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Research Article

On Syzygy Modules over Laurent Polynomial Rings

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In this paper, we present a dynamical method for computing the syzygy module of multivariate Laurent polynomials with coefficients in a Dedekind ring (with zero divisors) by reducing the computation over Laurent polynomial rings to calculations over a polynomial ring via an appropriate isomorphism.

1. Introduction

Our goal is to give a dynamical method for computing a finite basis for the syzygy module of finitely many multivariate Laurent polynomials with coefficients in a Dedekind ring $\mathbf{R}$. More precisely, given nonzero polynomials $f_1, \ldots, f_s \in \mathbf{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we will compute $s_1, \ldots, s_t \in \mathbf{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{s\times 1}$ generating the syzygy module $\mathrm{Syz}(f_1, \ldots, f_s) := \{{}^t(w_1, \ldots, w_s) \in \mathbf{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{s\times 1}$ such that $w_1 f_1 + \cdots + w_s f_s = 0\}$. The technique consists in reducing the computation over the Laurent polynomial ring $\mathbf{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ to a problem over a polynomial ring $\mathbf{R}[x_1, \ldots, x_n, y]$ via an appropriate isomorphism. One advantage of this indirect approach is that we can use techniques over polynomial rings more efficient than the corresponding methods in Laurent polynomial rings. Such kind of algorithm can be used in signal processing for the computation of the inverse FIR filter of a given multidimensional FIR filter. Our approach is inspired by the theory developed in the papers [1–13] and outlined in the next section.

2. Computing Dynamically a Basis for Syzygies of Polynomials over Dedekind Rings

Let S be a multiplicative subset of a ring $\mathbf{R}$; then, the localization of $\mathbf{R}$ at S is the ring $S^{-1}\mathbf{R} = \{(x/s), x \in \mathbf{R}, s \in S\}$ and the elements of S are forced to be invertible. If $x \in \mathbf{R}$, the localization of $\mathbf{R}$ at the multiplicative subset $\mathcal{M}(x)$ generated by x will be denoted by $\mathbf{R}_x$. Moreover, by induction, for each $x_1, \ldots, x_k \in \mathbf{R}$, we define $\mathbf{R}_{x_1 . x_2 . \ldots . x_k} := (\mathbf{R}_{x_1 . x_2 . \ldots . x_{k-1}})_{x_k}$.

Now, let $\mathbf{R}$ be a Dedekind ring and consider $f_1, \ldots, f_s \in \mathbf{R}[x_1, \ldots, x_n]\backslash\{0\}$. First of all, we need to present a dynamical process [6] for computing a basis for $\mathrm{Syz}(f_1, \ldots, f_s)$. This method works like the case where the basic ring is a Noetherian valuation ring [11]. The Noetherian hypothesis is added so that a dynamical Gröbner basis $G = \{(S_1, G_1), \ldots, (S_k, G_k)\}$ for the ideal $\langle f_1, \ldots, f_s\rangle$ of $\mathbf{R}[x_1, \ldots, x_n]$ can be computed. The only difference is when one has to handle two incomparable (under division) elements a, b in $\mathbf{R}$. In this situation, one should first compute $u, v, w \in \mathbf{R}$ such that

$$\begin{cases} ub = va, \\ wb = (1-u)a. \end{cases} \quad (1)$$

Henceforth, one opens two branches: the computations are pursued in $\mathbf{R}_u$ and $\mathbf{R}_{1+u\mathbf{R}} := \{(x/y), x \in \mathbf{R}$ and $\exists\, z \in \mathbf{R}$ such that $y = 1 + zu\}$. Note that contrary to [11], the localization $\mathbf{R}_{1+u\mathbf{R}}$ instead of $\mathbf{R}_{1-u}$ is used in order to avoid redundancies. The Dedekind ring $\mathbf{R}$ is forced to behave like a valuation ring and this situation will produce a binary tree in which leaves correspond to localizations $S_j^{-1}\mathbf{R}$, $1 \le j \le k$, of $\mathbf{R}$ at comaximal multiplicative subsets $S_1, \ldots, S_k$.

The fact that a basis for $\mathrm{Syz}(f_1,\ldots,f_s)$ can be computed at each leaf together with Lemma 1 will yield the desired one.

Let $H_j = \left\{h_{j,1},\ldots,h_{j,p_j}\right\}$ denote a basis for $\mathrm{Syz}(f_1,\ldots,f_s)$ over $(S_j^{-1}\mathbf{R})[x_1,\ldots,x_n]$, $1\le j\le k$. There exists a $d_j\in S_j$ such that $d_jh_{j,i}\in\mathbf{R}[x_1,\ldots,x_n]$, for each $1\le i\le p_j$, and $\left\{d_jh_{j,1},\ldots,d_jh_{j,p_j}\right\}$ is a generator for $\mathrm{Syz}(f_1,\ldots,f_s)$ over $(S_j^{-1}\mathbf{R})[x_1,\ldots,x_n]$. As explained in Theorem II.3.6 [10], we have the following concrete local-global principle for coherent modules.

Lemma 1. *(syzygy, coherent modules). Let A be a ring, $S_1,\ldots,S_n$ be comaximal monoids, M be a A–module, and $a=(a_1,\ldots,a_m)\in M^m$.*

(1) The syzygy module $N\subseteq A^m$ of the vector whose elements a_i are seen as vectors in M is finitely generated if and only if each syzygy module $N_i\subseteq A^m_{S_i}$ of the a_i's vector (a_i are seen as vector in M) is finitely generated.

(2) M is coherent if and only if each M_{S_i} is coherent.

(3) The ring A is coherent if and only if each A_{S_i} is coherent.

Proof.

(1) Let S be a monoid in A and N' be the syzygy module of the vector whose elements a_i are considered in M_S. We will prove that $N_S=N'$. It is clear that $N_S\subseteq N'$. Conversely, if $\sum_{j=1}^m(x_j/s_j)a_j=0$ in M_S, let us denote $u=\prod_i s_i$ and $u_j=\prod_{i\neq j}s_i$, such that $\sum_{j=1}^m x_ju_ja_j=0$ in M_S and $\sum_{j=1}^m sx_ju_ja_j=0$ in M for $s\in S$. We have $y=(y_1,\ldots,y_m)=(sx_1u_1,\ldots,sx_mu_m)\in N$ and $((x_1/s_1),\ldots,(x_m/s_m))=(1/su)y$ in A_S.

(2) Let $a=(a_1,\ldots,a_m)\in M^m$ and $N\subseteq A^m$ be the module of relations for a. For all monoids S, N_S is the module of relations for a in M_S.

(3) Is a particular case of 2. □

Theorem 1. *Let $a=(a_1,\ldots,a_m)\in\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]^m$ and $\widetilde{a}=u(a_1,\ldots,a_m)$, where $u=x_1^{k_1}\ldots x_n^{k_n}$ is a polynomial in $\mathbf{R}[x_1,\ldots,x_n]$ such that $\widetilde{a}=(\widetilde{a}_1,\ldots,\widetilde{a}_m)\in\mathbf{R}[x_1,\ldots,x_n]^m$. The syzygy module N of a over $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ is the same than the syzygy module of $\widetilde{a}$ over $\mathbf{R}[x_1,\ldots,x_n]$.*

Proof. Let us denote by $N\subseteq\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]^m$ the syzygy module of a over $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ and $N'\subseteq\mathbf{R}[x_1,\ldots,x_n]^m$ that of $\widetilde{a}$ in $\mathbf{R}[x_1,\ldots,x_n]$.

If $\sum_{j=1}^m x_j\widetilde{a}_j=0$ over $\mathbf{R}[x_1,\ldots,x_n]$, then $\sum_{j=1}^m x_j(\widetilde{a}_j/u)=\sum_{j=1}^m x_ja_j=0$, so we have $N'\subseteq N$.

Conversely if $\sum_{j=1}^m x_ja_j=0$ over $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$, we can suppose that $x_i\in\mathbf{R}[x_1,\ldots,x_n]^m$ (the set of generators of a over $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]^m$ can be supposed with only elements of $\mathbf{R}[x_1,\ldots,x_n]$) and we can write $\sum_{j=1}^m x_jua_j=\sum_{j=1}^m x_j\widetilde{a}_j=0$. Finally, $N=N'$. □

3. Syzygies of Laurent Polynomials over Dedekind Ring

Now, we can give a method for computing a set of generators for the syzygy module, $\mathrm{Syz}(f_1,\ldots,f_s)$, over the Laurent polynomial ring $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$. For this, we need the following.

Lemma 2 (see [14]). *Let I be the ideal of $\mathbf{R}[x_1,\ldots,x_n,y]$ generated by $x_1\ldots x_ny-1$; then,*

$$\mathbf{R}\left[x_1^{\pm1},\ldots,x_n^{\pm1}\right]\cong\frac{\mathbf{R}[x_1,\ldots,x_n,y]}{I}. \tag{2}$$

Proof. Let ϕ be an $\mathbf{R}$-algebra homomorphism defined from $\mathbf{R}[x_1,\ldots,x_n,y]$ to $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ by

$$\phi(x_i)=x_i,\qquad \phi(y)=x_1^{-1}\ldots x_n^{-1} \tag{3}$$

and extended to $\mathbf{R}[x_1,\ldots,x_n,y]$ as follows: for $\alpha\in\mathbf{R}$ and $f,g\in\mathbf{R}[x_1,\ldots,x_n,y]$,

$$\begin{aligned}\phi(\alpha f)&=\alpha\phi(f),\\ \phi(f+g)&=\phi(f)+\phi(g),\\ \phi(fg)&=\phi(f)\phi(g).\end{aligned} \tag{4}$$

Considering such isomorphism, our problem is reduced to a computation over the polynomial ring $\mathbf{R}[x_1,\ldots,x_n,y]$. More precisely, we are going to rely on such isomorphism between $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ and $(\mathbf{R}[x_1,\ldots,x_n,y]/I)$ to obtain an algorithm computing syzygy basis in $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$.

The previous isomorphism is well defined, as it does not depend on the way elements of $\mathbf{R}[x_1,\ldots,x_n,y]/I$ and $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ are represented. First, every Laurent polynomial can be written as a polynomial in the variables $x_1,\ldots,x_n,x_1^{-1}\ldots x_n^{-1}$, and to get an element of $\mathbf{R}[x_1,\ldots,x_n,y]$ from a Laurent polynomial by this isomorphism, we have to consider the relation $x_1\ldots x_ny=1$ on I.

Note that if $t=x_1^{e_1}\ldots x_n^{e_n}$ with $e_i\in\mathbb{Z}$ being a term of $\mathbf{R}[x_1^{\pm1},\ldots,x_n^{\pm1}]$ and $e=-\min\{0,e_1,\ldots,e_n\}$, we have

$$\phi^{-1}(t)=x_1^{e_1+e}\ldots x_n^{e_n+e}\cdot y^e\in\mathbf{R}[x_1,\ldots,x_n,y]. \tag{5}$$

Conversely, each element of $\mathbf{R}[x_1,\ldots,x_n,y]/I$ is the image of a polynomial in $\mathbf{R}[x_1,\ldots,x_n,y]$ through the canonical (surjective) homomorphism of $\mathbf{R}$-algebras:

$$\mathbf{R}[x_1,\ldots,x_n,y]\longrightarrow\frac{\mathbf{R}[x_1,\ldots,x_n,y]}{I}, \tag{6}$$

which converts $f\in\mathbf{R}[x_1,\ldots,x_n,y]$ into $\overline{f}=f\,(\mathrm{mod}\,(I))$.

Given $f\in\mathbf{R}[x_1,\ldots,x_n,y]$, we obtain $\overline{f}$ by replacing x_i by $\overline{x}_i$ and y by $\overline{y}$, bearing in mind that $\overline{x}_1\cdots\overline{x}_n.\overline{y}=1$. Each $\overline{f}\in\mathbf{R}[x_1,\ldots,x_n,y]/I$ can be expressed as algebraic combination of $\overline{x},\ldots,\overline{x}_n,\overline{y}$, with coefficients in R. By taking this expression of $\overline{f}$ without the bars over variables, we get a polynomial $p\in\mathbf{R}[x_1,\ldots,x_n,y]$ such that $\overline{p}=\overline{f}$. Also, each g with $\overline{f}$ as image can be written as $g=p+(x_1\ldots x_ny-1)q$ where $q\in\mathbf{R}[x_1,\ldots,x_n,y]$.

By the isomorphism between $\mathbf{R}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $\mathbf{R}[x_1, \dots, x_n, y]/I$, we get the image of $\overline{f}$ replacing $\overline{x}_i$ by x_i and $\overline{y}$ by $x_1^{-1} \cdots x_n^{-1}$. □

Theorem 2. *Let $\{s_1, \dots, s_m\}$ be a set of generators for $\mathrm{Syz}(f_1, \dots, f_r, x_1 \dots x_n y - 1)$; then, $\{\overline{s}_1, \dots, \overline{s}_m\}$ is a set of generators for $\mathrm{Syz}(\overline{f}_1, \dots, \overline{f}_r)$.*

Proof. Since $\overline{q}_1\overline{f}_1 + \cdots + \overline{q}_r\overline{f}_r = 0$ is equivalent to $q_1 f_1 + \cdots + q_r f_r + q \cdot (x_1 \dots x_n y - 1) = 0$ with $q \in \mathbf{R}[x_1, \dots, x_n, y]$, the $\mathbf{R}[x_1, \dots, x_n, y]$-homomorphism

$$\begin{aligned} &\mathrm{Syz}(f_1, \dots, f_r, x_1 \dots x_n y - 1) \longrightarrow \mathrm{Syz}(\overline{f}_1, \dots, \overline{f}_r), \\ &s = (q_1, \dots, q_r, q) \longmapsto \overline{s} = (\overline{q}_1, \dots, \overline{q}_r) \end{aligned} \tag{7}$$

is surjective. We conclude that if $\{s_1, \dots, s_m\}$ is a set of generators for $\mathrm{Syz}(f_1, \dots, f_r, x_1 \dots x_n y - 1)$, then $\{\overline{s}_1, \dots, \overline{s}_m\}$ is a set of generators for $\mathrm{Syz}(\overline{f}_1, \dots, \overline{f}_r)$. □

4. Illustrative Examples

Example 1. Let I be the ideal of $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ generated by

$$\begin{aligned} f_1 &= 10y^{-1} + 1, \\ f_2 &= 6x^2 y + 3x^{-1}y^{-1}, \\ f_3 &= 12x - y + 6x^{-2}y^{-2} - 10. \end{aligned} \tag{8}$$

To compute a set of generators for $\mathrm{Syz}(f_1, f_2, f_3)$, we rely on the isomorphism between $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and $\mathbb{Z}[x, y, t]/\langle xyt - 1\rangle$. It is equivalent to compute a set of generators for $\mathrm{Syz}(\overline{g}_1, \overline{g}_2, \overline{g}_3)$ with

$$\begin{aligned} \overline{g}_1 &= 10\overline{x}\overline{t} + 1, \\ \overline{g}_2 &= 6\overline{x}^2\overline{y} + 3\overline{t}, \\ \overline{g}_3 &= 12\overline{x} - \overline{y} + 6\overline{t}^2 - 10. \end{aligned} \tag{9}$$

First of all, let us compute in $\mathbb{Z}[x, y, t]$ a set of generators for $\mathrm{Syz}(g_1, g_2, g_3, g_4)$ with

$$\begin{aligned} g_1 &= 10xt + 1, \\ g_2 &= 6x^2 y + 3t, \\ g_3 &= 12x - y + 6t^2 - 10, \\ g_4 &= xyt - 1. \end{aligned} \tag{10}$$

Let us use the lexicographic order with $x > t > y$ as monomial order to compute a dynamical Gröbner basis for $J = \langle g_1, g_2, g_3, g_4\rangle$ in $\mathbb{Z}[x, y, t]$. As the leading coefficients of g_1 and g_2 are not comparable under division in $\mathbb{Z}$ and $\mathrm{lcm}(10, 6) = 2$, we can open two following leaves to proceed:

$$\begin{array}{c} \mathbb{Z} \\ \swarrow \quad \searrow \\ \mathbb{Z}_5 \qquad \mathbb{Z}_3 \end{array} \tag{11}$$

In $\mathbb{Z}_5$, $S(g_1, g_2) = (3/5)xyg_1 - tg_2 = (3/5)xy - 3t^2 := g_5$. Since the leading coefficients of g_1 and g_5 are not comparable, we need to open two new leaves as

$$\begin{array}{c} \mathbb{Z}_5 \\ \swarrow \quad \searrow \\ \mathbb{Z}_{5.3} \qquad \mathbb{Z}_{5.2} \end{array} \tag{12}$$

In $\mathbb{Z}_{5.3}$, $G_{5.3} = \{y + 10, 2x + t^2, 1 - 5t^3\}$ is a special Gröbner basis for J at the leaf $\mathbb{Z}_{5.3}$. And over $\mathbb{Z}_{5.3}[x, y, t]$, we have $\mathrm{Syz}(g_1, g_2, g_3, g_4)$ as

$$\left\langle \begin{pmatrix} 2xy + yt^2 \\ \dfrac{ty + 10t}{3} \\ 0 \\ -10t^2 + 2xy \end{pmatrix}, \begin{pmatrix} yt^3 + 2 \\ \dfrac{t^2 y + 10t^2}{3} \\ 0 \\ -10t^3 + 2 + 2xyt + 20xt \end{pmatrix}, \begin{pmatrix} \dfrac{2x + t^2}{5} \\ -\dfrac{1}{15}t - \dfrac{2}{3}xt^2 \\ 0 \\ \dfrac{2}{5}x + 4x^2 t \end{pmatrix}, \begin{pmatrix} 3xyt^2 - 3t \\ -xyt + 1 \\ 0 \\ -30xt^2 + 6x^2 y \end{pmatrix}, \begin{pmatrix} y \\ -2t \\ 1 \\ -10 + 12x \end{pmatrix}, \begin{pmatrix} -xyt + 1 \\ 0 \\ 0 \\ 10xt + 1 \end{pmatrix} \right\rangle. \tag{13}$$

In $\mathbb{Z}_{5.2}$, we find $G_{5.2} = \{y + 10, 6x + 3t^2, (3/5) - 3t^3, 10xt + 1\}$ as a special Gröbner basis for J at the leaf $\mathbb{Z}_{5.2}$, and we have the following $\mathrm{Syz}(g_1, g_2, g_3, g_4)$ over $\mathbb{Z}_{5.2}[x, y, t]$:

$$\left\langle \begin{pmatrix} -\frac{3}{10}t^2 - \frac{3}{5}x \\ \frac{1}{10}t + 10xt^2 \\ 0 \\ -\frac{3}{5}x - 6x^2t \end{pmatrix}, \begin{pmatrix} 6xy + 3yt^2 \\ -ty - 10t \\ 0 \\ -30t^2 + 6xy \end{pmatrix}, \begin{pmatrix} -6 - 3yt^3 \\ t^2y + 10t^2 \\ 0 \\ -6 + 30t^3 - 6xyt - 60xt \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} \frac{6}{5}x + \frac{3}{5}t^2 \\ -\frac{1}{5}t - 2xt^2 \\ 0 \\ \frac{6}{5}x + 12x^2t \end{pmatrix}, \begin{pmatrix} 3xyt^2 - 3t \\ -xyt + 1 \\ 0 \\ -30xt^2 + 6x^2y \end{pmatrix}, \begin{pmatrix} y \\ -2t \\ 1 \\ -10 + 12x \end{pmatrix}, \begin{pmatrix} -xyt + 1 \\ 0 \\ 0 \\ 10xt + 1 \end{pmatrix} \right\rangle. \tag{14}$$

In $\mathbb{Z}_3$, we proceed as above, and we will open two leaves:

$$\begin{array}{ccc} & \mathbb{Z}_3 & \\ \swarrow & & \searrow \\ \mathbb{Z}_{3.2} & & \mathbb{Z}_{3.5} \end{array} \tag{15}$$

$G_{3.2} = \{y + 10, 2x + t^2, 1 - 5t^3\}$ is a special Gröbner basis for J at the leaf $\mathbb{Z}_{3.2}$, and we get $\mathrm{Syz}(g_1, g_2, g_3, g_4)$ as

$$\left\langle \begin{pmatrix} xy + \frac{1}{2}t^2 \\ \frac{-ty - 10t}{6} \\ 0 \\ -5t^2 + xy \end{pmatrix}, \begin{pmatrix} -5yt^3 + 10 \\ \frac{5t^2y + 50t^2}{3} \\ 0 \\ 50t^3 - 10 - 10xyt - 100xt \end{pmatrix}, \begin{pmatrix} -x - \frac{1}{2}t^2 \\ \frac{1}{6}t + \frac{5}{3}xt^2 \\ 0 \\ -x - 10x^2t \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 3xyt^2 - 3t \\ -xyt + 1 \\ 0 \\ -30xt^2 + 6x^2y \end{pmatrix}, \begin{pmatrix} y \\ -2t \\ 1 \\ -10 + 12x \end{pmatrix}, \begin{pmatrix} -xyt + 1 \\ 0 \\ 0 \\ 10xt + 1 \end{pmatrix} \right\rangle. \tag{16}$$

Note that $G_{3.5} = G_{5.3}$.

Finally, $\mathrm{Syz}(g_1, g_2, g_3, g_4)$ over $\mathbb{Z}[x, y, t]$ is

$$\left\langle \begin{pmatrix} 15(yt^3+2) \\ -5t^2(y+10) \\ 0 \\ -10t^3+2+2xyt+20xt \end{pmatrix}, \begin{pmatrix} 3y(2x+t^2) \\ -t(y+10) \\ 0 \\ -30t^2+6xy \end{pmatrix}, \begin{pmatrix} -6x+3t^2 \\ t(1+10xt) \\ 0 \\ -6x(1+10xt) \end{pmatrix}, \begin{pmatrix} y \\ -2t \\ 1 \\ -10+12x \end{pmatrix}, \begin{pmatrix} 3xyt^2-3t \\ -xyt+1 \\ 0 \\ -30xt^2+6x^2y \end{pmatrix}, \begin{pmatrix} -xyt+1 \\ 0 \\ 0 \\ 10xt+1 \end{pmatrix} \right\rangle. \tag{17}$$

Note that $\overline{x}\,\overline{y}\,\overline{t}=1$. Hence, a set of generators for $\mathrm{Syz}(\overline{g}_1,\overline{g}_2,\overline{g}_3)$ is

$$\left\{ \begin{pmatrix} 15(\overline{y}\overline{t}^3+2) \\ -5\overline{t}^2(\overline{y}+10) \\ 0 \end{pmatrix}, \begin{pmatrix} 3\overline{y}(2\overline{x}+\overline{t}^2) \\ -\overline{t}(\overline{y}+10) \\ 0 \end{pmatrix}, \begin{pmatrix} -6\overline{x}+3\overline{t}^2 \\ \overline{t}(1+10\overline{x}\overline{t}) \\ 0 \end{pmatrix}, \begin{pmatrix} \overline{y} \\ -2\overline{t} \\ 1 \end{pmatrix} \right\}. \tag{18}$$

Therefore, a set of generators for $\mathrm{Syz}(f_1,f_2,f_3)$ is

$$\left\{ \begin{pmatrix} 15y^{-2}x^{-3}+30 \\ -5x^{-2}y^{-1}-50x^{-2}y^{-2} \\ 0 \end{pmatrix}, \begin{pmatrix} 6xy+3x^{-2}y^{-1} \\ -x^{-1}-10x^{-1}y^{-1} \\ 0 \end{pmatrix}, \begin{pmatrix} -6x-3x^{-2}y^{-2} \\ x^{-1}y^{-1}+10x^{-1}y^{-2} \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ -2x^{-1}y^{-1} \\ 1 \end{pmatrix} \right\}. \tag{19}$$

Finally, a set of generators of $\mathrm{Syz}(f_1,f_2,f_3)$ over $\mathbb{Z}[x^{\pm1},y^{\pm1}]$ is

$$\left\{ \begin{pmatrix} 6xy+3x^{-2}y^{-1} \\ -x^{-1}-10x^{-1}y^{-1} \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ -2x^{-1}y^{-1} \\ 1 \end{pmatrix} \right\}. \tag{20}$$

Now, let us see an example where the basic ring is not principal.

Example 2. Let us consider the ring $\mathbb{Z}[\theta][y^{\pm1},z^{\pm1}]$, where $\theta^2=-5$ is its ideal I generated by

$$\begin{aligned} f_1 &= (8+4\theta)z^{-1}-y^{-1}+15y^{-1}z^{-1}, \\ f_2 &= 1+3z^{-1}, \\ f_3 &= (4+2\theta)y+9. \end{aligned} \tag{21}$$

To compute a set of generators for $\mathrm{Syz}(f_1,f_2,f_3)$, we rely on the isomorphism between $\mathbb{Z}[\theta][y^{\pm1},z^{\pm1}]$ and $\mathbb{Z}[\theta][x,y,z]/\langle xyz-1\rangle$. It is equivalent to compute a set of generators for $\mathrm{Syz}(\overline{g}_1,\overline{g}_2,\overline{g}_3)$ with

$$\begin{aligned} \overline{g}_1 &= (8+4\theta)\overline{x}\,\overline{y}-\overline{x}\,\overline{z}+15\overline{x}, \\ \overline{g}_2 &= 3\overline{x}\,\overline{y}+1, \\ \overline{g}_3 &= (4+2\theta)\overline{y}+9. \end{aligned} \tag{22}$$

First of all, let us compute in $\mathbb{Z}[\theta][x,y,z]$ a set of generators for $\mathrm{Syz}(g_1,g_2,g_3,g_4)$ with

$$\begin{aligned} g_1 &= (8+4\theta)xy-xz+15x, \\ g_2 &= 3xy+1, \\ g_3 &= (4+2\theta)y+9, \\ g_4 &= xyz-1. \end{aligned} \tag{23}$$

Using the lexicographic order with $x>y>z$ as monomial order, we proceed as above by computing a dynamical Gröbner basis for $J=\langle g_1,g_2,g_3,g_4\rangle$ in $\mathbb{Z}[x,y,z]$.

Let us denote by a and b the leading coefficients of g_2 and g_3, respectively. Since $a:=3$ and $b:=4+2\theta$ are not comparable, we have to find $u,v,w\in\mathbb{Z}[\theta]$ such that

$$\begin{cases} ub=va, \\ wb=(1-u)a. \end{cases} \tag{24}$$

With the solution of this system given by $u = 5 + 2\theta$, $v = 6\theta$, and $w = -3$, *we can open two leaves:*

$$\begin{array}{ccc} & \mathbb{Z}[\theta] & \\ \swarrow & & \searrow \\ \mathbb{Z}[\theta]_{4+2\theta} & & \mathbb{Z}[\theta]_{5+2\theta} \end{array} \quad (25)$$

In $\mathbb{Z}[\theta]_{4+2\theta}$:

$$\begin{aligned} S(g_2, g_3) &= g_2 - \frac{3}{4+2\theta} g_3 = -\frac{27}{4+2\theta} x + 1 := g_5, \\ S(g_2, g_4) &= z g_2 - 3t g_4 = z + 3 := g_6, \\ S(g_2, g_5) &= -\frac{9}{4+2\theta} g_2 - yt g_5 = -\frac{9}{4+2\theta} - y \xrightarrow{g_3} 0, \\ S(g_2, g_6) &= z g_2 - 3xy g_6 = z - 9xy \xrightarrow{g_2} g_6 \xrightarrow{g_6} 0, g_1 \xrightarrow{g_3} - x g_6 \xrightarrow{g_6} 0, \\ S(g_3, g_5) &= -\frac{27}{(4+2\theta)^2} x g_3 - y g_5 = -\frac{243}{(4+2\theta)^2} x - y = -\frac{1}{4+2\theta} g_3 + \frac{9}{4+2\theta} g_5 \xrightarrow{g_5} 0, \\ S(g_3, g_6) &= z g_3 - (4+2\theta) y g_6 = 9z - 3(4+2\theta) y = -3g_3 + 9g_6 \xrightarrow{g_3, g_6} 0, g_4 \xrightarrow{g_6} - g_2 \xrightarrow{g_2} 0, \\ S(g_5, g_6) &= z g_5 + \frac{27}{4+2\theta} x g_6 = -3 g_5 + g_6 \xrightarrow{g_5}, g_6 \; 0. \end{aligned} \quad (26)$$

Thus, $\{(4+2\theta)y + 9, (-27/4+2\theta)x + 1, z + 3\}$ is a special Gröbner basis for J at the leaf $\mathscr{M}(4+2\theta)^{-1}\mathbb{Z} = \mathbb{Z}_{4+2\theta}$.

And we have on $\mathbb{Z}_{4+2\theta}[x, y, z]$, $\mathrm{Syz}(g_1, g_2, g_3, g_4)$ as

$$\left\langle \begin{pmatrix} 0 \\ -\dfrac{(4+2\theta)y+9}{4+2\theta} \\ \dfrac{1+3xy}{4+2\theta} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4yz - 2\theta yz - 9z \\ z+3 \\ 12y + 6\theta y + 27 \end{pmatrix}, \begin{pmatrix} 0 \\ 12 + 6\theta + 27xz4 + 2\theta \\ \dfrac{-3xz - 9x}{4+2\theta} \\ \dfrac{-81x + 12 + 6\theta}{4+2\theta} \end{pmatrix}, \begin{pmatrix} 1 \\ xz \\ -2x \\ -3x \end{pmatrix} \right\rangle. \quad (27)$$

In $\underline{\mathbb{Z}[\theta]_{5+2\theta}}$, as 2 and 3 are not comparable under division in $\mathbb{Z}[\theta]_{5+2\theta}$, in order to pursue the computations, we need to open two new leaves:

$$\begin{array}{ccc} & \mathbb{Z}[\theta]_{5+2\theta} & \\ \swarrow & & \searrow \\ \mathbb{Z}[\theta]_{(5+2\theta).3} & & \mathbb{Z}[\theta]_{(5+2\theta).2} \end{array} \quad (28)$$

The final evaluation tree is given by

$$\begin{array}{ccccc} & \mathbb{Z}[\theta] & & & \\ \swarrow & & \searrow & & \\ \mathbb{Z}[\theta]_{4+2\theta} & & \mathbb{Z}[\theta]_{5+2\theta} & & \\ & \swarrow & & \searrow & \\ & \mathbb{Z}[\theta]_{(5+2\theta).3} & & \mathbb{Z}[\theta]_{(5+2\theta).2} & \end{array} \quad (29)$$

In $\underline{\mathbb{Z}[\theta]_{(5+2\theta).2}}$, we get $\{z + 3, (-6 + 3\theta/2)x + 1, (2\theta/5 + 2\theta)y + 1\}$ as a special Gröbner basis for

$$\langle (8+4a)xy - xz + 15x, 3xy + 1, (4+2\theta)y + 9, xyz - 1 \rangle, \tag{30}$$

at the leaf $\mathscr{M}(5+2\theta,2)^{-1}\mathbb{Z}[\theta] = \mathbb{Z}[\theta]_{(5+2\theta)\cdot 2}$. And over $\mathbb{Z}[\theta] \leq_{(5+2\theta)\cdot 2} [x,y,z]$,Syz$(g_1,g_2,g_3,g_4)$ is

$$\left\langle \begin{pmatrix} 1 \\ \dfrac{5xz + 2\theta xz - 90x - 36\theta x - 36\theta xy + 270x^2y + 108\theta x^2 y + 108\theta x^2 y^2}{5+2\theta} \\ -18x^3y^2 \\ -3x \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{(2-\theta)(-6\theta + 45x + 18x\theta + 6xyz\theta + 18xy\theta)}{2(5+2\theta)} \\ \dfrac{(\theta-2)(x^2yz + 3x^2y)}{2(5+2\theta)} \\ \dfrac{(2-\theta)(45x + 18x\theta - 6\theta)}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{(2-\theta)(18\theta xy^2 + 6xyz\theta - 6\theta y + 18xy\theta + 6\theta xy^2 z - 15 + 45xy + 15xyz)}{2(5+2\theta)} \\ \dfrac{(\theta-2)(5x^2y^2z + 2\theta x^2y^2z + 15x^2y^2 + 6\theta x^2y^2)}{2(5+2\theta)} \\ \dfrac{(2-\theta)(-6\theta y - 15 - 6\theta)}{2(5+2\theta)} \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{(1-4\theta)(-90\theta x^2y^2 + 180x^2y^2 - 45x^2y - 180\theta x^2 y)}{4(5+2\theta)^2} \\ \dfrac{(1-4\theta)(5x^2y + 2\theta x^2 y + 15x^3y^2 + 6\theta x^3y^2)}{4(5+2\theta)} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{-45 - 18\theta - 18\theta y + 135xy + 54\theta xy + 54\theta xy^2}{5+2\theta} \\ 1 - 9x^2y^2 \\ 0 \end{pmatrix} \right\rangle. \tag{31}$$

In $\mathbb{Z}[\theta]_{(5+2\theta)\cdot 3}$, we get $\{z+3, 3x-(2\theta/5+2\theta), (2\theta/5+2\theta)y+1\}$ as a special Gröbner basis for

$$\langle (8+4a)xy - xz + 15x, 3xy+1, (4+2\theta)y+9, xyz-1\rangle, \tag{32}$$

at the leaf $\mathscr{M}(5+2\theta,3)^{-1}\mathbb{Z}[\theta] = \mathbb{Z}[\theta]_{(5+2\theta).3}$. And on $\mathbb{Z}[\theta]_{(5+2\theta)\cdot 3}[x,y,z]$,Syz$(g_1,g_2,g_3,g_4)$ is

$$\left\langle \begin{pmatrix} 1 \\ \dfrac{5xz+2\theta xz-90x-36\theta x-36\theta xy+270x^2y+108\theta x^2y+108\theta x^2y^2}{5+2\theta} \\ -18x^3y^2 \\ -3x \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{2\theta-15x-6x\theta-2xyz\theta-6xy\theta}{5+2\theta} \\ \dfrac{x^2yz+3x^2y}{3} \\ \dfrac{-15x-6x\theta+2\theta}{5+2\theta} \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{-45-18\theta-18\theta y+135xy+54\theta xy+54\theta xy^2}{5+2\theta} \\ 1-9x^2y^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{18\theta xy^2+6xyz\theta-6\theta y+18xy\theta-6\theta xy^2z-15+45xy+15xyz}{5+2\theta} \\ -x^2y^2z-3x^2y^2 \\ \dfrac{-6\theta y-15-6\theta}{5+2\theta} \end{pmatrix}, \begin{pmatrix} 0 \\ \dfrac{6\theta x^2y^2+15x^2y+6\theta x^2y}{5+2\theta} \\ \dfrac{-x^2y-3x^3y^2}{3} \\ 0 \end{pmatrix} \right\rangle. \tag{33}$$

Hence, over $\mathbb{Z}[\theta][x,y,z]$, a generating set of Syz(g_1,g_2,g_3,g_4) is

$$\left\langle \begin{pmatrix} 0 \\ -4y-2\theta y-9 \\ 1+3xy \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4yz-2\theta yz-9z \\ z+3 \\ 12y+6\theta y+27 \end{pmatrix}, \begin{pmatrix} 1 \\ xz \\ -2x \\ -3x \end{pmatrix}, \begin{pmatrix} 0 \\ 12+6\theta+27xz \\ -3xz-9x \\ -81x+12+6\theta \end{pmatrix}, \right.$$
$$\begin{pmatrix} 5+2\theta \\ 5xz+2\theta xz-90x-36\theta x-36\theta xy+270x^2y+108\theta x^2y+108\theta x^2y^2 \\ -(90+36\theta)x^3y^2 \\ -(15+6\theta)x \end{pmatrix},$$
$$\begin{pmatrix} 0 \\ -6\theta+45x+18x\theta(1+y)+6xyz\theta \\ -(5+2\theta)\left(x^2yz+3x^2y\right) \\ 45x+18x\theta-6\theta \end{pmatrix}, \tag{34}$$
$$\begin{pmatrix} 0 \\ 18\theta xy^2+6xyz\theta-6\theta-6\theta y+18xy\theta+6\theta xy^2z-15+45xy+15xyz \\ -5x^2y^2z-2\theta x^2y^2z-15x^2y^2-6\theta x^2y^2 \\ -6\theta y-15-6\theta \end{pmatrix},$$
$$\left. \begin{pmatrix} 0 \\ -90\theta x^2y^2+180x^2y^2-45x^2y-180\theta x^2y \\ (5+2\theta)\left(5x^2y+2\theta x^2y+15x^3y^2+6\theta x^3y^2\right) \\ 0 \end{pmatrix} \right\rangle.$$

Note that we have $\overline{x}\,\overline{y}\,\overline{z} = 1$, and over $\mathbb{Z}[\theta][y^{\pm 1}, z^{\pm 1}]$, the generation is reduced to $\mathrm{Syz}(f_1, f_2, f_3)$ as

$$\left\langle \begin{pmatrix} 1 \\ \dfrac{1}{y} \\ -\dfrac{2}{yz} \end{pmatrix}, \begin{pmatrix} 0 \\ -4yz-2\theta yz-9z \\ z+3 \end{pmatrix}, \begin{pmatrix} 5+2\theta \\ \dfrac{5+2\theta}{y}-\dfrac{90+36\theta}{yz}-\dfrac{36\theta}{z}+\dfrac{270+108\theta}{yz^2}+\dfrac{108\theta}{y^2z^2} \\ \dfrac{-90-36\theta}{yz^3} \end{pmatrix} \right\rangle. \tag{35}$$

In fact the trick in Lemma 2 can be used for computing syzygies of a finite system in any module $M[1/u]$ over $A[1/u]$ when A is a coherent A–module. Here, $M = A = \mathbf{R}[x_1, \dots, x_n]$ and $u = x_1 \dots x_n$.

Note that a generator set for the syzygy module of $(f_1, \dots, f_s)$ over the Laurent polynomial ring $\mathbf{R}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ can be directly computed in $\mathbf{R}[x_1, \dots, x_n]$ by multiplying $(f_1, \dots, f_s)$ by a polynomial such that the new vector obtained is in $\mathbf{R}[x_1, \dots, x_n]$, by virtue of Theorem1. The syzygy module does not change and the problem is reduced to a computation of a generator set for the syzygy module in $\mathbf{R}[x_1, \dots, x_n]$.

Example 3. Let $I = \langle F_1 = 8Y^{-1} + X^{-2}Y^{-2}, F_2 = 10XY^{-2} -2X^{-2}Y^{-2} \rangle$ in $\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$. The problem of computing a generator for the syzygy module of I can be reduced to find a generator for the syzygy module of $\langle f_1 = 8X^2Y + 1, f_2 = 10X^3 - 2 \rangle$ in $\mathbb{Z}[X, Y]$.

(f_1, f_2) is obtained by multiplying (F_1, F_2) by X^2Y^2.

Let us consider the lexicographic order with $X > Y$.

As $8 \wedge 10 = 2$, $8 = 2 \times 4$, and $10 = 2 \times 5$, we have to open two leaves: $\mathbb{Z}_4$ and $\mathbb{Z}_5$.

In $\mathbb{Z}_4$, $S(f_1, f_2) = (5/4)Xf_1 - Yf_2 = (5/4)X + 2Y =: f_3$. The leading coefficients of f_1 and f_3 are comparable under division. As $8\wedge(5/4) = 2\wedge 5 = 1$, we open in $\mathbb{Z}_4$ two new leaves $\mathbb{Z}_{4.2}$ and $\mathbb{Z}_{4.5}$.

In $\mathbb{Z}_{4.2}$:

$$\begin{aligned}
&S(f_1, f_3) = (5/32) f_1 - XY f_3 = \left(\frac{5}{32}\right) - 2XY^2 := f_4,\\
&S(f_1, f_4) = Y f_1 + 4X f_4 = \left(\frac{1}{2}\right) f_3 \xrightarrow{f_3} 0,\\
&S(f_2, f_4) = Y^2 f_2 + 5X^2 f_4 = \left(\left(\frac{5}{8}\right)X - Y\right) f_3 \xrightarrow{f_3} 0,\\
&S(f_3, f_4) = Y^2 f_3 + (5/8) f_4 = 2Y^3 + \left(\frac{25}{256}\right) := f_5,\\
&S(f_1, f_5) = Y^2 f_1 - 4X^2 f_5 = \left(-\left(\frac{5}{16}\right)X + \left(\frac{1}{2}\right)Y\right) f_3 \xrightarrow{f_3} 0,\\
&S(f_2, f_5) = Y^3 f_2 - 5X^3 f_5 = \left(-\left(\frac{25}{64}\right)X^2 + \left(\frac{5}{8}\right)XY - Y^2\right) f_3 \xrightarrow{f_3} 0\ldots,\\
&S(f_3, f_5) = Y^3 f_3 - \left(\frac{5}{8}\right)X f_5 \xrightarrow{f_3} Y f_5 \xrightarrow{f_5} 0, S(f_4, f_5) = Y f_4 + X f_5 = \left(\frac{5}{64}\right) f_3 \xrightarrow{f_3} 0.
\end{aligned} \tag{36}$$

So, $G_1 = \{f_1, f_2, f_3, f_4, f_5\}$ is a special Gröbner basis for $\langle f_1, f_2\rangle$ over $\mathbb{Z}_{4.2}[X,Y]$.

We obtain

$$\operatorname{Syz}(F) = \left\langle \begin{pmatrix} 2 - 10X^3 \\ 1 + 8X^2 \end{pmatrix}, \begin{pmatrix} Y^2 - \frac{5}{8}XY - 5X^3Y^2 + \frac{25}{8}X^4Y \\ \frac{1}{2}Y^2 - \frac{5}{16}XY + 4X^2Y^3 - \frac{5}{2}X^3Y^2 \end{pmatrix} \right\rangle, \text{ over } \mathbb{Z}_{4.2}[X,Y]. \tag{37}$$

In $\mathbb{Z}_{4.5}[X,Y]$, we obtain $G_2 = \{1 + 8X^2, 10X^3 - 2, (5/4)X + 2Y, 1 - (64/5)XY^2, 2Y^3 + (25/256)\}$ as special Gröbner basis for $\langle f_1, f_2\rangle$ over $\mathbb{Z}_{4.5}[X,Y]$.

Also,

$$\operatorname{Syz}(F) = \left\langle \begin{pmatrix} 2 - 10X^3 \\ 1 + 8X^2Y \end{pmatrix}, \begin{pmatrix} Y^2 - \frac{5}{8}XY - 5X^3Y^2 + \frac{25}{8}X^4Y \\ -\frac{5}{16}XY + \frac{1}{2}Y^2 + 4X^2Y^3 - \frac{5}{2}X^3Y^2 \end{pmatrix} \right\rangle, \text{ over } \mathbb{Z}_{4.5}[X,Y]. \tag{38}$$

We find $G_3 = \{1 + 8X^2, 10X^3 - 2, X + (8/5)Y, 1 - (64/5)XY^2, -(512/25)Y^3 - 1\}$ as special Gröbner basis for $\langle f_1, f_2\rangle$ over $\mathbb{Z}_5[X,Y]$.

Also,

$$\operatorname{Syz}(F) = \left\langle \begin{pmatrix} 2 - 10X^3 \\ 1 + 8X^2Y \end{pmatrix}, \begin{pmatrix} \frac{8}{5}XY - \frac{64}{25}Y^2 - 8X^4Y + \frac{64}{5}X^3Y2 \\ \frac{4}{5}XY - \frac{32}{25}Y^2 - \frac{256}{25}X^2Y^3 + \frac{32}{5}X^3Y^2 \end{pmatrix} \right\rangle, \text{ over } \mathbb{Z}_5[X,Y]. \tag{39}$$

Finally, we obtain

$$\operatorname{Syz}(f_1, f_2) = \left\langle \begin{pmatrix} 2 - 10X^3 \\ 1 + 8X^2Y \end{pmatrix}, \begin{pmatrix} 16Y^2 - 10XY - 80X^3Y^2 + 50X^4Y \\ -5XY + 8Y^2 + 64X^2Y^3 - 40X^3Y^2 \end{pmatrix} \right\rangle, \text{ over } \mathbb{Z}[X,Y]. \quad (40)$$

The two approaches used give similar results. In the approach used in the last example, there is one less polynomial than using the isomorphism approach, but the calculations remain similar and relation $\overline{x_1} \dots \overline{x_n} = 1$ allows simplifications with the isomorphism approach.

Data Availability

The data used to support the findings are included within the article and are cited as references.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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