1, \dots, w_d\}$ is a standard y -eigenbasis of V_d .
- (iii) Let $v_0, v_1, \dots, v_d$ be as in Lemma 2. Then, $v = \{v_0, v_1, \dots, v_d\}$ is a standard z -eigenbasis of V_d [10].

The main results of this paper are the following theorems.

Theorem 10. Fix an integer $d \geq 3$; let x , y , and z be the equitable basis of sl_2 over $\mathcal{F}$. Let V denote an irreducible sl_2 -module with dimension $d + 1$; let $A \in \text{End}(V)$. Then, the following are equivalent:

- (i) Any two of the pairs A, x , A, y , and A, z are Leonard pairs
- (ii) All the pairs A, x , A, y , and A, z are Leonard pairs

Theorem 11. Fix an integer $d \geq 3$; let x , y , and z be the equitable basis of sl_2 over $\mathcal{F}$. Let V denote an irreducible sl_2 -module with dimension $d + 1$; let $A \in \text{End}(V)$ such that each of the pairs A, x , A, y , and A, z is a Leonard pair; then, all the pairs are of Krawtchouk type. Moreover, A is a linear combination of 1 , x , y , and z .

We remark here that the authors in [12] proved similar result with the generators of the quantum algebra $U_q(sl_2)$.

4. The Type of Leonard Pair

We start recalling some facts which help us to determine the type of the Leonard pairs.

Lemma 5 (see [17]). Let d denote a nonnegative integer, and let V denote a vector space over $\mathcal{F}$ with dimension $d + 1$. Let A, A^* denote a Leonard pair on V with fundamental parameter β . Let the scalars θ_i and θ_i^* be as in Definition 2; then, there exist scalars γ , γ^* , ϱ , and ϱ^* such that

- (i) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}$, $1 \leq i \leq d - 1$
- (ii) $\gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^*$, $1 \leq i \leq d - 1$
- (iii) $\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i)$, $1 \leq i \leq d$
- (iv) $\varrho^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*)$, $1 \leq i \leq d$

Theorem 12 (see [17]). Let d denote a nonnegative integer, and let V denote a vector space over $\mathcal{F}$ with dimension $d + 1$. Let A, A^* denote a Leonard pair on V . Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*, \omega, \eta$, and η^* denote a sequence of scalars taken from $\mathcal{F}$, which satisfy (12) and (13). Let the scalars $\theta_i, \theta_i^*, a_i$, and a_i^* be as in Definition 2. Then, the following hold:

- (i) $\omega = a_i^*(\theta_i - \theta_{i+1}) + a_{i-1}^*(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_i + \theta_{i-1})$, $1 \leq i \leq d$
- (ii) $\omega = a_i(\theta_i^* - \theta_{i+1}^*) + a_{i-1}(\theta_{i-1}^* - \theta_{i-2}^*) - \gamma(\theta_i^* + \theta_{i-1}^*)$, $1 \leq i \leq d$
- (iii) $\eta = a_i^*(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^*\theta_i^2 - \omega\theta_i$, $0 \leq i \leq d$
- (iv) $\eta^* = a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - \gamma\theta_i^{*2} - \omega\theta_i^*$, $0 \leq i \leq d$

Lemma 6. With reference to Lemma 1, if each of the pairs A, x , A, y , and A, z is a Leonard pair, then all the pairs are of dual Hahn type or all the pairs of Krawtchouk type.

Proof. Note that, from Definition 4 and Lemma 3, all the pairs are of classical type. Now, the result holds by Theorem 5 and Theorem 6–Theorem 9.

We have two cases to check, the first case, is there A such that all the pairs A, x , A, y , and A, z are Leonard pairs of dual Hahn type?, and the second case, is there A such that all the pairs A, x , A, y , and A, z are Leonard pairs of Krawtchouk type? We start with the dual Hahn case.

For the rest of the paper, fix an integer $d \geq 3$; let V_d denote an irreducible sl_2 -module with dimension $d + 1$; let $A \in \text{End}(V_d)$; Let $[T]_b$ be the matrix that represents the linear map T with respect to the basis b . $\square$

Lemma 7. With reference to Lemma 3, let $\xi \in \{x, y, z\}$, assume that A, ξ is a Leonard pair of dual Hahn type; let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ be the parameter array associated with the pair A, ξ ; then,

$$\begin{aligned}
\theta_i &= \theta_0 + hi(i+1+s), \quad 0 \leq i \leq d, \\
\theta_i^* &= \theta_0^* + h_\xi i, \quad 0 \leq i \leq d, \\
\varphi_i &= hh_\xi i(i-d-1)(i+r_\xi), \quad 1 \leq i \leq d, \\
\phi_i &= hh_\xi i(i-d-1)(i+r_\xi-s-d-1), \quad 1 \leq i \leq d,
\end{aligned} \tag{23}$$

where h and h_ξ are nonzero scalars, none of $r_\xi, s-r_\xi$ equal to $-i$ for $1 \leq i \leq d$, and $s \neq i$ for $2 \leq i \leq 2d$.

Proof. Clear from Theorem 8. $\square$

Lemma 8. Let $A \in \text{End}(V_d)$ such that A, z is a Leonard pair of dual Hahn type; then, the basis $\{e_i\}_{i=0}^d$ for V_d with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing z is diagonal is $\{k_i v_i\}_{i=0}^d$ or $\{k_{d-i} v_{d-i}\}_{i=0}^d$ for some nonzero scalars $k_i \in \mathcal{F}$.

Proof. Clear from Lemma 3 and the paragraph after Definition 2.

Since the case with basis $\{k_{d-i} v_{d-i}\}_{i=0}^d$ can be treated similar to the case with basis $\{k_i v_i\}_{i=0}^d$, we shall prove our results only for the case $\{k_i v_i\}_{i=0}^d$. Let $\{e_i = k_i v_i\}_{i=0}^d$. $\square$

Lemma 9. With reference to Lemmas 2 and 8,

$$\begin{aligned}
(x + dI)e_0 &= 0, \\
(x + (d-2i)I)e_i &= \frac{2(d-i+1)k_i}{k_{i-1}} e_{i-1}, \quad (1 \leq i \leq d), \\
(y + (d-2i)I)e_i &= \frac{-2(i+1)k_i}{k_{i+1}} e_{i+1}, \quad (0 \leq i \leq d-1), \\
(y - dI)e_d &= 0, \\
ze_i &= (d-2i)e_i, \quad (0 \leq i \leq d).
\end{aligned} \tag{24}$$

Lemma 10. With reference to Lemmas 7 and 8, let A, z be a Leonard pair of dual Hahn type; then,

$$\begin{aligned}
Ae_i &= \beta_{i-1}^* e_{i-1} + \alpha_i^* e_i + \lambda_{i+1}^* e_{i+1}, \quad 1 \leq i \leq d-1, \\
Ae_0 &= \alpha_0^* e_0 + \lambda_1^* e_1, \\
Ae_d &= \beta_{d-1}^* e_{d-1} + \alpha_d^* e_d,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\beta_i^* &= h(d-i)(i+1+r_z), \quad 0 \leq i \leq d-1, \\
\alpha_i^* &= \theta_0 + h(is + (d-2i)r_z + 2i(d-i) + d), \quad 0 \leq i \leq d, \\
\lambda_i^* &= hi(i-1-d+r_z+s), \quad 1 \leq i \leq d.
\end{aligned} \tag{26}$$

Proof. The result is held by Theorem 3 and Lemmas 7 and 9. $\square$

Lemma 11. With reference to Lemma 5 and Theorem 12, let $\xi \in \{x, y, z\}$; assume A, ξ is a Leonard pair of dual Hahn type; let $\theta_i = 0$ and $\theta_i^* = 0$; then,

$$\begin{aligned}
\gamma &= 2h, \\
\gamma^* &= 0, \\
\varrho &= sh^2(2+s), \\
\varrho^* &= h_\xi^2, \\
\omega &= hh_\xi(2r_\xi - s - 2d), \\
\eta &= -dh^2 h_\xi s(1+r_\xi), \\
\eta^* &= -dh h_\xi^2(1+r_\xi).
\end{aligned} \tag{27}$$

Proof. Clear from Lemma 7. $\square$

Lemma 12. Let $A \in \text{End}(V_d)$; assume A, z is a Leonard pair of dual Hahn type; then, the pair A, y is not a Leonard pair of dual Hahn type.

Proof. Let

$$\begin{aligned}
H &= A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) \\
&\quad - \varrho^*A - \gamma A^{*2} + \omega A^* + \eta^*I.
\end{aligned} \tag{28}$$

Since the pair A, z is a Leonard pair, the action of A on the basis $e = \{e_i\}_{i=0}^d$ is given in Lemma 10, and the action of y is given in Lemma 9. Assume that the pair A, y is a Leonard pair and let $y = A^*$; hence, by Theorem 4, there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \rho^*, \omega, \eta$, and η^* as in Lemma 11 such that $[H]_e = 0$. Now, if we solve $[H]_e[1, 1]$ for r_y , we find that $r_y = k_1 r_z - k_0(r_z + 1)/k_1$. Substitute r_y ; then, for $1 \leq i \leq d-2$, solve the $(i+1, i)$ -entries of $[H]_e$ for k_i ; we find that $k_i = k_{i+1}(r_z + i + 2)/k_{i+2}(r_z + i + 1)$. Substitute k_i to find that

$$[H]_e[4, 1] = \frac{48k_{d-1}^2 h(r_z + d)^2}{k_d(r_z + 1)(r_z + 2)(r_z + 3)}. \tag{29}$$

By Lemmas 7 and 8, $h \neq 0$, $r_z + d \neq 0$, and $k_{d-1} \neq 0$, so $[H]_e[4, 1] \neq 0$. Hence, the pair A, y is not a Leonard pair. $\square$

Lemma 13. Let A, z be a Leonard pair of dual Hahn type; then, the pair A, x is not a Leonard pair of dual Hahn type.

Proof. Similar to proof of Lemma 12. $\square$

5. Leonard Pair of Krawtchouk Type

In this section, we describe a linear map A such that all the pairs A, x, A, y , and A, z are Leonard pairs of Krawchouch type.

Lemma 14. With reference to Lemma 3, let $\xi \in \{x, y, z\}$, assume that A, ξ is a Leonard pair of Krawtchouk type; let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ be the parameter array

associated with the pair A, ξ ; then, there exist a nonzero scalars h, h_ξ , and r_ξ such that $r_\xi - hh_\xi \neq 0$, and

$$\begin{aligned}\theta_i &= \theta_0 + hi, & 0 \leq i \leq d, \\ \theta_i^* &= \theta_0^* + h_\xi i, & 0 \leq i \leq d, \\ \varphi_i &= r_\xi i(i-d-1), & 1 \leq i \leq d, \\ \phi_i &= (r_\xi - hh_\xi)i(i-d-1), & 1 \leq i \leq d,\end{aligned}\quad (30)$$

Proof. Clear from Theorem 9. $\square$

Lemma 15. With reference to Lemma 5 and Theorem 12, assume A, ξ is a Leonard pair of Krawtchouk type; let $\theta_i = 0$ and $\theta_i^* = 0$; then,

$$\begin{aligned}\gamma &= 0, \\ \gamma^* &= 0, \\ \varrho &= h^2, \\ \varrho^* &= h_\xi, \\ \omega &= -(hh_\xi - 2r_\xi), \\ \eta &= -dh r_\xi, \\ \eta^* &= -dh_\xi r_\xi.\end{aligned}\quad (31)$$

Proof. Clear from Lemma 14. $\square$

Lemma 16. With reference to Lemmas 14 and 8, Let A, z be a Leonard pair of Krawtchouk type; then, $Ae_i = \beta_{i-1}e_{i-1} + \alpha_i e_i + \lambda_{i+1}e_{i+1}$ $1 \leq i \leq d-1$, $Ae_0 = \alpha_0 e_0 + \lambda_1 e_1$, and $Ae_d = \beta_{d-1}e_{d-1} + \alpha_d e_d$, where

$$\begin{aligned}\beta_i &= \frac{(d-i)r_z}{2}, & 0 \leq i \leq d-1, \\ \alpha_i &= \theta_0 + ih - \frac{(d-2i)r_z}{2}, & 0 \leq i \leq d, \\ \lambda_i &= \frac{-i(r_z + 2h)}{2}, & 1 \leq i \leq d.\end{aligned}\quad (32)$$

Proof. The result holds by Theorem 3 and Lemmas 3 and 14. $\square$

Lemma 17. With reference to Lemma 16, the pair $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type in $\text{Mat}_{d+1}(\mathcal{F})$ if and only if $h \neq 0$, $r_z \neq 0$, and $r_z + 2h \neq 0$.

Proof. Let P be $(d+1) \times (d+1)$ matrix indexed $1, 2, \dots, d+1$ such that the (i, j) -entry is

$$P_{ij} = \begin{cases} \frac{(-2)^{j-1}(i-1)!}{(i-j)!}, & j \leq i, \\ 0, & j > i. \end{cases}\quad (33)$$

Then, the matrices represent $P^{-1}[A]_e P$ and $P^{-1}[z]_e P$ as in (1), where $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$ and $\{\phi_i\}_{i=1}^d$ are as in Lemma 14. Hence, by Theorem 2, $P^{-1}[A]_e P$, $P^{-1}[z]_e P$ is a Leonard pair of Krawtchouk type if and only if $h \neq 0$, $r_z \neq 0$, and $r_z + 2h \neq 0$ hold, which implies that the pair $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type if and only if $h \neq 0$, $r_z \neq 0$, and $r_z + 2h \neq 0$. $\square$

Lemma 18. With reference to Lemmas 8 and 14, assume $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type. If the pair $[A]_e, [y]_e$ is a Leonard pair, then there exists a nonzero scalar $t \in \mathcal{F}$ such that $k_i/k_{i-1} = t$ ($1 \leq i \leq d$). Moreover, $r_y = r_z(1-t)/t$.

Proof. Routine calculations using Theorem 4 and Lemma 15. $\square$

Lemma 19. With reference to Lemmas 8 and 14, assume $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type. If the pair $[A]_e, [x]_e$ is a Leonard pair, then there exists a nonzero scalar $t \in \mathcal{F}$ such that $k_i/k_{i-1} = t$ ($1 \leq i \leq d$). Moreover, $r_x = t(r_z + 2h) - r_z$.

Proof. Routine calculations using Theorem 4 and Lemma 15. $\square$

Lemma 20. Suppose that the pairs A, z and A, y are Leonard pairs of Krawtchouk type; let $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$ and $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ be the parameter array associated with the pair A, y . Then, there exist a nonzero scalars h, r_z , and t such that $t \neq 1$, $r_z + 2h \neq 0$, $r_z - t(r_z + 2h) \neq 0$, and

$$\begin{aligned}\theta_i &= \theta_0 + hi, & 0 \leq i \leq d, \\ \theta_i^* &= 2i - d, & 0 \leq i \leq d, \\ \varphi_i &= \frac{r_z(1-t)i(i-d-1)}{t}, & 1 \leq i \leq d, \\ \phi_i &= \frac{(r_z - t(r_z + 2h))i(i-d-1)}{t}, & 1 \leq i \leq d.\end{aligned}\quad (34)$$

Proof. Let $k_0 = 1$ in Lemma 8, so the basis $e = \{t^i v_i\}_{i=0}^d$. Hence, the result holds by Lemmas 14 and 18. $\square$

Lemma 21. Suppose that the pairs A, z and A, x are Leonard pairs of Krawtchouk type, let $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$ and $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ be the parameter array associated with the pair A, x . Then, there exist nonzero scalars h, r_z , and t such that $t \neq 1$, $r_z + 2h \neq 0$, $r_z - t(r_z + 2h) \neq 0$, and

$$\begin{aligned}\theta_i &= \theta_0 + hi, & 0 \leq i \leq d, \\ \theta_i^* &= 2i - d, & 0 \leq i \leq d, \\ \varphi_i &= (t(r_z + 2h) - r_z)i(i-d-1), & 1 \leq i \leq d, \\ \phi_i &= ((t-1)(r_z + 2h))i(i-d-1), & 1 \leq i \leq d.\end{aligned}\quad (35)$$

Proof. Let $k_0 = 1$ in Lemma 8, so the basis $e = \{t^i v_i\}_{i=0}^d$. Hence, the result holds by Lemmas 14 and 19. $\square$

Lemma 22. *With reference to Lemma 2, let $t \in \mathcal{F}$ be a nonzero scalar; let $e = \{t^i v_i\}_{i=0}^d$. Let A be as in Lemma 16; assume that $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type; then, the pair $[A]_e, [y]_e$ is a Leonard pair of Krawtchouk type if and only if $t \neq 1$ and $r_z - t(r_z + 2h) \neq 0$.*

Proof. The pair $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type; then, by Lemma 17, $h \neq 0$, $r_z \neq 0$, and $r_z + 2h \neq 0$.

Let P be $(d+1) \times (d+1)$ matrix indexed $1, 2, \dots, d+1$ such that the (i, j) -entry is

$$P_{ij} = \begin{cases} \frac{(2)^{j-1} (t-1)^{j-1} (i-1)!}{t^{j-1} (i-j)!}, & j \leq i, \\ 0, & j > i. \end{cases} \quad (36)$$

Then, the matrices represent $P^{-1}[A]_e P, P^{-1}[y]_e P$ as in (1), where $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ and $\{\phi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d$ are as in Lemma 20. Hence, by Theorem 2, $P^{-1}[A]_e P, P^{-1}[y]_e P$ is a Leonard pair of Krawtchouk type, if and only if $t \neq 1$ and $r_z - t(r_z + 2h) \neq 0$ hold, which implies that the pair $[A]_e, [y]_e$ is a Leonard pair of Krawtchouk type if and only if $t \neq 1$ and $r_z - t(r_z + 2h) \neq 0$. $\square$

Lemma 23. *With reference to Lemma 2, let $t \in \mathcal{F}$ be a nonzero scalar; let $e = \{t^i v_i\}_{i=0}^d$. Let A be as in Lemma 16; assume, that $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type; then, the pair $[A]_e, [x]_e$ is a Leonard pair of Krawtchouk type if and only if $t \neq 1$ and $r_z - t(r_z + 2h) \neq 0$.*

Proof. The pair $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type; then, by Lemma 17, $h \neq 0$, $r_z \neq 0$, and $r_z + 2h \neq 0$.

Let P be $(d+1) \times (d+1)$ matrix indexed $1, 2, \dots, d+1$ such that the (i, j) -entry is

$$P_{ij} = \begin{cases} \frac{(2)^{j-d-1} (t-1)^{j-1} (d-i+1)!}{t^d d! (d-i-j+1)!}, & j \leq d-i+2, \\ 0, & j > d-i+2. \end{cases} \quad (37)$$

Then,

$$P^{-1}[x]_e P = \begin{pmatrix} \theta_d & & & & \\ & 1 & \theta_{d-1} & & \\ & & 1 & \theta_{d-2} & \\ & & & \ddots & \ddots \\ & & & & 1 & \theta_0 \end{pmatrix}, \quad (38)$$

$$P^{-1}[A]_e P = \begin{pmatrix} \theta_0^* & \phi_1 & & & \\ & \theta_1^* & \phi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \phi_d \\ & & & & \theta_d^* \end{pmatrix},$$

where $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ and $\{\phi_i\}_{i=1}^d, \{\phi_i\}_{i=1}^d$ are as in Lemma 21. Let $G = T^{-1}KT^*$, where T, K , and T^* are in $\text{Mat}_{d+1}(\mathcal{F})$ such that $T_{i+1,j+1} = \prod_{h=0}^{j-1} (\theta_i^* - \theta_h^*)$, $T_{i+1,j+1}^* = \prod_{h=0}^{j-1} (\theta_{d-i}^* - \theta_{d-h}^*)$ for $0 \leq i, j \leq d$, and $K_{ij} = 1$ if $i+j = d+2$ and 0 otherwise. Then, the matrices represent $GP^{-1}[A]_e PG^{-1}$, $GP^{-1}[x]_e PG^{-1}$ as in (1). The rest of the proof will be similar to proof of Lemma 22.

We remark here that the existence of G that appears in proof of Lemma 23 was proved by Terwilliger in [18]. $\square$

Lemma 24. *With reference to Lemma 2, let $t \in \mathcal{F}$ be a nonzero scalar; let $e = \{t^i v_i\}_{i=0}^d$. Let A be as in Lemma 16; assume that $[A]_e, [z]_e$ is a Leonard pair of Krawtchouk type; then, $[A]_e, [x]_e$ is a Leonard pair of Krawtchouk type if and only if $[A]_e, [y]_e$ is a Leonard pair of Krawtchouk type.*

Proof. Clear from Lemmas 22 and 23. $\square$

Lemma 25. *Fix an integer $d \geq 3$, let $t \in \mathcal{F}$ be a nonzero scalar; let x, y, z denote the equitable basis for the Lie algebra sl_2 . Let V_d denote a finite-dimensional irreducible sl_2 -module, let $\{v_i\}_{i=0}^d$ be a basis of V_d as in Lemma 3, let $e = \{t^i v_i\}_{i=0}^d$, and let $A \in \text{End}(V_d)$ such that A acts on e as in Lemma 16. Then, each of the pairs $[A]_e, [x]_e, [A]_e, [y]_e, [A]_e, [z]_e$ is a Leonard pair of Krawtchouk type if and only if $r_z \neq 0, h \neq 0, t \neq 1, r_z + 2h \neq 0$ and $r_z - t(r_z + 2h) \neq 0$.*

Proof. The result holds from Lemmas 17, 22, and 23. $\square$

6. A Basis of A

Let $U(sl_2)$ denote the universal enveloping algebra of sl_2 . Thus, $U(sl_2)$ is the associative $\mathcal{F}$ -algebra with generators x, y, z and relations $[x, y] = 2x + 2y$, $[y, z] = 2y + 2z$, and $[z, x] = 2z + 2x$, where $[n, m] = nm - mn$ is commutator of n and m . In this section, we write A as a linear combination of the generators of $U(sl_2)$ [16, 20–23].

Definition 7. Let $Y \in U(sl_2)$ denote any linear combination of 1, x, y , and z , and write

$$Y = a_I I + a_x x + a_y y + a_z z, \quad (39)$$

where $a_I, a_x, a_y, a_z \in \mathcal{F}$.

Lemma 26. *With reference to Lemma 3 and Definition 7, let t be a nonzero scalar in $\mathcal{F}$ and let $e_i = t^i v_i$ ($0 \leq i \leq d$); then,*

$$\begin{aligned} Y e_0 &= \kappa_0 e_0 + \eta_1 e_1, \\ Y e_d &= \mu_{d-1} e_{d-1} + \kappa_d e_d, \\ Y e_i &= \mu_{i-1} e_{i-1} + \kappa_i e_i + \eta_{i+1} e_{i+1}, \quad (1 \leq i \leq d-1), \end{aligned} \quad (40)$$

where

$$\begin{aligned}
\kappa_i &= a_I + (d-2i)(a_z - a_x - a_y), \quad (0 \leq i \leq d), \\
\mu_i &= 2(d-i)ta_x, \quad (0 \leq i \leq d-1), \\
\eta_i &= \frac{-2ia_y}{t}, \quad (1 \leq i \leq d).
\end{aligned} \tag{41}$$

Proof. Expand the action of x , y , and z on $e = \{e_i\}_{i=0}^d$ using Lemma 8. $\square$

Lemma 27. Fix an integer $d \geq 3$; let x , y , z denote the generators of $U(\mathfrak{sl}_2)$, let $A \in U(\mathfrak{sl}_2)$ such that $A = a_I I + a_x x + a_y y + a_z z$, let V_d denote a finite-dimensional irreducible $\mathfrak{sl}_2$ -module, and let $e = \{t^i v_i\}_{i=0}^d$ be a basis of V_d , assume that A acts on e as in Lemma 16. Then,

$$\begin{aligned}
a_I &= \theta_0 + \frac{dh}{2}, \\
a_x &= \frac{r_z}{4t}, \\
a_y &= \frac{t(r_z + 2h)}{4}, \\
a_z &= \frac{t(r_z + 2h)}{4} - \frac{r_z + h}{2} + \frac{r_z}{4t}, \\
r_z &= 4ta_x, \\
h &= \frac{2}{t}(a_y - t^2 a_x).
\end{aligned} \tag{42}$$

Proof. Compare the action of A on e with the action of Y in Lemma 26 to get the result. $\square$

Lemma 28. With reference to Lemma 27, $a_z = t(1-t)(ta_x - a_y)$.

Proof. Routine calculations show that

$$t(1-t)(ta_x - a_y) = \frac{t(r_z + 2h)}{4} - \frac{r_z + h}{2} + \frac{r_z}{4t}. \tag{43}$$

Lemma 29. Fix an integer $d \geq 3$; let x , y , z denote the generators of $U(\mathfrak{sl}_2)$, let $A \in U(\mathfrak{sl}_2)$ such that $A = a_I I + a_x x + a_y y + a_z z$, where a_I, a_x, a_y , and a_z in $\mathcal{F}$, and let V_d denote a finite-dimensional irreducible $\mathfrak{sl}_2$ -module. Then, each of the pairs A, x , A, y , and A, z is a Leonard pair of Krawtchouk type if and only if a_x, a_y, a_z are nonzero scalars in $\mathcal{F}$ and there exists a nonzero scalar $t \in \mathcal{F}$ such that $a_z = t(1-t)(ta_x - a_y)$ and $a_y - t^2 a_x \neq 0$.

Proof. Assume all the pairs A, x , A, y , and A, z are Leonard pairs of Krawtchouk type; then, by Lemma 25, there exist nonzero scalars $r_z, h, t \in \mathcal{F}, t \neq 1$ such that A acts on the basis $\{t^i v_i\}_{i=0}^d$ as in Lemma 16. Moreover, $r_z + 2h \neq 0$ and $r_z - t(r_z + 2h) \neq 0$. Hence, A acts on $\{t^i v_i\}_{i=0}^d$ as

$a_I I + a_x x + a_y y + a_z z$, where a_I, a_x, a_y , and a_z are as in Lemma 27, and $a_z = t(1-t)(ta_x - a_y)$.

Note that $r_z, r_z + 2h$, and h are nonzero scalars if and only if a_x, a_y , and $a_y - t^2 a_x$ are nonzero scalars, respectively, and since $r_z - t(r_z + 2h) = 4(ta_x - a_y)$, then, by Lemma 28, $r_z - t(r_z + 2h) \neq 0$ if and only if $a_z \neq 0$.

In the another direction, if $A = a_I I + a_x x + a_y y + a_z z$ for $a_I \in \mathcal{F}$, a_x, a_y, a_z are nonzero scalars in $\mathcal{F}$, $a_z = t(1-t)(ta_x - a_y)$ and $a_y - t^2 a_x \neq 0$ for nonzero scalar $t \in \mathcal{F}$, then $a_z \neq 0$ implies $t \neq 1$; hence, we can find $r_z = 4ta_x$, $h = (2/t)(a_y - t^2 a_x)$ nonzero scalars such that A acts on $\{t^i v_i\}_{i=0}^d$ as in Lemma 16. Hence, the pairs A, x , A, y , and A, z are Leonard pairs of Krawtchouk type by Lemma 25. $\square$

Proof of Theorem 10. Note that, from Lemma 4, $[x]_u = [y]_w = [z]_v$. Hence, the result holds from Lemmas 16, 17, and 24. $\square$

Proof of Theorem 11. Clear from Lemmas 12, 25, and 29. $\square$

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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