Research Article

A Relation between D -Index and Wiener Index for r -Regular Graphs

Ahmed Mohammed Ali and Asmaa Salah Aziz

Department of Mathematics, College of Computer Sciences and Mathematics, Mosul University, Mosul, Iraq

Correspondence should be addressed to Asmaa Salah Aziz; asmaas982@uomosul.edu.iq

Received 25 November 2019; Accepted 13 January 2020; Published 22 February 2020

Academic Editor: Vladimir V. Mityushev

Copyright © 2020 Ahmed Mohammed Ali and Asmaa Salah Aziz. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

For any two distinct vertices u and v in a connected graph G , let $l_p(u, v) = l(P)$ be the length of $u - v$ path P and the D -distance between u and v of G is defined as: $d^D(u, v) = \min_p \{l(P) + \sum_{y \in V(P)} \deg y\}$, where the minimum is taken over all $u - v$ paths P and the sum is taken over all vertices of $u - v$ path P . The D -index of G is defined as $W^D(G) = (1/2) \sum_{u, v \in V(G)} d^D(u, v)$. In this paper, we found a general formula that links the Wiener index with D -index of a regular graph G . Moreover, we obtained different formulas of many special irregular graphs.

1. Introduction

Throughout this paper, the nontrivial connected graphs G were considered to be finite without multiple edges and loops [1]. As usual, $V = V(G)$ which denotes the set of vertices of G and $E = E(G)$ represents the set of edges of G , the order of G is $p = p(G) = |V(G)|$ and the size of G is $q = q(G) = |E(G)|$. The vertex degree $\deg_G v$ or $\deg v$ is referred to the number of vertices incident to the vertex v .

The standard distance between any two arbitrary vertices u and v of G is the length of a shortest path joining u to v , and it is denoted by $d_G(u, v)$ or $d(u, v)$. The diameter of a connected graph G denoted by $\text{diam } G$ is the length of any longest geodesic [2]. Many researchers have determined different concepts of distances as well as ordinary distance, such as Steiner distance [3], width distance [4], superior distance [5], and signal distance [6]. All distances mentioned above depend on the path length in a connected graph G .

In the present work, the concept of D -distance between distinct vertices of G was considered, in which it depends on the length of a path P as well as the degrees of the vertices that lie on P [7].

“The Wiener index of a graph G is represented by $W(G)$ and defined as the sum of distances between all pairs of vertices in a simple graph G ”:

$$W(G) = \left(\frac{1}{2}\right) \sum_{u, v \in V(G)} d(u, v). \quad (1)$$

Based on the Wiener index, Hosoya introduced the Wiener polynomial (now called Hosoya polynomial) in 1988 [8]. The Wiener polynomial $W(G; x)$ is defined as

$$W(G; x) = \left(\frac{1}{2}\right) \sum_{u, v \in V(G)} d(u, v) x^{d(u, v)}. \quad (2)$$

There are a lot of papers on finding general formulas for Wiener polynomials for different types of distances (see [9, 10]).

Definition 1 (see [7]). “The D -distance $d^D(u, v)$ between two vertices u and v of a connected graph G is defined as

$d^D(u, v) = \min_p \{l(P) + \sum_{y \in V(P)} \deg y\}$, where $l(P)$ is the length of $u - v$ path P and the minimum is taken over all $u - v$ paths P in G ”.

The total D -distance or D -index of a connected graph G is defined as

$$W^D(G) = \left(\frac{1}{2}\right) \sum_{u, v \in V(G)} d^D(u, v). \quad (3)$$

Definition 2 (see [1]). It can be said that the graph G is r -regular if every vertex in G has r degree, that is, $\deg v = r$, for all $v \in V(G)$, otherwise the graph G is irregular.

Definition 3 (see [1]). A friendship graph F_n is a graph of odd order n , $n \geq 7$, which consists of $(n-1)/2$ triangles with a common vertex.

Let G_1 and G_2 be nontrivial disjoint connected graphs, then the Cartesian product graph $G_1 \times G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set:

$$E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2): u_1 u_2 \in E(G_1) \text{ and } v_1 = v_2, \text{ or } v_1 v_2 \in E(G_2) \text{ and } u_1 = u_2\}, \quad (4)$$

and the strong product graph $G_1 * G_2$ has vertex set $V(G_1) \times V(G_2)$ and edge set:

$$E(G_1 * G_2) = \{(u_1, v_1)(u_2, v_2): u_1 u_2 \in E(G_1) \text{ and } v_1 v_2 \in E(G_2), \text{ or } u_1 u_2 \in E(G_1) \text{ and } v_1 = v_2, \text{ or } v_1 v_2 \in E(G_2) \text{ and } u_1 = u_2\}. \quad (5)$$

There are several papers about the D -distance and the average of this distance, see References [11, 12, 13, 14, 15].

In this work, a relationship between the Wiener index and D -index of any regular graph G was found. Furthermore, the D -index of some special graphs was obtained.

2. Preliminaries

In this section, we will mention some of the results that were found previously and others that will be established.

Theorem 1 (see [16])

- (1) $W(K_n) = \binom{n}{2}$.
- (2) $W(P_n) = (1/6)n(n^2 - 1)$.
- (3) $W(C_n) = (n/8) \begin{cases} n^2, & \text{if } n \text{ is an even,} \\ (n^2 - 1), & \text{if } n \text{ is an odd.} \end{cases}$
- (4) $W(K_{n,n}) = n(3n^2 - 2)$.
- (5) $W(Q_n) = n2^{2n-2}$.

Theorem 2 (see [16, 17]). If G_1 and G_2 are nontrivial disjoint connected graphs, then

- (1) $W(G_1 \times G_2; x) = 2W(G_1; x)W(G_2; x) - p(G_1)W(G_2; x) - p(G_2)W(G_1; x) + p(G_1)p(G_2)$.
- (2) If $W(G_1; x) = \sum_{k=0}^{\alpha} a_k x^k$ and $W(G_2; x) = \sum_{k=0}^{\beta} b_k x^k$ in which $\alpha = \text{diam } G_1$ and $\beta = \text{diam } G_2$. Then, $W(G_1 * G_2; x) = \sum_{k=0}^{\delta} c_k x^k$, where $\delta = \text{diam}(G_1 * G_2) = \max\{\alpha, \beta\}$, $c_0 = a_0 b_0$, $c_1 = a_0 b_1 + a_1 b_0 + 2a_1 b_1$, and $c_k = a_0 b_k + a_k b_0 + 2a_k \sum_{i=1}^k b_i + 2b_k \sum_{i=1}^{k-1} a_i$.

Lemma 1. For $n \geq 4$, we have

- (1) $W(K_2 \times C_n) = \begin{cases} (1/2)n^2(n+2), & \text{if } n \text{ is an even,} \\ (1/2)n(n^2+2n-1), & \text{if } n \text{ is odd.} \end{cases}$

$$(2) W(K_2 * C_n) = \begin{cases} (1/2)n^2(n+2), & \text{if } n \text{ is an even,} \\ (1/2)n(n^2+1), & \text{if } n \text{ is an odd.} \end{cases}$$

Proof

- (1) From Theorem 2 (18), if $G_1 = K_2$ and $G_2 = C_n$, $n \geq 4$, then

$$\begin{aligned} W(K_2 \times C_n; x) &= 2(2+x)W(C_n; x) \\ &\quad - 2W(C_n; x) - n(2+x) + 2n \\ &= 2W(C_n; x) + 2xW(C_n; x) - nx. \end{aligned} \quad (6)$$

Taking the derivative with respect to x of both sides and then putting $x = 1$, we get

$$W(K_2 \times C_n) = 4W(C_n) + n^2. \quad (7)$$

From Theorem 1 (20), we get

$$W(K_2 \times C_n) = \begin{cases} \left(\frac{1}{2}\right)n^2(n+2), & \text{if } n \text{ is an even,} \\ \left(\frac{1}{2}\right)n(n^2+2n-1), & \text{if } n \text{ is odd.} \end{cases} \quad (8)$$

- (2) From Theorem 2 (19), if $G_1 = K_2$ and $G_2 = C_n$, $n \geq 4$, then

$$\begin{aligned} W(K_2 * C_n; x) &= 2n + 5nx + \sum_{k=2}^{\delta(C_n)} 4b_k x^k \\ &= 4W(C_n; x) - 2n + nx. \end{aligned} \quad (9)$$

Taking the derivative with respect to x of both sides and then putting $x = 1$, we get

$$W(K_2 * C_n) = 4W(C_n) + n. \quad (10)$$

From Theorem 1 (19), we get

$$W(K_2 * C_n) = \begin{cases} \left(\frac{1}{2}\right)n(n^2+2), & \text{if } n \text{ is an even,} \\ \left(\frac{1}{2}\right)n(n^2+1), & \text{if } n \text{ is odd.} \end{cases} \quad (11)$$

□

3. The Index of D -Distance for a Connected r -Regular Graph

Theorem 3. For any connected r -regular graph G of order n , we have

$$W^D(G) = (r+1)W(G) + r \binom{n}{2}. \quad (12)$$

Proof

$$\begin{aligned}
 W^D(G) &= \left(\frac{1}{2}\right) \sum_{u,v \in V(G)} d^D(u, v) \\
 &= \frac{1}{2} \sum_{u,v \in V(G)} \left(\min_p \left\{ l(P) + \sum_{y \in V(P)} \deg y \right\} \right) \\
 &= \frac{1}{2} \sum_{u,v \in V(G)} \left\{ d(u, v) + \sum_{y \in V(P: u-v \text{ geodesic})} \deg y \right\} \quad (\text{because is regular}) \\
 &= \frac{1}{2} \sum_{u,v \in V(G)} \{d(u, v)\} + \frac{1}{2} \sum_{u,v \in V(G)} \left\{ \sum_{y \in V(P: u-v \text{ geodesic})} \deg y \right\} \\
 &= W(G) + \frac{1}{2} \sum_{u,v \in V(G)} \left\{ \sum_{y \in V(P: u-v \text{ geodesic})} \deg y \right\}.
 \end{aligned} \tag{13}$$

Since G is r -regular, then $\deg y = r, \forall y \in V(G)$, and every shortest path between u and v contains $(d(u, v) + 1)$ vertices with every vertex of degree r .

$$\begin{aligned}
 W^D(G) &= W(G) + \frac{1}{2} \sum_{u,v \in V(G)} (d(u, v) + 1)r \\
 &= W(G) + rW(G) + \frac{r}{2} \sum_{u,v \in V(G)} 1 \\
 &= (r+1)W(G) + r \binom{n}{2},
 \end{aligned} \tag{14}$$

$$\text{Hence } W^D(G) = (r+1)W(G) + r \binom{n}{2}.$$

□

Example 1. Let G be a Petersen graph of ten orders, as shown in Figure 1.

It is clear that $W(P_{10}) = 75$, then $W^D(P_{10}) = 4(75) + 3 \binom{10}{2} = 435$.

Corollary 1

$$\begin{aligned}
 (1) \quad W^D(K_n) &= (2n-1) \binom{n}{2}. \\
 (2) \quad W^D(C_n) &= n(n-1) + (3n/8) \begin{cases} n^2, & \text{if } n \text{ is an even,} \\ n^2-1, & \text{if } n \text{ is an odd.} \end{cases} \\
 (3) \quad W^D(K_{n,n}) &= n(5n^2-2). \\
 (4) \quad W^D(Q_n) &= n2^n[2^{n-1}(n+3)-1].
 \end{aligned}$$

Proof. From Theorems 1 and 3, we get results listed in this corollary. □

Corollary 2. For $n \geq 4$, we have

$$(1) \quad W^D(K_2 \times C_n) = \begin{cases} (1/2)n(4n^2 + 11n - 3), & \text{if } n \text{ is an even,} \\ (1/2)n(4n^2 + 11n - 7), & \text{if } n \text{ is an odd.} \end{cases}$$

$$(2) \quad W^D(K_2 * C_n) = \begin{cases} (1/2)n(6n^2 + 5n + 7), & \text{if } n \text{ is an even,} \\ (1/2)n(6n^2 + 5n + 1), & \text{if } n \text{ is an odd.} \end{cases}$$

Proof. Since from $K_2 \times C_n$ and $K_2 * C_n$ are all 3-regular and 5-regular, respectively, shown Figure 2, then from Lemma 1 and Theorem 3, we get the formulas given in Corollary 2. □

4. The Index of D -Distance for Some Special Graph

In this section, we find the D -index of some special irregular graphs, such as the path graph P_n , $n \geq 2$, the star graph S_n , $n \geq 4$, the friendship graph F_n , of order odd n , $n \geq 7$, the wheel graph W_n , $n \geq 6$, and the bipartite complete graph $K_{m,n}$, $m \neq n$, $m, n \geq 2$.

Theorem 4

$$\begin{aligned}
 (1) \quad W^D(P_n) &= (1/2)n^3 + n^2 - (7/2)n + 2. \\
 (2) \quad W^D(S_n) &= (1/2)(n-1)^2(n+4).
 \end{aligned}$$

Proof

(1) Let $P_n = (v_1, v_2, v_3, \dots, v_{n-1}, v_n)$ be a path graph of order $n \geq 2$.

Since $d(v_1, v_i) = i-1$, then $d^D(v_1, v_i) = 1 + 3(i-1)$, $i = 2, 3, \dots, n-1$. Also, by symmetric, $d^D(v_n, v_i) = d^D(v_1, v_i) = 1 + 3(i-1)$, for all $i = 2, 3, \dots, n-1$, and $d^D(v_1, v_n) = 3(n-1)$.

Since $d(v_i, v_j) = j-i$, $i = 2, 3, \dots, n-2$, $j = i+1, \dots, n-1$, then $d^D(v_i, v_j) = 3(j-i) + 2$, for all $i = 2, 3, \dots, n-2$, $j = i+1, \dots, n-1$. Hence,

FIGURE 1

$$\begin{aligned}
 W^D(P_n) &= \sum_{i=2}^{n-1} d^D(v_1, v_i) + \sum_{i=2}^{n-1} d^D(v_n, v_i) \\
 &\quad + d^D(v_1, v_n) + \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} d^D(v_i, v_j) \\
 &= 3n^2 - 4n - 1 + \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} [3(j-i) + 2] \\
 &= 3n^2 - 4n - 1 + 3 \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} (j-i) + 2 \sum_{i=2}^{n-2} \sum_{j=i+1}^{n-1} 1 \\
 &= 3n^2 - 4n - 1 + \frac{1}{2}(n^3 - 6n^2 + 11n - 6) + n^2 - 5n + 6 \\
 &= \frac{1}{2}n^3 + n^2 - \frac{7}{2}n + 2.
 \end{aligned}$$

(15)

- (2) Let S_n be a star graph with order n and let v_1 be the center of S_n . It is obvious that

$$\begin{aligned}
 d(v_1, v) &= 1, \\
 \deg v_1 &= n-1, \\
 \deg v &= 1, \forall v \in V(S_n - \{v_1\}),
 \end{aligned}$$

(16)

then $d^D(v_1, v) = n+1$.

Now, since the distance for each pair of end vertices is equal two, then $d^D(u, v) = n+3$.

$$\begin{aligned}
 \text{Hence, } W^D(S_n) &= (n-1)(n+1) + (1/2)(n-1)(n-2) \\
 (n+3) &= (1/2)(n-1)^2(n+4).
 \end{aligned}$$

Theorem 5

- (1) $W^D(F_n) = (1/2)(n-1)(n^2 + 4n - 6)$.
 (2) $W^D(W_n) = (n-1)\{n+3 + ((n-1)/2 - 1)(2((n-1)/2 + 3))\} + \alpha(n-1)$,
 where $\alpha = \begin{cases} (2n-1), & n \text{ is an even,} \\ (n+7)/2, & n \text{ is odd.} \end{cases}$
 (3) $W^D(K_{m,n}) = m^3 + n^3 + (3/2)mn(n+m) - (n+m)$.

The graph $K_2 \times C_n$

(a)

The graph $K_2 * C_n$

(b)

FIGURE 2

Proof

- (1) Let $V(F_n) = \{v_1, v_2, v_3, \dots, v_n\}$, where $\deg v_1 = n-1$ and $\deg v_i = 2$, for all $i = 2, 3, \dots, n$. Then, there are three cases:

Case 1: if $v_1 v_i \in E(F_n)$, $i = 2, 3, \dots, n$, then $d^D(v_1, v_i) = n+2$, for all $i = 2, 3, \dots, n$.

Case 2: if $v_{2i} v_{2i+1} \in E(F_n)$, $i = 1, 2, \dots, (n-1)/2$, then $d^D(v_{2i}, v_{2i+1}) = 5$, and the number of pairs of (v_{2i}, v_{2i+1}) , when $v_{2i} v_{2i+1} \in E(F_n)$, for all $i = 1, 2, \dots, (n-1)/2$, is $(n-1)/2$.

Case 3: if $v_{2i-1} v_{2i} \notin E(F_n)$, $i = 2, 3, \dots, (n-1)/2$, then the path P consists from v_{2i-1}, v_1, v_{2i} . Hence, $d^D(v_{2i-1}, v_{2i}) = n+5$. The number of pairs of (v_{2i-1}, v_{2i}) when $v_{2i-1} v_{2i} \notin E(F_n)$, $i = 2, 3, \dots, (n-1)/2$, is $(n-1)(n-3)/2$.

From the three previous cases, we get

$$\begin{aligned}
 W^D(F_n) &= (n-1)(n+2) + 5 \frac{(n-1)}{2} \\
 &\quad + \frac{(n+5)(n-1)(n-3)}{2} \\
 &= (n-1)(n^2 + 4n - 6)/2.
 \end{aligned}$$

(17)

- (2) Let $V(W_n) = \{v_1, v_2, v_3, \dots, v_n\}$, where $\deg v_1 = n-1$ and $\deg v_i = 3$, for all $i = 2, 3, \dots, n$.

It is clear that

$$d^D(v_1, v_i) = n + 3. \quad (18)$$

Now, let $i < j$, for all $i, j = 2, 3, \dots, n$, then there are two cases:

Case 1: if $j - i = 1, 2, \dots, ((n - 1)/2) - 1$, then

$$\begin{aligned} d^D(v_i, v_j) &= d(v_i, v_j) + \sum_{k=i}^j \deg v_k \\ &= j - i + 3 \sum_{k=i}^j 1 = 4(j - i) + 3, \end{aligned} \quad (19)$$

where the number of such pairs (v_i, v_j) is $(n - 1)$.

Case 2: if $j - i = ((n - 1)/2)$ and n is an even, then

$$d^D(v_i, v_j) = 4 \left(\frac{(n - 1)}{2} \right) + 3. \quad (20)$$

Also, the number of such pairs (v_i, v_j) is $(n - 1)$.

But when n is odd, then

$$d^D(v_i, v_j) = n + 7, \quad (21)$$

and the number of such pairs is $((n - 1)/2)$.

Hence, from (18)–(21), we have the formula in Theorem 5 part 2.

- (3) Let $V(K_{m,n}) = U \cup W$, where $U = \{u_1, u_2, \dots, u_m\}$ and $W = \{w_1, w_2, \dots, w_n\}$.

The following cases are true in this graph:

Case 1: if $v, v' \in U$, then $d^D(v, v') = 2 + 2n + m$, this is true for all $m((m - 1)/2)$ pairs of vertices.

Case 2: if $v, v' \in W$, then $d^D(v, v') = 2 + 2m + n$, and also this is true for all $n((n - 1)/2)$ pairs of vertices.

Case 3: if $v \in U$ and $v' \in W$, then $d^D(v, v') = 1 + m + n$ and there are mn pairs of vertices in this case that is true.

From three cases, we conclude that

$$\begin{aligned} W^D(K_{m,n}) &= \frac{1}{2} \left\{ m^2(m + 1) + n^2(n + 1) \right. \\ &\quad \left. + 2(m + n)(2mn - 1) - 2mn \right\}. \end{aligned} \quad (22)$$

□

5. Conclusion

In this paper, we find a general formula giving a relation between the Wiener index and D -index for any regular graph, also we find the D -index for some special graphs.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors called for a lack of conflicts of interest to publish this paper.

Acknowledgments

This paper was supported by the College Computer Sciences and Mathematics, University of Mosul, Republic of Iraq.

References

- [1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Wadsworth and Brooks/Cole, Pacific Grove, CA, USA, 6th edition, 2016.
- [2] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Boston, MA, USA, 1990.
- [3] A. A. Ali and W. A. Saeed, "Wiener polynomials of steiner distance of graphs," *Journal of Applied Sciences*, vol. 8, no. 2, pp. 64–71, 2006.
- [4] D. F. Hsu, "On container width and length in graphs, groups and networks," *IEICE Transactions on Fundamentals of Electronics Communications and Computer Sciences*, vol. E77-A, no. 4, pp. 668–680, 1994.
- [5] K. M. Kathiresan and G. Marimnthn, "Superior distance in graphs," *Journal of combinatorial mathematics and combinatorial computing*, vol. 61, pp. 73–80, 2007.
- [6] K. M. Kathiresan and R. Sumathi, *A Study Signal Distance in Graphs, Algebra, Graph Theory and Their Applications*, pp. 50–54, Narosa Publishing House Pvt. Ltd, New Delhi, India, 2010.
- [7] D. Reddy Babu and P. L. N. Varma, " D -distance in graphs," *Golden Research Thoughts*, vol. 2, pp. 53–58, 2013.
- [8] H. Hosoya, "On some counting polynomials in chemistry," *Discrete Applied Mathematics*, vol. 19, no. 1–3, pp. 239–257, 1988.
- [9] M. A. Ahmed and N. M. Haitham, "Schultz and modified schultz polynomials of some cog-special graphs," *Open Access Library Journal*, vol. 6, no. 8, pp. 1–13, 2019.
- [10] A. A. Ali and M. A. Ahmed, "Hosoya polynomials of pentachains," *MATCH Communications in Mathematical and in Computer Chemistry*, vol. 65, pp. 807–819, 2011.
- [11] A. Anto Kinsley and P. Siva Ananthi, " Dd -distance in graphs," *Imperial Journal of Interdisciplinary Research*, vol. 3, pp. 1457–1459, 2017.
- [12] D. Reddy Babu and P. L. N. Varma, "Average D -distance between vertices of a graph," *Italian Journal of Pure and Applied Mathematics*, vol. 33, pp. 293–298, 2014.
- [13] D. Reddy Babu and P. L. N. Varma, "A note on radius and diameter of a graph W.R.T. D -distance," *International journal of chemical science*, vol. 14, no. 3, pp. 1725–1729, 2016.
- [14] J. Veeranjanyulu and P. L. N. Varma, "Circular D -distance and path graphs," *International Journal of Recent Technology and Engineering*, vol. 7, pp. 219–223, 2019.
- [15] V. Venkateswara Rao, P. L. N. Varm, and D. Reddy Babu, "Average eccentricity of graphs W.R.T. D distance,"

International Journal of Recent Technology and Engineering,
vol. 7, pp. 208–210, 2019.

- [16] B. E. Sagan, Y.-N. Yeh, and P. Zhang, “The wiener polynomial of a graph,” *International Journal of Quantum Chemistry*, vol. 60, no. 5, pp. 959–969, 1996.
- [17] W. A. Saeed, *Wiener polynomials of graphs*, Ph.D. thesis, Mosul University, Mosul, Iraq, 1999.