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Research Article

Estimation of the Value at Risk Using the Stochastic Approach of Taylor Formula

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The aim of this paper is to provide an approximation of the value-at-risk of the multivariate copula associated with financial loss and profit function. A higher dimensional extension of the Taylor–Young formula is used for this estimation in a Euclidean space. Moreover, a time-varying and conditional copula is used for the modeling of the VaR.

1. Introduction

A copula function is an instrument of probability theory that makes able to characterize joint dependence. The relationship between the marginal distributions of two or more random variables and the cumulate joint distribution is clarified by the associated copula independently of stochastic behavior of their marginal laws.

As shown by Sklar (1959), see Nelsen [2], a multivariate cumulated distribution function (cdf) $F = (F_1, \ldots, F_n)$ of a random continuous vector $X = (X_1, \ldots, X_n)$ is canonically linked to a unit uniform cdf C_F which can be written as for all $(u_1, \ldots, u_n) \in [0,1]^n$,

$$C_F(u_1, \ldots, u_n) = F_t\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right), \tag{1}$$

where $F_i^{-1}(u) = \inf\{x \in \mathbb{R}, F_i(x) \geq u\}$, for $1 \leq i \leq n$, is the quantile function of F_i. Joe and Xu [1] and Nelsen [2] provided the key standard references for copulas analysis. They provided detailed and readable introductions to copulas and their statistical and mathematical foundations. In the same vein, Bouyé et al. [3], Cherubini et al. [4], and Beirlant et al. [5] dealt with applications of copulas to different levels of financial issues and derivative pricing. The relation (1) is justified since probability integral transformation returns every univariate variable X to the unit uniform random variable, $U = F(X)$.

The value-at-risk (VaR) is also intrinsically linked in its expression to this function, making it possible to bridge the copula function with the VaR. More generally in the multivariate study, for a random vector X satisfying the regularity conditions, one defines the multidimensional VaR at probability level α by

$$\mathrm{VaR}_\alpha(X) = \mathbb{E}[X \mid X \in \partial L(\alpha)], \tag{2}$$

where $\partial L(\alpha)$ is the boundary of the $\alpha-$level set of F_t, and the univariate component of the vector $\mathrm{VaR}_\alpha(X)$ is, for all portfolio X,

$$\mathrm{VaR}_\alpha(X) = \{z \mid F_t(z) \geq \alpha\} = F_t^{-1}(\alpha), \tag{3}$$

where F_t^{-1} being here the right continuous inverse of F_t.

As stated in the summary, the purpose of this paper is to propose another form of the VaR approximation. We went around on the estimates that were proposed, given that there are several methods of estimating the copula. One of the most famous methods is the marginal inference function (IFM), proposed by Carreau and Bengio [6] and Joe and Xu [1]. It consists in separating the estimation of the parameters of the marginal laws and those of the copula. Thus, the global log likelihood can be expressed as follows:

$$L(x,\theta)=\sum_{t=1}^{T}\log\{D_n[C_t(F_1(x_1,\theta_1);\ldots,F_n(x_n,\theta_n);\alpha)]\}$$
$$+\sum_{t=1}^{T}\sum_{i=1}^{n}\log[f_i(x_i,\theta_i)],$$
$$D_n[C_\theta(F_1(x_1,\theta_1);\ldots;F_n(x_n,\theta_n);\alpha)]$$
$$=\frac{\partial^n C_\theta(F_1(x_1,\theta_1),\ldots,F_n(x_n,\theta_n);\alpha)}{\partial u_1\times\cdots\times\partial u_n},\tag{4}$$

with $u_i=F_i(x_i,\theta_i)$, where $\theta=(\theta_1,\ldots,\theta_n;\alpha)$ is the vector of the parameters containing the parameter θ_i of each marginal F_i and the parameter α of the copula. At first, the parameters of each marginal law are estimated independently by the method of maximum likelihood $\widehat{\theta}_i=\arg\max\sum_{i=1}^{T}\ln f_i(x_i,\theta_i)$. Then, we estimate α taking into account the estimates obtained, so we have

$$\widehat{\alpha}=\arg\left\{\max\left(\sum_{i=1}^{T}\log D_n\left[C\left(F_1\left(x_1,\widehat{\theta}_1\right),\ldots,F_n\left(x_n,\widehat{\theta}_n\right);\alpha\right)\right]\right)\right\}.\tag{5}$$

This estimate of the copula is a good approximation. It is widely used in numerical simulations of the copula. We could have used this approximation of the copula. Another idea is to use the limited development formula to get an estimate. Our approach would be better and much more precise. It is in this logic we thought to use the Taylor–Young formula. We will later use a method of limited development theory (Taylor's formula) in this paper to provide an estimate of the value at risk. In calculus, Taylor's theorem gives an approximation of a k-time differentiable function around a given point by a k-th order Taylor polynomial. For analytic functions, the Taylor polynomials at a given point are finite-order truncations of its Taylor series, which completely determines the function in some neighborhood of the point. It can be thought as the extension of linear approximation to higher order polynomials, and in the case of k equals to 2, it is often referred to as a quadratic approximation.

Modeling the risk of multivariate portfolio, it is also on the one hand to choose the appropriate mathematical tools for interpreting the data of a portfolio and, on the other hand, to provide an interface to the financial world. But between these two stages a rigorous and precise work should be carried out.

In this paper, we have given an approximation of the VaR using the Taylor–Young formula. So, we first tested at order 2 before and, subsequently, extended this approximation to higher orders.

2. Preliminaries

The works of Clauss [7], Coles [8], Dean [9], and Nelsen [2] have inspired us a lot on the theory of copulas. Before trying to model our different relationships on VaR, the works of Artzner et al. [10], Lee and Prékopa [11], and Yves Bernadin Loyara et al. [12] make us understand that VaR and its derived measures are measured under consistent risk. The use of copulas in finance has been well illustrated in the works of Böcker and Klüppelberg [13], Cherubini et al. [4], Marius Hofert [14], and Lee and Prékopa [11]. All of these authors have allowed us to understand and develop the relationships that appear in the following sections.

In this section, we have an overview of the key statements (definitions, propositions, and theorems) which will be useful thereafter. Thus, the conditional approach of Sklar's theorem [15] will play a central role as far the concept of scalar product which allows us to highlight a link between the conditional copula and the notion of norms in the metric spaces.

2.1. Scalar Product and Copulas Applications on the VaR. Using the above relation (1) (positiveness of the volume of any hyper-rectangle of $\overline{\mathbb{R}}^n$), Yves Bernadin Loyara and Barro [16] provided the following result by extending a proposition of Patton [15] both to space-varying case and to higher dimensional framework.

Proposition 1. *Let F_{t,w_t} denote the joint distribution of $(\widetilde{X}_{t,n-1},W_t)$, $t\in T$ with $\widetilde{X}_{t,n-1}=(X_{t,1},\ldots,X_{t,n-1})$, and then the conditional time-varying distribution of $(\widetilde{X}_{t,n-1},W_t)$ is given, for all $\widetilde{y}_t\in(\overline{\mathbb{R}})^{n-1}$ by*

$$H_{t,w_t}\left(x_t\mid w_t\right)=f_w^{-1}(w_t)\frac{\partial H_{t,w_t}\left(x_{t,1},\ldots,x_{t,n-1},w_t\right)}{\partial w_t},\tag{6}$$

where f_w is the spatial density of the law of W_t. Moreover, the following properties are satisfied: $H_{t,w_t}(x_{t,1};\ldots;-\infty;\ldots;x_{t,n-1},w_t)=0$ for all $\widetilde{y}_t\in(\overline{\mathbb{R}})^{n-1}$ and $H(\infty,\ldots,\infty/w_t)=1$ for all $\widetilde{y}_t\in(\overline{\mathbb{R}})^{n-1}$.

For all $\widetilde{x}_t^{(1)}=(x_{t,1}^{(1)},\ldots,x_{t,n-1}^{(1)})\in(\overline{\mathbb{R}})^{n-1}$ and $\widetilde{y}_t^{(1)}=(x_{t,1}^{(2)},\ldots,x_{t,n-1}^{(2)})\in(\overline{\mathbb{R}})^{n-1}$, $x_{t,j}^{(1)}\le x_{t,j}^{(2)}$, then

$$\sum_{(i_1,\ldots,i_n)\in\{1,2\}^n}(-1)^{[j=1]n\Sigma j_j}H_{n-1,w_t}\left(x_{t,1}^{(i_1)};\ldots;x_{t,n-1}^{(i_{n-1})},w_t\right)\ge 0.\tag{7}$$

Let us consider a linear portfolio consisting of n different financial instruments (risks and actions) $X=(X_1,\ldots,X_n)$. Furthermore, let $p_0=(p_{0,1},\ldots,p_{0,n})$; the initial value of the portfolio is given by $V_0=\sum_{i=1}^{n}x_ip_{0,i}$ for a realization $x=(x_1,\ldots,x_n)$ of X. At the next date at time t, the uncertain profit and loss functions of the portfolio are given by

$$F_t(x_1,\ldots,x_n)=\sum_{i=1}^{n}x_i\left(p_{0,i}-p_{t,i}\right)=\sum_{i=1}^{n}x_ip_{t,i}\left(e^{z_{t,i}}-1\right),\tag{8}$$

where $Z_t=(z_{t,i},\ldots,z_{t,n})$ is the log price vector, where $z_{t,i}=\log p_{t,i}$.

Particularly, from the integral probability transforms, one can associate with F_t a parametric copula C_t as, for all $(u_1:\ \ldots;u_n)\in[0,1]^n$,

$$C_t(u_1,\ldots,u_n) = P(F_1(X_1)\le u_1;\ldots;F_n(X_n)\le u_n), \quad (9)$$

where $u_i = F_i(x_i)$, for all $i \in \{1,\ldots,n\}$.

Let us consider the particular case where X denotes a portfolio of risks X_i of potential losses in independent lines of business for an assurance company and suppose that $P_t = (p_{1,t}e^{z_{t,1}};\ldots;p_{n,t}e^{z_{n,1}})$. In the same way, Loyara et al. [16] proposed the following result which extended the conditional copula to VaR.

Theorem 1. *For a realization $x = (x_1,\ldots,x_n)$ of X, at the next date at time t, the uncertain profit and loss functions of the portfolio are given by*

$$F_t(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i\left(p_{0,i} - p_{t,i}\right) = \sum_{i=1}^{n} x_i p_{t,i}\left(e^{z_{t,i}} - 1\right). \quad (10)$$

Then,

$$C_t\left(u_{t,1};\ldots;u_{t,n}\right) = \left\|\mathrm{VaR}_u(X)\right\|\left\|P_t - p_t\right\|, \quad (11)$$

$$C_t\left(u_{t,1},\ldots,u_{t,n}\right) = \frac{1}{4}\left(\left\|\mathrm{VaR}_u(X) + P_t - p_t\right\|^2 + \left\|\mathrm{VaR}_u(X) - (P_t - p_t)\right\|^2\right), \quad (12)$$

where $P_t = (p_{1,t}e^{z_{t,1}};\ldots;p_{n,t}e^{z_{n,1}})$ and

$$\mathrm{VaR}_u(X) = \left(\mathrm{VaR}_{u_1}(X_1),\ldots,\mathrm{VaR}_{u_n}(X_n)\right) \quad (13)$$

is a value at risk of X and $\|\cdot\|$ Euclidean norm.

Even in stochastic analysis, the Taylor–Young formula plays a key role in the estimation of the number of functions.

2.2. Taylor–Young Formula in Standardized Vector Spaces. The exact content of "Taylor's theorem" is not universally agreed upon. Indeed, there are several versions of it applicable in different situations, and some of them contain explicit estimates on the approximation error of the function by its Taylor polynomial.

The statement of the most basic version of Taylor's theorem is as follows: let $k \ge 1$ be an integer and let the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be k times differentiable at the point $a \in \mathbb{R}$. Then, there exists a function $h_k\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) = \sum_{k=0}^{n} \frac{(x-a)^k}{k!} D_x^k f(a) + h_k(x)(x-a)^k, \quad (14)$$

with $D_x^k f(a) = \partial^k f(a)/\partial x^k$ and $\lim_{x\longrightarrow a} h_k(x) = 0$.

Let E and F be two normalized vector spaces. If a function $f\colon E \longrightarrow F$ is n times differentiable into a point $a \in E$, then it admits at this point a development limited to the order n, given by

$$f(a+h) = f(a) + D_a f(h) + \frac{1}{2!}D_a^2 f\left(h^2\right) + \cdots + \frac{1}{n!}D_a^n f\left(h^n\right) + o\left(\left\|h^n\right\|\right), \quad (15)$$

where h^k denotes the k-tuple $(h,\ldots,h) \in E^k$.

In the next section, we will use the relation (15) to have a limited development of the time-varying conditional copula function. Then, using the relation (11), we will express an approximation of the value at risk.

3. Main Results

In this subsection, we develop the second-order relation established by Yves Bernadin Loyara and Barro [][16], and we took as a basic notion, probability metric of Rachev et al. [17]. We also use the Taylor–Young relation to obtain an estimate of the value-at-risk norm.

3.1. Our Working Assumptions

(H_1) The space of work is a Euclidean one [12]: so, we can use the characteristic elements of metric spaces and scalar product to clarify the magnitude of the dependence between the VaR and the conditional time-varying copula.

(H_2) In all the following, ϕ denotes a bilinear application mapping $\mathscr{L}([0,1]^2)$ such that for all (u,v) and (h,k) in $[0,1]^n$,

$$\phi((u,v),(h,k)) = (u+h, v+k). \quad (16)$$

(H_3) The copula differentiating formula (1) shows that the density function of the copula is equal to the ratio of the joint density h of H to the product of marginal densities h_i such that for all $(u_1,\ldots,u_n) \in [0,1]^n$,

$$c(u_1,\ldots,u_n) = \frac{\partial^n C(u_1,\ldots,u_n)}{\partial u_1,\ldots,\partial u_m} = \frac{h\left[H_1^{-1}(u_1),\ldots,H_n^{-1}(u_n)\right]}{h_1\left[H_1^{-1}(u_1)\right]\times\cdots\times h_n\left[H_n^{-1}(u_n)\right]}. \quad (17)$$

More preciously, the consequence of the relation (17) is that the copula satisfies the Taylor formula.

3.2. Value-At-Risk Modeling with Time-Varying Conditional Copulas. Let us consider a linear portfolio consisting of 2 different financial instruments (risks and actions) $X = (X_1, X_2)$ and let $p_t = (p_{1,t}, p_{2,t})$ at a given date measured at given time t. Furthermore, let $p_0 = (p_{0,1}; p_{0,2})$, and the initial value of the portfolio is given by $V_0 = x_1 p_{0,1} + x_2 p_{0,2}$ for a realization $x = (x_1, x_2)$ of X. At the next date at time t, the uncertain profit and loss functions of the portfolio are given by

$$F_t(x_1,x_2) = x_1\left(p_{0,1} - p_{t,1}\right) + x_2\left(p_{0,1} - p_{t,1}\right) = x_1 p_{t,1}\left(e^{z_{t,1}} - 1\right) + x_2 p_{t,2}\left(e^{z_{t,2}} - 1\right), \quad (18)$$

where $Z_t = (z_{t,1}, z_{t,2})$ is the log price vector such that $z_{t,i} = \log p_{t,i}$ with $i \in \{1,2\}$. Particularly, from the integral probability transforms, we can associate with F_t a parametric copula C_t such that, for all $(u_1, u_2) \in [0,1]^2$ and $u_i = F_i(x_i)$, $\forall i \in \{1,2\}$,

$$C_t(u_1, u_2) = P(F_1(X_1) \le u_1, F_2(X_2) \le u_2). \tag{19}$$

Proposition 2. *For a realization $x = (x_1, x_2)$ of $X = (X_1, X_2)$, at the next date at time t, the uncertain profit and loss functions of the portfolio are given by relation (18); then*

$$C_t^2(u_1, u_2) = \left(\mathrm{VaR}_{u_1}^2(X_1) + \mathrm{VaR}_{u_2}^2(X_2)\right) \times \left[\left(p_{t,1}\left(e^{z_{t,1}} - 1\right)\right)^2 + \left(p_{t,2}\left(e^{z_{t,2}} - 1\right)\right)^2\right], \tag{20}$$

is a value at risk of X and $\|\cdot\|$ Euclidean norm.

Proof. Consider the following relation:

$$C_t(u_1, u_2) = F_t\left(F_1^{(-1)}(u_1), F_2^{(-1)}(u_2)\right). \tag{21}$$

The relation (3) allows us to write that

$$\mathrm{VaR}_{u_i}(X_i) = F_i^{-1}(u_1), \quad \forall i \in \{1,2\}. \tag{22}$$

If we consider relation (18), we obtain

$$C_t(u_1, u_2) = \mathrm{VaR}_{u_1}(X_1)\left(p_{t,1}e^{z_{t,1}} - p_{t,1}\right) + \mathrm{VaR}_{u_2}(X_2)\left(p_{t,2}e^{z_{t,2}} - p_{t,2}\right). \tag{23}$$

Consider $P_t = (p_{1,t}e^{z_{t,1}}, p_{2,t}e^{z_{t,2}})$ and $\mathrm{VaR}_u(X) = (\mathrm{VaR}_{u_1}(X_1), \mathrm{VaR}_{u_2}(X))$. Then,

$$C_t(u_1, u_2) = \langle \mathrm{VaR}_u(X), P_t - p_t \rangle. \tag{24}$$

The application

$$x \longmapsto \|x\| = \sqrt{\langle x, x \rangle}, \tag{25}$$

defines on E a norm, called Euclidean norm and noted $\| \ \|$. $\forall (x, y) \in E^2$, we have the following:

Cauchy–Schwartz inequality:

$$|\langle x, y \rangle| \le \|x\| \|y\|. \tag{26}$$

Cauchy–Schwartz equality (in the case where (x, y) do not form a free family):

$$|\langle x, y \rangle| = \|x\| \|y\|. \tag{27}$$

Value at risk is intrinsically linked to the portfolio and therefore to the initial amount and the amount at a given time t. We will suppose linked vector $\mathrm{VaR}_u(X)$ and vector $p_t - P_t$.

Based on the relation (27), we obtain

$$\left|\langle \mathrm{VaR}_u(X), P_t - p_t \rangle\right| = \left\|\mathrm{VaR}_u(X)\right\| \left\|P_t - p_t\right\|. \tag{28}$$

Then,

$$C_t(u_1, u_2) = \left\|\mathrm{VaR}_u(X)\right\| \left\|P_t - p_t\right\|. \tag{29}$$

□

3.3. Estimation of the VaR via Taylor–Young Formula of Order 2. We introduce the linear application to avoid having the aspect of the development formula limited in our relation.

Let us consider $\phi \in \mathscr{L}([0,1]^2)$ such that

$$\phi \in \mathscr{L}\left([0,1]^2\right), \quad \text{such that } ((u, v), (h, k)) \longmapsto (u + h, v + k). \tag{30}$$

In short, ϕ is a bilinear application. Under the key assumption (we are in a Euclidean space [][1]), let us consider here $Z_t = (z_{t,1}; z_{t,2})$ is the log price vector such that $z_{t,i} = \log p_{t,i}$ with $i \in \{1,2\}$. Particularly, from the integral probability transforms, we can associate with F_t a parametric copula C_t such that, for all $(u_1, u_2) \in [0,1]^2$ and $u_i = F_i(x_i)$, $\forall i \in \{1,2\}$,

$$C_t(u_1, u_2) = P(F_1(X_1) \le u_1, F_2(X_2) \le u_2). \tag{31}$$

Proposition 3. *Let C_t be a copula of order 2 and $I = [0,1]^2 \subset U$ a subset of $\mathbb{R}^2$ $(u_1, u_2) \in I$, and for any $h \in [0,1]$,*

$$\begin{aligned}\left\|\mathrm{VaR}_u(X)\right\| = {} & \frac{1}{\left[\left(p_{t,1}\left(e^{z_{t,1}} - 1\right)\right)^2 + \left(p_{t,2}\left(e^{z_{t,2}} - 1\right)\right)^2\right]^{1/2}} \\ & \times \left[C_t(\phi((u_1, u_2), (h, h))) - C_\pi(h, 1)\left[D_1 C_t \cdot (u_1, u_2) - DC_t(u_1, u_2)\right] - \frac{1}{2}C_\pi(h, h)\left[\Delta C_t \cdot (u_1, u_2) + c_t(u_1, u_2)\right]\right],\end{aligned} \tag{32}$$

where $\Delta C_t(u_1, u_2)$ and $D_i C_t(u_1, u_2)$, $i = 1,2$, denote, respectively, the Laplacian and the partial derivatives of the copula C_t.

Proof. If we refer to Taylor–Young, α exists such that $\forall (h, k)$ of I with $\|(h, k)\| < \alpha$,

$$\begin{aligned}C_t(u_1 + h, u_2 + k) = {} & C_t(u_1, u_2) + h\frac{\partial C_t}{\partial u}(u_1, u_2) + k\frac{\partial C_t}{\partial v}(u_1, u_2) + \frac{1}{2}\left[h^2\frac{\partial^2 C_t}{\partial u^2}(u_1, u_2) + hk\frac{\partial^2 C_t}{\partial u \partial v}(u_1, u_2) + k^2\frac{\partial^2 C_t}{\partial v^2}(u_1, u_2)\right] \\ & + o\left(\|(h, k)\|^2\right).\end{aligned} \tag{33}$$

Consider $c_t(u_1, u_2) = (\partial^2 C_t / \partial u_1 \partial u_2)(u_1, u_2)$, $D_1 C_t(u_1, u_2) = (\partial C_t / \partial u_1)(u_1, u_2)$ and $D_2 C_t = (\partial C_t / \partial u_2)(u_1, u_2)$, so we have;

$$C_t(u_1+h,u_2+k)=C_t(u_1,u_2)+hD_1C_t(u_1,u_2)+kD_2C_t \cdot (u_1,u_2)+\frac{1}{2}\left[h^2\frac{\partial^2 C_t}{\partial u_1^2}(u_1,u_2)+hk\frac{\partial^2 C_t}{\partial u_1\partial u_2}(u_1,u_2)+k^2\frac{\partial^2 C_t}{\partial u_2^2}(u_1,u_2)\right]+o\left(\|(h,k)\|^2\right). \tag{34}$$

Now, we will use the independent copula $C_\pi(h,k)=hk$ making the composition of ϕ application and the copula C_t. We get the following relationship:

$$C_t(\phi((u,v),(h,k)))=\left(\mathrm{VaR}_{u_1}^2(X_1)+\mathrm{VaR}_{u_2}^2(X_2)\right)^{1/2} \times\left[\left(p_{t,1}(e^{z_{t,1}}-1)\right)^2+\left(p_{t,2}(e^{z_{t,2}}-1)\right)^2\right]^{1/2} +C_\pi(h,1)D_1C_t(u_1,u_2)+C_\pi(1,k)D_2C_t(u_1,u_2) +\frac{1}{2}\left[C_\pi(h,h)\frac{\partial^2 C_t}{\partial u_1^2}(u_1,u_2)+C_\pi(h,k)c_t(u_1,u_2) +C_\pi(k,k)\frac{\partial^2 C_t}{\partial u_2^2}(u_1,u_2)\right]+o\left(\|(h,k)\|^2\right). \tag{35}$$

Moreover, if we consider that $h=k$, it follows

$$C_t(\phi((u,v),(h,h)))=\left(\mathrm{VaR}_{u_1}^2(X_1)+\mathrm{VaR}_{u_2}^2(X_2)\right)^{1/2} \times\left[\left(p_{t,1}(e^{z_{t,1}}-1)\right)^2+\left(p_{t,2}(e^{z_{t,2}}-1)\right)^2\right]^{1/2} +C_\pi(h,1)D_1C_t(u_1,u_2)+C_\pi(1,k)D_2C_t(u_1,u_2) +\frac{1}{2}\left[C_\pi(h,h)\frac{\partial^2 C_t}{\partial u_1^2}(u_1,u_2)+C_\pi(h,k)c_t(u_1,u_2) +C_\pi(h,h)\frac{\partial^2 C_t}{\partial u_2^2}(u_1,u_2)\right]+o\left(\|(h,k)\|^2\right). \tag{36}$$

Let us introduce now the notion of the Laplacian Δ. For definition,

$$\Delta C_t(u_1,u_2)=\frac{\partial^2 C_t}{\partial u_1^2}(u_1,u_2)+\frac{\partial^2 C_t}{\partial u_2^2}(u_1,u_2)$$

$$C_t(\phi((u,v),(h,h)))=\left(\mathrm{VaR}_{u_1}^2(X_1)+\mathrm{VaR}_{u_2}^2(X_2)\right)^{1/2} \times\left[\left(p_{t,1}(e^{z_{t,1}}-1)\right)^2+\left(p_{t,2}(e^{z_{t,2}}-1)\right)^2\right]^{1/2} +C_\pi(h,1)D_1C_t(u_1,u_2)+C_\pi(1,k)D_2C_t(u_1,u_2) +\frac{1}{2}C_\pi(h,h)[\triangle C_t(u_1,u_2)+c_t(u_1,u_2)+]+o\left(\|(h,k)\|^2\right), \tag{37}$$

where

$$c_t(u_1,u_2)=\frac{\partial^2 C_t}{\partial u_1\partial u_2}(u_1,u_2),\quad D_1C_t(u_1,u_2)=\frac{\partial C_t}{\partial u_1}(u_1,u_2),\quad D_2C_t=\frac{\partial C_t}{\partial u_2}(u_1,u_2). \tag{38}$$

From or by drawing the expression of value at risk, we have

$$\mathrm{VaR}_u(X)\simeq\frac{1}{\left[\left(p_{t,1}(e^{z_{t,1}}-1)\right)^2+\left(p_{t,2}(e^{z_{t,2}}-1)\right)^2\right]^{1/2}} \times[C_t(\phi((u_1,u_2),(h,h)))-C_\pi(h,1)[D_1C_t \cdot (u_1,u_2)-DC_t(u_1,u_2)]-\frac{1}{2}C_\pi(h,h)[\Delta C_t \cdot (u_1,u_2)+c_t(u_1,u_2)]], \tag{39}$$

so we get the relation (32). □

3.4. VaR Estimation for Several Variables. In this section, from the integral probability transforms, we can associate with F_t a parametric copula C_t such that, for all $(u_1,\ldots,u_n)\in[0,1]^n$ and $u_i=F_i(x_i)$, $\forall i\in\{1,\ldots,n\}$,

$$C_t(u_{t,1},\ldots,u_{t,n})=P(F_1(X_1)\le u_1,\ldots,F_n(X_n)\le u_n). \tag{40}$$

Theorem 2. *Let copula C_t: $[0,1]^p\longrightarrow[0,1]$ be p times differentiable into a point $u\in[0,1]^p$; then it admits at this point a development limited to the order n, given by*

$$C_t(u+h)=C(u)+\sum_{k=1}^{p}\frac{1}{p!}D_u^kC(h^k)+o\left(\|h^p\|\right), \tag{41}$$

where h^k denotes the k-tuple $(h,\ldots,h)\in E^k$. Then,

$$\|\mathrm{VaR}_u(X)\|=\frac{1}{\|P_t-p_t\|}\times\left[C_t(u+h)-\sum_{k=1}^{p}\frac{1}{k!}D_u^kC(h^k)\right], \tag{42}$$

with $D_u^kC(h^k)=(\partial^kC(h^k)/\partial u^k)$.

Proof. For the proof action of Theorem 1, we will skip the relation (41), i.e.,

$$C_t(u+h)=C(u)+\sum_{k=1}^{p}\frac{1}{p!}D_u^kC(h^k)+o\left(\|h^p\|\right), \tag{43}$$

and $u=(u_1,\ldots,u_p)\in[0,1]^p$.

If we use the relation (11), then the following equation is obtained:

$$C_t(u+h) = \|\mathrm{VaR}_u(X)\|\|P_t - p_t\| + \sum_{k=1}^{p} \frac{1}{p!} D_u^k C\left(h^k\right) + o\left(\|h^p\|\right). \tag{44}$$

Under the key assumption, it follows

$$\|\mathrm{VaR}_u(X)\| \simeq \frac{1}{\|P_t - p_t\|} \times \left[C_t\left(\phi_p(u,h)\right) - \sum_{k=1}^{p} \frac{1}{k!} D_u^k C\left(h^k\right)\right]. \tag{45}$$

Moreover, by pulling the value at risk, the following equation is obtained:

$$\|\mathrm{VaR}_u(X)\| \simeq \frac{1}{\|P_t - p_t\|} \times \left[C_t\left(\phi_p(u,h)\right) - \nabla C_t(u)h - \frac{1}{2}h^T \mathscr{H}(u)h\right]. \tag{46}$$

Let us take another variant of the previous relation (15). For a function C_t: $[0,1]^p \longrightarrow [0,1]$ twice differentiable and $a \in [0,1]^p$, we have

$$C_t(a+h) = C_t(a) + \nabla C_t(a) + h + \frac{1}{2}h^T \mathscr{H}(a)h + o\left(\|h^2\|\right), \tag{47}$$

where ∇C_t is a gradient of C_t and $\mathscr{H}(a)$ is the Hessian matrix at point a.

By considering our third key assumption (H_3), the following characterization of the VaR (47) can be obtained. □

Corollary 1. *Let copula C_t: $[0,1]^p \longrightarrow [0,1]$ be twice differentiable and $u \in [0,1]^p$, and we have*

$$C_t(u+h) = C_t(u) + \nabla C_t(u) + h + \frac{1}{2}h^T \mathscr{H}(u)h + o\left(\|h^2\|\right), \tag{48}$$

where ∇f is a gradient of f and $\mathscr{H}(a)$ Hessian matrix evaluated in a, with the condition $u + h \in [0,1]^p$. Then,

$$\|\mathrm{VaR}_u(X)\| = \frac{1}{\|P_t - p_t\|} \times \left[C_t\left(\phi_p(u,h)\right) - \nabla C_t(u)h - \frac{1}{2}h^T \mathscr{H}(u)h\right]. \tag{49}$$

Proof. For the proof action of Theorem 1, we will skip the relation (48), i.e.,

$$C_t(u+h) = C_t(u) + \nabla C_t(u) + h + \frac{1}{2}h^T \mathscr{H}(u)h + o\left(\|h^2\|\right), \tag{50}$$

with

$$\mathscr{H}(u) = \begin{bmatrix} \frac{\partial^2 C_t}{\partial u_1^2} & \frac{\partial^2 C_t}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 C_t}{\partial u_1 \partial u_p} \\ \frac{\partial^2 C_t}{\partial u_2 \partial u_1} & \frac{\partial^2 C_t}{\partial u_2^2} & & \frac{\partial^2 C_t}{\partial u_2 \partial u_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 C_t}{\partial u_p \partial u_1} & \frac{\partial^2 C_t}{\partial u_p \partial u_2} & & \frac{\partial^2 C_t}{\partial u_p^2} \end{bmatrix}, \tag{51}$$

and $u = (u_1, \ldots, u_p) \in [0,1]^p$.

If we used the relation (11), the following equation is obtained:

$$\begin{aligned} C_t(u+h) &= \|\mathrm{VaR}_u(X)\|\|P_t - p_t\| + \nabla C_t(u) + h \\ &\quad + \frac{1}{2}h^T \mathscr{H}(u)h + o\left(\|h^2\|\right). \end{aligned} \tag{52}$$

We are going to suppose later that we are in the conditions suggested by Loyara et al. That is to say, the conditions to verify by a Euclidean space, we have

$$\begin{aligned} C_t\left(\phi_p(u,h)\right) &= \|\mathrm{VaR}_u(X)\|P_t - p_t + \nabla C_t(u) + h \\ &\quad + \frac{1}{2}h^T \mathscr{H}(u)h + o\left(\|h^2\|\right). \end{aligned} \tag{53}$$

By pulling the value at risk, it comes that

$$\|\mathrm{VaR}_u(X)\| \simeq \frac{1}{\|P_t - p_t\|} \times \left[C_t\left(\phi_p(u,h)\right) - \nabla C_t(u)h - \frac{1}{2}h^T \mathscr{H}(u)h\right]. \tag{54}$$

□

4. Conclusion and Discussion

In this paper, we have introduced the notion of product in the stochastic modeling of the copula and the VaR. Under the first working hypothesis, we have estimated the VaR via the copula of both in the dimensional and higher dependence considerations. It becomes quite easy to calculate the values of VaR when we know the value of the copula and the type of copula.

The main limitation of the VaR lies in the fact that whatever the method used, the data injected into the calculation algorithm always come more or less from the market values found in the past, which are not necessarily a reflection of the evolutions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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