Research Article

θ - ϕ -Contraction on (α, η) -Complete Rectangular b -Metric Spaces

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In this paper, we present some fixed point results for generalized θ - ϕ -contraction in the framework of (α, η) -complete rectangular b -metric spaces. Further, we establish some fixed point theorems for this type of mappings defined on such spaces. Our results generalize and improve many of the well-known results. Moreover, to support our main results, we give an illustrative example.

1. Introduction

The well-known Banach contraction theory is one of the methods used, which states that if (X, d) is a complete metric space and $T: X \rightarrow X$ is self-mapping with contraction, then T has a unique fixed point [1].

In 2000, Branciari [2] introduced the notion of generalized metric spaces, for example, the triangle inequality is replaced by the inequality $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all pairwise distinct points x, y, u , and $v \in X$. Since then, several results have been proposed by many mathematicians on such spaces (see [3–8]).

The concept of metric space, as an ambient space in fixed point theory, has been generalized in several directions, such as, b -metric spaces [9] and generalized metric spaces.

Combining conditions are used for definitions of b -metric and generalized metric spaces. Roshan et al. [10] announced the notion of rectangular b -metric space.

Hussain et al. [11] introduced the concept of α - η -complete rectangular b -metric space and proved certain results of fixed point theory on such spaces.

In this paper, we provide some fixed point results for generalized θ - ϕ -contraction in the framework of (α, η) -complete rectangular b -metric spaces, and also we give two examples to support our results.

2. Preliminaries

Definition 1 (see [10]). Let X be a nonempty set, $s \geq 1$ be a given real number, and let $d: X \times X \rightarrow [0, +\infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y :

- (1) $d(x, y) = 0$, if only if $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ (b -rectangular inequality)

Then, (X, d) is called a b -rectangular metric space.

Lemma 1 (see [10]). Let (X, d) be a rectangular b -metric space.

- (a) Suppose that sequences (x_n) and (y_n) in X are such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, with $x \neq y$, $x_n \neq x$, and $y_n \neq y$ for all $n \in \mathbb{N}$. Then, we have

$$\frac{1}{s}d(x, y) \leq \lim_{n \rightarrow \infty} \inf d(x_n, y_n) \leq \lim_{n \rightarrow \infty} \sup d(x_n, y_n) \leq sd(x, y).$$

(1)

- (b) If $y \in X$ and (x_n) is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}$, $m \neq n$, converging to $x \neq y$, then

$$\frac{1}{s}d(x, y) \leq \lim_{n \rightarrow \infty} \inf d(x_n, y) \leq \lim_{n \rightarrow \infty} \sup d(x_n, y) \leq sd(x, y), \quad (2)$$

for all $x \in X$.

Zheng et al. [12] introduced a new type of contractions called $\theta - \phi$ -contractions in metric spaces and proved a new fixed point theorems for such mapping.

Definition 2 (see [13]). We denote by Θ the set of functions $\theta:]0, \infty[\rightarrow]1, \infty[$, satisfying the following conditions:

- (θ_1) θ is increasing
- (θ_2) for each sequence $(x_n) \in]0, \infty[$, $\lim_{n \rightarrow \infty} \theta(x_n) = 1 \Leftrightarrow \lim_{n \rightarrow \infty} x_n = 0$
- (θ_3) θ is continuous on $]0, \infty[$

Definition 3 (see [12]). We denote by Φ the set of functions $\phi: [1, \infty[\rightarrow [1, \infty[$ satisfying the following conditions:

- (Φ_1) $\phi: [1, \infty[\rightarrow [1, \infty[$ is nondecreasing
- (Φ_2) for each $t > 1$, $\lim_{n \rightarrow \infty} \phi^n(t) = 1$
- (Φ_3) ϕ is continuous on $[1, \infty[$

Lemma 2 (see [12]). If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for each $t > 1$.

In 2014, Hussain et al. [14] introduced a weaker notion than the concept of completeness and called it α -completeness for metric spaces.

Definition 4 (see [14]). Let $T: X \rightarrow X$ and $\alpha, \eta: X \times X \rightarrow [0, +\infty[$. We say that T is a triangular (α, η) -admissible mapping if

- (T_1) $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$, $x, y \in X$
- (T_2) $\eta(x, y) \leq 1 \Rightarrow \eta(Tx, Ty) \leq 1$, $x, y \in X$
- (T_3) $\begin{cases} \alpha(x, y) \geq 1 \\ \alpha(y, z) \geq 1 \end{cases} \Rightarrow \alpha(x, z) \geq 1$ for all $x, y, z \in X$
- (T_4) $\begin{cases} \eta(x, y) \leq 1 \\ \eta(y, z) \leq 1 \end{cases} \Rightarrow \eta(x, z) \leq 1$ for all $x, y, z \in X$

Definition 5 (see [14]). Let (X, d) be a b -rectangular metric space and let $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ be two mappings. The space is said to be as follows:

- (a) T is α -continuous mapping on (X, d) , if for given point $x \in X$ and sequence (x_n) in X , $x_n \rightarrow x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ imply that $Tx_n \rightarrow Tx$.
- (b) T is η subcontinuous mapping on (X, d) , if for given point $x \in X$ and sequence (x_n) in X , $x_n \rightarrow x$ and $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ imply that $Tx_n \rightarrow Tx$.

- (c) T is (α, η) -continuous mapping on (X, d) , if for given point $x \in X$ and sequence (x_n) in X , $x_n \rightarrow x$ and $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ imply that $Tx_n \rightarrow Tx$.

The following definitions were given by Hussain et al. [11].

Definition 6 (see [11]). Let $d(X, d)$ be a rectangular b -metric space and let $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ be two mappings. The space X is said to be

- (a) α -complete, if every Cauchy sequence (x_n) in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X
- (b) η -subcomplete, if every Cauchy sequence (x_n) in X with $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ converges in X
- (c) (α, η) -complete, if every Cauchy sequence (x_n) in X with $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ converges in X

Definition 7 (see [11]). Let (X, d) be a rectangular b -metric space and let $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ be two mappings. The space X is said to be

- (a) (X, d) is α -regular, if $x_n \rightarrow x$, where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ implies $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$
- (b) (X, d) is η -subregular, if $x_n \rightarrow x$, where $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ implies $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$
- (c) (X, d) is (α, η) -regular, if $x_n \rightarrow x$, where $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ implies that $\alpha(x_n, x) \geq 1$ or $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

3. Main Results

Definition 8. Let $d(X, d)$ be a (α, η) -rectangular b -metric space with parameter $s > 1$ and let T be a self-mapping on X . Suppose that $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ are two functions. We say that T is an $(\alpha, \eta) - \theta - \phi$ -contraction, if for all $x, y \in X$ with $(\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1)$ and $d(Tx, Ty) > 0$, we have

$$\theta[s^2 d(Tx, Ty)] \leq \phi[\theta(\beta_1 d(x, y) + \beta_2 d(Tx, x) + \beta_3 d(Ty, y) + \beta_4 d(y, Tx))], \quad (3)$$

where $\theta \in \Theta$, $\phi \in \Phi$, $\beta_i \geq 0$ for $i \in \{1, 2, 3, 4\}$, $\sum_{i=0}^4 \beta_i \leq 1$, and $\beta_3 < (1/s)$.

Definition 9. Let (X, d) be a (α, η) -complete rectangular b -metric space and $T: X \rightarrow X$ be a mapping.

- (1) T is said to be a $\theta - \phi$ -Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$\theta[s^2 d(Tx, Ty)] \leq \phi\left[\theta\left(\frac{(d(Tx, x) + d(Ty, y))}{2s}\right)\right]. \quad (4)$$

- (2) T is said to be a $\theta - \phi$ -Reich-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$\theta[s^2 d(Tx, Ty)] \leq \phi \left[\theta \left(\frac{d(x, y) + d(Tx, x) + d(Ty, y)}{3s} \right) \right]. \quad (5)$$

- (3) T is said to be a Kannan-type mapping, that is, if there exists $\alpha \in]0, (1/2s)[$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$s^2 d(Tx, Ty) \leq \alpha (d(Tx, x) + d(y, Ty)). \quad (6)$$

- (4) T is said to be a Reich-type mapping, that is, if there exists $\lambda \in]0, (1/3s)[$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$s^2 d(Tx, Ty) \leq \lambda [d(x, y) + d(Tx, x) + d(Ty, y)]. \quad (7)$$

Theorem 1. Let (X, d) be a (α, η) -complete rectangular b-metric and let $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ be two functions. Let $T: X \times X \rightarrow X$ be a self-mapping satisfying the following conditions:

- (i) T is a triangular (α, η) -admissible mapping
- (ii) T is an $(\alpha, \eta) - \theta - \phi$ -contraction
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$
- (iv) T is a (α, η) -continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$.

Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$. Since T is a triangular (α, η) -admissible mapping, then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \implies \alpha(Tx_0, Tx_1) \geq 1 = \alpha(x_1, x_2)$ or $\eta(x_0, x_1) = \eta(x_0, Tx_0) \leq 1 \implies \alpha(Tx_0, Tx_1) \leq 1 = \alpha(x_1, x_2)$.

Continuing this process, we have $\alpha(x_{n-1}, x_n) \geq 1$ or $\eta(x_{n-1}, x_n) \leq 1$, for all $n \in \mathbb{N}$. By (T_3) and (T_4) , one has

$$\alpha(x_m, x_n) \geq 1 \quad \text{or} \quad \eta(x_m, x_n) \leq 1, \quad \forall m, n \in \mathbb{N}, \quad m \neq n. \quad (8)$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$. Then, x_{n_0} is a fixed point of T and the proof is finished. Hence, we assume that $x_n \neq Tx_n$, i.e., $d(x_{n-1}, x_n) > 0$ for all $n \in \mathbb{N}$. We have

$$x_n \neq x_m, \quad \forall m, n \in \mathbb{N}, \quad m \neq n. \quad (9)$$

Indeed, suppose that $x_n = x_m$ for some $m = n + k > n$, so we have

$$x_{n+1} = Tx_n = Tx_m = x_{m+1}. \quad (10)$$

Denote $d_m = d(x_m, x_{m+1})$. Then, (3) and Lemma 2 imply that

$$\begin{aligned} \theta(d_n) &= \theta(d_m) \leq \theta(s^2 d_m) = \theta(s^2 d(Tx_{m-1}, Tx_m)) \\ &\leq \phi(\theta(\beta_1 d_{m-1} + \beta_2 d_{m-1} + \beta_3 d_m)) \\ &< \theta(\beta_1 d_{m-1} + \beta_2 d_{m-1} + \beta_3 d_m). \end{aligned} \quad (11)$$

As θ is increasing, so

$$d_n = d_m < \beta_1 d_{m-1} + \beta_2 d_{m-1} + \beta_3 d_m. \quad (12)$$

Hence,

$$d_m < \frac{\beta_1 + \beta_2}{1 - \beta_3} d_{m-1}. \quad (13)$$

Since

$$\beta_1 + \beta_2 + \beta_3 \leq 1. \quad (14)$$

Thus

$$d_m < d_{m-1}. \quad (15)$$

Continuing this process, we can prove that $d_n = d_m < d_n$, which is a contradiction. Thus, in the following, we can assume that (8) and (9) hold.

We shall prove that

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} d(n_n, x_{n+2}) = 0. \quad (16)$$

Since T is $(\alpha, \eta) - \theta - \phi$ -contraction, we get

$$\begin{aligned} \theta(d_n) &= \theta(d(Tx_{n-1}, Tx_n)) \leq \theta(s^2 d(Tx_{n-1}, Tx_n)) \\ &\leq \phi(\theta(\beta_1 d_{n-1} + \beta_2 d_{n-1} + \beta_3 d_n)) \\ &< \theta(\beta_1 d_{n-1} + \beta_2 d_{n-1} + \beta_3 d_n). \end{aligned} \quad (17)$$

Since θ is increasing, we deduce that $d_n < \beta_1 d_{n-1} + \beta_2 d_{n-1} + \beta_3 d_n$, and thus

$$d_n < \frac{\beta_1 + \beta_2}{1 - \beta_3} d_{n-1}. \quad (18)$$

Since $\beta_1 + \beta_2 / 1 - \beta_3 \leq 1$, then

$$d_n < d_{n-1}. \quad (19)$$

Therefore, $d(x_n, x_{n+1})$ is monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geq 0$, such that

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = \alpha, \quad (20)$$

which again by (3) and (19) and property of (θ) , we have

$$\begin{aligned} 1 &< \theta(d_n) \leq \phi(\theta(\beta_1 d_{n-1} + \beta_2 d_{n-1} + \beta_3 d_n)) \\ &\leq \phi(\theta(d_{n-1})) \leq \phi^2(\theta(d_{n-2})) \leq \dots \leq \phi^n(\theta(d_0)) \\ &= \phi^n(\theta(d(x_0, x_1))). \end{aligned} \quad (21)$$

By taking the limit as $n \rightarrow \infty$ in (21) and using (Φ_2) , we have

$$1 \leq \lim_{n \rightarrow +\infty} \theta(d(x_n, x_{n+1})) \leq \phi^n(\theta(d(x_0, x_1))). \quad (22)$$

Then, $\lim_{n \rightarrow +\infty} \theta(d(x_n, x_{n+1})) = 1$, by Θ_2 , we obtain

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0. \quad (23)$$

On the other hand,

$$\begin{aligned} \theta(s^2 d(x_n, x_{n+2})) &\leq \phi[\theta(\beta_1 d(x_{n-1}, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n))] \\ &\leq \phi[\theta(s\beta_1 d(x_{n-1}, x_{n+2}) + s\beta_1 d(x_{n+2}, x_n) + s\beta_1 d(x_n, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n))] \\ &\leq \phi[\theta(s^2 \beta_1 d(x_{n-1}, x_n) + s^2 \beta_1 d(x_n, x_{n+1}) + s^2 \beta_1 d(x_{n+1}, x_{n+2}) + s\beta_1 d(x_{n+2}, x_n) + s\beta_1 d(x_n, x_{n+1}) \\ &\quad + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n)]. \end{aligned} \quad (24)$$

By θ_1 and Lemma 2, we obtain

$$\begin{aligned} s^2 d(x_n, x_{n+2}) &< s^2 \beta_1 d(x_{n-1}, x_n) + s^2 \beta_1 d(x_n, x_{n+1}) + s^2 \beta_1 d(x_{n+1}, x_{n+2}) + s\beta_1 d(x_{n+2}, x_n) \\ &\quad + s\beta_1 d(x_n, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n). \end{aligned} \quad (25)$$

Therefore,

$$\begin{aligned} (s^2 - s\beta_1) d(x_n, x_{n+2}) &< s^2 \beta_1 d(x_{n-1}, x_n) + s^2 \beta_1 d(x_n, x_{n+1}) + s\beta_1 d(x_{n+1}, x_{n+2}) \\ &\quad + s\beta_1 d(x_n, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n). \end{aligned} \quad (26)$$

Taking the limit as $n \rightarrow \infty$ in (28) and using (23), since $s^2 - s\beta_1 > 0$, we have

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+2}) = 0. \quad (27)$$

Hence, (16) is proved.

Next, we show that $\{x_n\}$ is an Cauchy sequence in X , if otherwise there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of (x_n) such that, for all positive integers k , $n(k) > m(k) > k$,

$$d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \quad (28)$$

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \quad (29)$$

From (30) and using the rectangular inequality, we get

$$\begin{aligned} \varepsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq sd(x_{m(k)}, x_{m(k)+1}) + sd(x_{m(k)+1}, x_{n(k)+1}) \\ &\quad + sd(x_{n(k)+1}, x_{n(k)}). \end{aligned} \quad (30)$$

Taking the upper limit as $k \rightarrow \infty$ in (32) and using (16), we get

$$\frac{\varepsilon}{s} \lim_{n \rightarrow +\infty} \sup d(x_{m(k)+1}, x_{n(k)+1}). \quad (31)$$

Moreover,

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &\leq sd(x_{m(k)}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{n(k)+1}) \\ &\quad + sd(x_{n(k)+1}, x_{n(k)}). \end{aligned} \quad (32)$$

Then, from (23) and (31), we get

$$\lim_{n \rightarrow +\infty} \sup d(x_{m(k)}, x_{n(k)}) \leq s\varepsilon. \quad (33)$$

On the other hand, we have

$$\begin{aligned} d(x_{n(k)}, x_{m(k)+1}) &\leq sd(x_{n(k)}, x_{n(k)-1}) + sd(x_{n(k)-1}, x_{m(k)}) \\ &\quad + sd(x_{x_{m(k)}}, x_{m(k)+1}). \end{aligned} \quad (34)$$

Then, from (23) and 31 we get

$$\lim_{n \rightarrow +\infty} \sup d(x_{n(k)}, x_{m(k)+1}) \leq s\varepsilon. \quad (35)$$

Applying (3) with $x = x_{m(k)}$ and $y = x_{n(k)}$, we have

$$\begin{aligned} \theta(s^2 d(x_{m(k)+1}, x_{n(k)+1})) &= \theta(s^2 d(Tx_{m(k)}, Tx_{n(k)})) \\ &\leq \phi \left(\theta \left(\begin{aligned} &\beta_1 d(x_{m(k)}, x_{n(k)}) + \beta_2 d(x_{m(k)}, Tx_{m(k)}) \\ &+ \beta_3 d(x_{n(k)}, Tx_{n(k)}) + \beta_4 d(x_{n(k)}, Tx_{m(k)}) \end{aligned} \right) \right). \end{aligned} \quad (36)$$

Now taking the upper limit as $k \rightarrow \infty$ in (38) and using (θ_1) , (θ_3) , (ϕ_3) , (23), (33), (35), (37), and Lemma 2, we have

$$\begin{aligned}
\theta\left(s^2 \cdot \frac{\varepsilon}{s}\right) &= \theta(\varepsilon s) \leq \theta\left(s^2 \lim_{k \rightarrow +\infty} \sup d\left(x_{m(k)+1}, x_{n(k)+1}\right)\right) \\
&\leq \phi\left(\theta\left(\beta_1 \lim_{k \rightarrow +\infty} \sup d\left(x_{m(k)}, x_{n(k)}\right)\right.\right. \\
&\quad \left.+\beta_2 \lim_{k \rightarrow +\infty} \sup d\left(x_{m(k)}, Tx_{m(k)}\right)\right. \\
&\quad \left.+\beta_3 \lim_{k \rightarrow +\infty} \sup d\left(x_{n(k)}, Tx_{n(k)}\right)\right. \\
&\quad \left.+\beta_4 \lim_{k \rightarrow +\infty} \sup d\left(x_{n(k)}, Tx_{m(k)}\right)\right)\Bigg) \\
&\leq \phi(\theta(\beta_1 s\varepsilon + \beta_4 s\varepsilon)) < \theta(s\varepsilon(\beta_1 + \beta_4)).
\end{aligned} \tag{37}$$

Therefore, $\varepsilon s < s\varepsilon(\beta_1 + \beta_4)$ implies $s < \beta_1 + \beta_4$, which is a contradiction.

Consequently, $\{x_n\}$ is a Cauchy sequence in $\alpha - \eta$ -complete rectangular b -metric space (X, d) . Since $\alpha(x_{n-1}, x_n) \geq 1$ or $\eta(x_{n-1}, x_n) \leq 1$, for all $n \in \mathbb{N}$.

This implies that the sequence $\{x_n\}$ converges to some $z \in X$. Suppose that $z \neq Tz$. Then, we have all the assumption of Lemma 1 and T is (α, η) -continuous, then $Tx_n \rightarrow Tz$ as $n \rightarrow \infty$. Therefore,

$$\frac{1}{s}d(z, Tz) \leq \lim_{n \rightarrow +\infty} \sup d(x_n, Tx_n) = 0. \tag{38}$$

Hence, we have $d(z, Tz) = 0$ and so $Tz = z$. Thus, z is a fixed point of T . $\square$

3.1. Uniqueness. Let $z, u \in \text{Fix}(T)$ where $z \neq u$ and $\alpha(z, u) \geq 1$ or $\eta(z, u) \leq 1$.

Applying (3) with $x = z$ and $y = u$, we have

$$\begin{aligned}
\theta(d(z, u)) &= \theta(d(Tz, Tu)) \leq \theta(s^2 d(Tz, Tu)) \\
&\leq \phi(\theta(\beta_1 d(z, u) + \beta_2 d(z, Tz) + \beta_3 d(u, Tu) \\
&\quad + \beta_4 d(Tz, u))) \\
&\leq \phi(\theta(\beta_1 d(z, u) + \beta_4 d(Tz, u))) \leq \phi(\theta(d(z, u))).
\end{aligned} \tag{39}$$

Since θ is increasing, therefore

$$d(z, u) < d(z, u), \tag{40}$$

which is a contradiction. Hence, $z = u$ and T have a unique fixed point.

Recall that a self-mapping T is said to have the property P , if $\text{Fix}(T) = \text{Fix}(T^n)$ for every $n \in \mathbb{N}$.

Theorem 2. Let $\alpha, \eta: X \times X \rightarrow \mathbb{R}^+$ be two functions and let (X, d) be an (α, η) -complete rectangular b -metric space. Let $T: X \rightarrow X$ be a mapping satisfying the following conditions:

- (i) T is a triangular (α, η) -admissible mapping
- (ii) T is an $(\alpha, \eta) - \theta - \phi$ -contraction
- (iii) $\alpha(z, Tz) \geq 1$ or $\eta(z, Tz) \leq 1$, for all $z \in \text{Fix}(T)$

Then T has the property P .

Proof. Let $z \in \text{Fix}(T^n)$ for some fixed $n > 1$. As $\alpha(z, Tz) \geq 1$ or $\eta(z, Tz) \leq 1$ and T is a triangular (α, η) -admissible mapping, then

$$\alpha(Tz, T^2z) \geq 1 \text{ or } \eta(T^2z, Tz) \leq 1. \tag{41}$$

Continuing this process, we have

$$\alpha(T^n z, T^{n+1} z) \geq 1 \text{ or } \eta(T^{n+1} z, T^n z) \leq 1, \tag{42}$$

for all $n \in \mathbb{N}$. By (T_3) and (T_4) , we get

$$\begin{aligned}
\alpha(T^m z, T^n z) \geq 1 \quad \text{or} \quad \eta(T^m z, T^n z) \leq 1, \quad \forall m, n \in \mathbb{N}, \\
n \neq m.
\end{aligned} \tag{43}$$

Assume that $z \notin \text{Fix}(T)$, i.e., $d(z, Tz) > 0$.

Applying (3) with $x = T^{n-1}z$ and $y = z$, we get

$$d(z, Tz) = d(T^n z, Tz) = d(T^{n-1} z, Tz) \leq s^2 d(T^{n-1} z, Tz), \tag{44}$$

which implies that

$$\begin{aligned}
\theta(d(T^{n-1} z, Tz)) &\leq \phi[\theta(\beta_1 d(T^{n-1} z, z) + \beta_2 d(T^{n-1} z, T^n z) + \beta_3 d(z, Tz) + \beta_4 d(z, T^n z))] \\
&< \theta[\beta_1 d(T^{n-1} z, z) + \beta_2 d(T^{n-1} z, T^n z) + \beta_3 d(z, Tz) + \beta_4 d(z, T^n z)] \\
&= \theta[\beta_1 d(T^{n-1} z, T^n z) + \beta_2 d(T^{n-1} z, T^n z) + \beta_3 d(z, Tz)].
\end{aligned} \tag{45}$$

Since θ is increasing, therefore,

$$d(z, Tz) < \frac{\beta_1 + \beta_2}{1 - \beta_3} d(T^{n-1} z, T^n z) \leq d(T^{n-1} z, T^n z), \tag{46}$$

which is a contradiction as $d(T^{n-1} z, T^n z) \rightarrow 0$ and $d(z, Tz) > 0$.

Assuming the following conditions, we prove that Theorem 2 still holds for T not necessarily continuous. $\square$

Theorem 3. Let $\alpha, \eta: X \times X \longrightarrow \mathbb{R}^+$ be two functions and let $d(X, d)$ be an (α, η) -complete rectangular b -metric space.

Let $T: X \longrightarrow X$ be a mapping satisfying the following assertions:

- (i) T is triangular (α, η) -admissible
- (ii) T is $(\alpha, \eta) - \theta - \phi$ -contraction
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$
- (iv) (X, d) is an (α, η) -regular rectangular b -metric space

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(z, u) \geq 1$ or $\eta(z, u) \leq 1$ for all $z, u \in \text{Fix}(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$. Similar to the proof of Theorem 3, we can conclude that

$$(\alpha(x_n, x_{n+1}) \geq 1 \text{ or } \eta(x_n, x_{n+1}) \leq 1), \text{ and } x_n \longrightarrow z \text{ as } n \longrightarrow \infty, \quad (47)$$

where $x_{n+1} = Tx_n$.

From (iv), $\alpha(x_{n+1}, z) \geq 1$ or $\eta(x_{n+1}, z) \leq 1$ holds for $n \in \mathbb{N}$.

Suppose that $Tz = x_{n_0+1} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$. From Theorem 3, we know that the members of the sequence $\{x_n\}$ are distinct. Hence, we have $Tz \neq Tx_n$, i.e., $d(Tz, Tx_n) > 0$ for all $n > n_0$. Thus, we can apply (3), to x_n and z for all $n > n_0$ to get

$$\begin{aligned} \theta(d(Tx_n, Tz)) &\leq \theta(s^2 d(Tx_n, Tz)) \leq \phi(\theta(\beta_1 d(x_n, z) \\ &\quad + \beta_2 d(x_n, Tx_n) + \beta_3 d(z, Tz) + \beta_4 d(z, Tx_n))). \end{aligned} \quad (48)$$

By Lemma 2 and (θ_1) , we obtain

$$\begin{aligned} d(Tx_n, Tz) &< (\beta_1 d(x_n, z) + \beta_2 d(x_n, Tx_n) + \beta_3 d(z, Tz) \\ &\quad + \beta_4 d(z, Tx_n)). \end{aligned} \quad (49)$$

By taking the limit as $n \longrightarrow \infty$ in (51), we have

$$\lim_{n \longrightarrow \infty} \sup d(Tx_n, Tz) \leq \beta_3 d(z, Tz). \quad (50)$$

Assume that $z \neq Tz$. Then, from Lemma 1,

$$\frac{1}{s} d(z, Tz) \leq \lim_{n \longrightarrow \infty} \sup d(Tx_n, Tz) \leq \beta_3 d(z, Tz). \quad (51)$$

By assumption $\beta_3 < 1/s$, we have $d(z, Tz) = 0$ and so $z = Tz$. Thus, z is a fixed point of T .

The proof of the uniqueness is similarly to that of Theorem 3.

above theorems, if we take $\phi(t) = t^k$, for some fixed $k \in]0, 1[$, where $\beta_1 = 1$ and $\beta_2 = \beta_3 = \beta_4 = 0$. We obtain the following extension of Jamshaid et al. result (Theorem 1) [13] of (α, η) -complete rectangular b -metric space. $\square$

Corollary 1. Let $\alpha, \eta: X \times X \longrightarrow [0, +\infty[$ be two functions and $d(X, d)$ be an (α, η) -complete rectangular b -metric space and let $T: X \longrightarrow X$ be self-mapping. Suppose for all

$x, y \in X$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ and $d(Tx, Ty) > 0$, we have

$$\theta[s^2 d(Tx, Ty) \leq [\theta(d(x, y))]^k], \quad (52)$$

where $\theta \in \Theta$ and $k \in]0, 1[$. If the mapping T satisfies the following assertions: point, if

- (i) T is a triangular (α, η) -admissible mapping
- (ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$
- (iii) T is (α, η) -continuous or
- (iv) is an (α, η) -regular rectangular b -metric space

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(z, u) \geq 1$ or $\eta(z, u) \leq 1$ for all $z, u \in \text{Fix}(T)$.

Proof. Let $\phi(t) = t^k$, we prove that T is an $(\alpha, \eta) - \theta - \phi$ -contraction. Hence, T satisfies in assumption of Theorem 3 or 2 and is the unique fixed point of T .

It follows from Theorem 3; we obtain the following fixed point theorems for $\theta - \phi$ -Kannan-type contraction and $\theta - \phi$ -Reich-type contraction. $\square$

Theorem 4. Let (X, d) be a (α, η) -complete rectangular b -metric space and let $\alpha, \eta: X \times X \longrightarrow [0, +\infty[$ be two functions. Let $T: X \times X \longrightarrow X$ be a self-mapping satisfying the following conditions:

- (i) T is a triangular (α, η) -admissible mapping
- (ii) T is a $(\alpha, \eta) - \theta - \phi$ -Kannan-type contraction
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$
- (iv) T is a (α, η) -continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$.

Proof. If T is a $(\alpha, \eta) - \theta - \phi$ -Kannan-type contraction, thus there exist $\theta \in \Theta$ and $\phi \in \Phi$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$\theta[s^2 d(Tx, Ty)] \leq \phi \left[\theta \left(\frac{(d(Tx, x) + d(Ty, y))}{2s} \right) \right]. \quad (53)$$

Therefore,

$$\begin{aligned} \theta[s^2 d(Tx, Ty)] &\leq \phi[\theta(\beta_1 d(x, y) + \beta_2 d(Tx, x) + \beta_3 d(Ty, y) \\ &\quad + \beta_4 d(Tx, y))], \end{aligned} \quad (54)$$

where $\beta_1 = \beta_4 = 0$, $\beta_2 = \beta_3 = 1/2s$, which implies that T is a $(\alpha, \eta) - \theta - \phi$ contraction. Therefore, from Theorem 2, T has a unique fixed point. $\square$

Theorem 5. Let (X, d) be a (α, η) -complete rectangular b -metric space and let $\alpha, \eta: X \times X \longrightarrow [0, +\infty[$ be two functions. Let $T: X \times X \longrightarrow X$ be a self-mapping satisfying the following conditions:

- (i) T is a triangular (α, η) -admissible mapping
- (ii) T is a $(\alpha, \eta) - \theta - \phi$ -Reich-type contraction
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$
- (iv) T is a (α, η) -continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$.

Proof. If T is a $(\alpha, \eta) - \theta - \phi$ -Reich-type contraction, thus there exist $\theta \in \Theta$ and $\phi \in \Phi$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$\theta[s^2 d(Tx, Ty)] \leq \phi \left[\theta \left(\frac{(d(x, y) + d(Tx, x) + d(Ty, y))}{3s} \right) \right]. \quad (55)$$

Therefore,

$$\theta[s^2 d(Tx, Ty)] \leq \phi[\theta(\beta_1 d(x, y) + \beta_2 d(Tx, x) + \beta_3 d(Ty, y) + \beta_4 d(Tx, y))], \quad (56)$$

where $\beta_1 = \beta_2 = \beta_3 = (1/3s)$ and $\beta_4 = 0$, which implies that T is a $(\alpha, \eta) - \theta - \phi$ contraction. Therefore, from Theorem 3, T has a unique fixed point. $\square$

Corollary 2. Let (X, d) be a (α, η) -complete rectangular b -metric space and let $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ be two functions. Let $T: X \times X \rightarrow X$ be a self-mapping satisfying the following conditions:

- (i) T is a triangular (α, η) -admissible mapping
- (ii) T is a (α, η) -Kannan-type mapping
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$
- (iv) T is a (α, η) -continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$. Then T has a unique fixed point $x \in X$.

Proof. Let $\theta(t) = e^t$ for all $t \in]0, +\infty[$, and $\phi(t) = t^{2s\alpha}$ for all $t \in [1, +\infty[$.

It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$. We prove that T is a $\theta - \phi$ -Kannan-type contraction:

$$\begin{aligned} \theta(s^2 d(Tx, Ty)) &= e^{s^2 d(Tx, Ty)} \leq e^{\alpha(d(Tx, x) + d(y, Ty))} \\ &= e^{2s\alpha \left(\frac{d(Tx, x) + d(y, Ty)}{2s} \right)} \\ &= \left[e^{\left(\frac{d(Tx, x) + d(y, Ty)}{2s} \right)} \right]^{2s\alpha} \\ &= \phi \left[\theta \left(\frac{d(Tx, x) + d(y, Ty)}{2s} \right) \right]. \end{aligned} \quad (57)$$

Therefore, from Theorem 3, T has a unique fixed point $x \in X$. $\square$

Corollary 3. Let (X, d) be a (α, η) -complete rectangular b -metric space and let $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ be two functions. Let $T: X \times X \rightarrow X$ be a self-mapping satisfying the following conditions:

- (i) T is a triangular (α, η) -admissible mapping
- (ii) T is a (α, η) -Reich-type mapping
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$
- (iv) T is a (α, η) -continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$.

Proof. Let $\theta(t) = e^t$ for all $t \in]0, +\infty[$, and $\phi(t) = t^{3s\lambda}$ for all $t \in [1, +\infty[$.

We prove that T is a $\theta - \phi$ -Reich-type contraction:

$$\begin{aligned} \theta(s^2 d(Tx, Ty)) &= e^{s^2 d(Tx, Ty)} \leq e^{3\lambda s ((d(x, y) + d(Tx, x) + d(y, Ty))/3s)} \\ &= \left[e^{\frac{(d(x, y) + d(Tx, x) + d(y, Ty))}{3s}} \right]^{3\lambda s} \\ &= \phi \left[\theta \left(\frac{(d(x, y) + d(Tx, x) + d(y, Ty))}{3} \right) \right]. \end{aligned} \quad (58)$$

Therefore, from Theorem 3, T has a unique fixed point $x \in X$. $\square$

Example 1. Consider the set $X = \{1, 2, 3, 4\}$. It is easy to check that the mapping $d: X \times X \rightarrow [0, +\infty[$ given by

- (i) $d(x, y) = d(y, x)$, $d(x, x) = 0$ for all $x, y \in X$
- (ii) $d(1, 2) = 1/24$, $d(1, 3) = 3$, $d(1, 4) = 4$
- (iii) $d(2, 3) = 5$, $d(2, 4) = 6$, and $d(3, 4) = 18$

Clearly (X, d) is a rectangular b -metric space with parameter $s = 2$.

Define mapping $T: X \rightarrow X$ and $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ by

$$\begin{cases} T(1) = 1, \\ T(2) = 1, \\ T(3) = 1, \\ T(4) = 2. \end{cases} \quad (59)$$

$$\alpha(x, y) = \frac{x + y}{\max\{x, y\}}, \quad (60)$$

$$\eta(x, y) = \frac{|x - y|}{\max\{x, y\}}. \quad (61)$$

Then, T is an (α, η) -continuous triangular (α, η) -admissible mapping.

Let $\theta(t) = \sqrt{t} + 1$, $\phi(t) = 2t + 1/3$, and $\beta_1 = 4/10$, $\beta_2 = 1/10$, $\beta_3 = 3/10$, and $\beta_4 = 2/10$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$. Evidently, $(\alpha(x, y) \geq 1$ or $(x, y) \leq 1)$ and

$d(Tx, Ty) > 0$ are when, or $\{x, y\} = \{3, 4\}$. Consider the following four possibilities:

For $x = 1$ and $y = 4$, then

$$\theta(s^2 d(T1, T4)) = \sqrt{\frac{1}{6}} + 1 = 1.4, \quad (62)$$

$$\begin{aligned} & \phi(\theta(\beta_1 d(1, 4) + \beta_2 d(1, T1)) + \beta_3 d(4, T4) + \beta_4 d(4, T1)) \\ &= \phi\left(\theta\left(\frac{21}{5}\right)\right) = 2.36. \end{aligned} \quad (63)$$

Then

$$\begin{aligned} \theta(s^2 d(T1, T4)) &\leq \phi(\theta(\beta_1 d(1, 4) + \beta_2 d(1, T1)) \\ &\quad + \beta_3 d(4, T4) + \beta_4 d(4, T1)). \end{aligned} \quad (64)$$

For $x = 2$ and $y = 4$, then

$$\theta(s^2 d(T2, T4)) = \sqrt{\frac{1}{6}} + 1 = 1.4, \quad (65)$$

$$\begin{aligned} & \phi(\theta(\beta_1 d(2, 4) + \beta_2 d(2, T2)) + \beta_3 d(4, T4) + \beta_4 d(4, T2)) \\ &= \phi(\theta(3.65)) = 2.27. \end{aligned} \quad (66)$$

Then

$$\begin{aligned} \theta(s^2 d(T2, T4)) &\leq \phi(\theta(\beta_1 d(2, 4) + \beta_2 d(2, T2)) \\ &\quad + \beta_3 d(4, T4) + \beta_4 d(4, T2)). \end{aligned} \quad (67)$$

For $x = 3$ and $y = 4$, then

$$\theta(s^2 d(T3, T4)) = \sqrt{\frac{1}{6}} + 1 = 1.4, \quad (68)$$

$$\begin{aligned} & \phi(\theta(\beta_1 d(3, 4) + \beta_2 d(3, T3)) + \beta_3 d(4, T4) + \beta_4 d(4, T3)) \\ &= \phi(\theta(10.1)) = 3.13. \end{aligned} \quad (69)$$

Then

$$\begin{aligned} \theta(s^2 d(T3, T4)) &\leq \phi(\theta(\beta_1 d(3, 4) + \beta_2 d(3, T3)) + \beta_3 d(4, T4) \\ &\quad + \beta_4 d(4, T3)). \end{aligned} \quad (70)$$

Hence, T satisfying the assumption of Theorems 3 and 1 is the unique fixed point of T .

Example 2. Let $X = A \cup B$, where $A = \{(1/n): n \in \{2, 3, 4, 5, 6, 7\}\}$ and $B = [1, 2]$. Define $d: X \times X \rightarrow [0, +\infty[$ as follows:

$$\begin{cases} d(x, y) = d(y, x) & \text{for all } x, y \in X, \\ d(x, y) = 0 \Leftrightarrow y = x, \end{cases} \quad (71)$$

$$\begin{cases} d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = d\left(\frac{1}{6}, \frac{1}{7}\right) = 0.05, \\ d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{3}, \frac{1}{7}\right) = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0.08, \\ d\left(\frac{1}{2}, \frac{1}{6}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{7}\right) = 0.4, \\ d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{6}\right) = d\left(\frac{1}{4}, \frac{1}{7}\right) = 0.24, \\ d\left(\frac{1}{2}, \frac{1}{7}\right) = d\left(\frac{1}{3}, \frac{1}{5}\right) = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0.15, \\ d(x, y) = (|x - y|)^2 & \text{otherwise.} \end{cases} \quad (72)$$

Then, (X, d) is a rectangular b -metric space with coefficient $s = 3$. However, we have the following:

- (1) (X, d) is not a metric space, as $d((1/5), (1/7)) = 0.4 > 0.29 = d((1/5), (1/4)) + d((1/4), (1/7))$
- (2) (X, d) is not a b -metric space for $s = 3$, as $d((1/3), (1/4)) = 0.4 > 0.39 = 3[d((1/3), (1/2)) + d((1/2), (1/4))]$
- (3) (X, d) is not a rectangular metric space, as $d((1/5), (1/7)) = 0.4 > 0.28 = d((1/5), (1/4)) + d((1/4), (1/2)) + d((1/2), (1/7))$

Define mapping $T: X \rightarrow X$ and $\alpha, \eta: X \times X \rightarrow [0, +\infty[$ by

$$T(x) = \begin{cases} \sqrt[3]{x}, & \text{if } x \in [1, 2], \\ 1, & \text{if } x \in A, \end{cases} \quad (73)$$

$$\alpha(x, y) = \begin{cases} \sinh(x + y), & \text{if } x, y \in [1, 2], \\ \frac{1}{e^{x+y}}, & \text{otherwise,} \end{cases} \quad (74)$$

$$\eta(x, y) = \begin{cases} \frac{x + y}{4}, & \text{if } x, y \in [1, 2], \\ 1 + e^{-(x+y)}, & \text{otherwise.} \end{cases} \quad (75)$$

Then, T is an (α, η) -continuous triangular (α, η) -admissible mapping.

Let $\theta(t) = \sqrt{t} + 1$, $\phi(t) = t + 1/2$ and taking $\beta_1 = 1$ and $\beta_2 = \beta_3 = \beta_4 = 0$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$.

Evidently, $(\alpha(x, y) \geq 1 \text{ or } (x, y) \leq 1)$ and $d(Tx, Ty) > 0$ are when $\{x, y\} \in [1, 2]$ with $x \neq y$.

Consider two cases:

Case 1: $x > y$:

$$\theta s^2 d(Tx, Ty) = 3(\sqrt[6]{x} - \sqrt[6]{y}) + 1, \quad (76)$$

$$\phi[\theta d(x, y)] = \frac{x - y}{2} + 1. \quad (77)$$

On the other hand

$$\begin{aligned} \theta(s^2 d(Tx, Ty)) - \phi[\theta(d(x, y))] &= \frac{6(\sqrt[6]{x} - \sqrt[6]{y}) - (x - y)}{2} \\ &= \frac{1}{2}((\sqrt[6]{x} - \sqrt[6]{y})) \left[6 - \left(\sqrt[6]{x^5} + \sqrt[6]{x^4 y} + \sqrt[6]{x^3 y^2} + \sqrt[6]{x^2 y^3} \right. \right. \\ &\quad \left. \left. + \sqrt[6]{6xy^4} + \sqrt[6]{y^5} \right) \right]. \end{aligned} \quad (78)$$

Since $x, y \in [1, 2]$, then

$$\left[6 - \left(\sqrt[6]{x^5} + \sqrt[6]{x^4 y} + \sqrt[6]{x^3 y^2} + \sqrt[6]{x^2 y^3} + \sqrt[6]{6xy^4} + \sqrt[6]{y^5} \right) \right] \leq 0, \quad (79)$$

which implies that

$$\begin{aligned} \theta s^2 d(Tx, Ty) &\leq \phi[\theta(d(x, y))] \\ &= \phi[\theta(\beta_1 d(x, y)) + \beta_2 d(x, Tx) + \beta_3 d(y, Ty) \\ &\quad + \beta_4 d(y, Tx)]. \end{aligned} \quad (80)$$

Case 2: $y > x$:

$$\theta s^2 d(Tx, Ty) = 3(\sqrt[6]{y} - \sqrt[6]{x}) + 1, \quad (81)$$

$$\phi[\theta(d(x, y))] = \frac{y - x}{2} + 1. \quad (82)$$

Similarly for Case 2, we conclude that

$$\begin{aligned} \theta s^2 d(Tx, Ty) &\leq \phi[\theta(\beta_1 d(x, y)) + \beta_2 d(x, Tx) + \beta_3 d(y, Ty) \\ &\quad + \beta_4 d(y, Tx)]. \end{aligned} \quad (83)$$

Hence, condition (3) is satisfied. Therefore, T has a unique fixed point $z = 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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