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Research Article

On the Construction of the Reflexive Vertex k-Labeling of Any Graph with Pendant Vertex

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A total k-labeling is a function f_e from the edge set to first natural number k_e and a function f_v from the vertex set to non negative even number up to $2k_v$, where $k = \max\{k_e, 2k_v\}$. A *vertex irregular reflexive k-labeling* of a simple, undirected, and finite graph G is total k-labeling, if for every two different vertices x and x' of G, $wt(x) \neq wt(x')$, where $wt(x) = f_v(x) + \Sigma_{xy\in E(G)} f_e(xy)$. The minimum k for graph G which has a vertex irregular reflexive k-labeling is called the reflexive vertex strength of the graph G, denoted by $\mathrm{rvs}(G)$. In this paper, we determined the exact value of the reflexive vertex strength of any graph with pendant vertex which is useful to analyse the reflexive vertex strength on sunlet graph, helm graph, subdivided star graph, and broom graph.

1. Introduction

We consider a simple and finite graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. We motivate the readers to refer Chartrand et al. [1], for detailed definition of the graph. A topic in graph theory which has grown fast is the labeling of graphs. The concept of graph labeling, firstly, was introduced by Wallis in [2]. He defined a labeling of G is a mapping that carries a set of graph elements into a set of integers called labels. By this definition, we can have a vertex label, edge label, or both of them. Baca et al. [3] introduced the total labeling, and they defined the vertex weight as the sum of all incident edge labels along with the label of the vertices. Many types of labeling have been studied by researchers, namely, graceful labeling, magic labeling, antimagic labeling, irregular labeling, and irregular reflexive labeling.

Furthermore, labeling known as a vertex irregular total k-labeling and total vertex irregularity strength of graph is the minimum k for which the graph has a vertex irregular total k-labeling. The bounds for the total vertex irregularity strength are given in [3]. In [4], Tanna et al. identified the concept of vertex irregular reflexive labeling of graphs. In this paper, we continue to study a vertex irregular reflexive labeling as there are still many open problems. By irregular reflexive labeling, we mean a labeling of graph which the vertex labels are assigned by even numbers from $0, 2, \ldots, 2k$ and the edge labels are assigned by $1, 2, 3, \ldots, k$, where k is positive integer. The weight of each vertex, under a total labeling, is determined by summing the incident edge labels and the label of the vertex itself.

A k-labeling assigns numbers $\{1, 2, \ldots, k\}$ to the elements of graph. Let k be a natural number, a function $f\colon V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, k\}$ is called total k-irregular labeling. Hinding et al. [5] defined that a total labeling $\phi\colon V(G) \cup E(G) \longrightarrow \{1, 2, 3, \ldots, k\}$ is called vertex irregular total k-labeling of graph G if the vertex weight $wt_\phi(x) = \phi(x) + \Sigma_{xy\in E(G)} \phi(xy)$ is distinct for every two different vertices, $w_{t_\phi}(x) \neq w_{t_\phi}(y)$ for $x, y \in V(G), x \neq y$. The minimum k for which graph G has a vertex irregular total k-labeling is called total vertex irregularity strength, denoted by $\mathrm{tvs}(G)$.

The concept of vertex irregular total k-labeling extends to a vertex irregular reflexive total k-labeling. The definition of total k-labeling is a function f_e from the edge set to the first natural number k_e and a function f_v from the vertex set to the nonnegative even number up to $2k_v$, where $k = \max\{k_e, 2k_v\}$. A *vertex irregular reflexive k-labeling* of the graph G is the total k-labeling, for every two different vertices x and x' of G, $w_t(x) \neq w_t(x')$, where $w_t(x) = f_v(x) + \Sigma_{xy\in E(G)} f_e(xy)$. The minimum k for graph G which has a vertex irregular reflexive k-labeling is called the reflexive vertex strength of the graph G, denoted by $\text{rvs}(G)$.

Some results related to vertex irregular reflexive labeling have been studied by several researchers. Tanna et al. [4] have studied the vertex irregular reflexive of prism and wheel graphs, Ahmad and Bača [6] have studied the total vertex irregularity strength for two families of graphs, namely, Jahangir graphs and circulant graphs, and Agustin et al. [7] also study the concept of vertex irregular reflexive labeling of cycle graph and generalized friendship. Another results of irregular labeling can be seen on [8–15]. In this paper, we have found the lower bound of vertex irregular reflexive strength of any graph G and determined the vertex irregular reflexive strength of graphs with pendant vertex. Our results are started by showing one lemma and theorem which describe a general construction of the existence of vertex irregular reflexive k-labeling of graph with pendant vertex.

2. Result and Discussion

The following lemma and theorem will be used as a base construction of analysing the reflexive vertex strength of any graph with pendant vertex, namely, sunlet graph, helm graph, subdivided star graph, and broom graph.

Lemma 1. *For any graph G of order p, the minimum degree δ, and the maximum degree Δ,*

$$\text{rvs}(G) \geq \left\lceil \frac{p+\delta-1}{\Delta+1} \right\rceil. \quad (1)$$

Proof. Let G be a graph of order p, the minimum degree δ, and the maximum degree Δ. The total k-labeling which labeling f defined $f_e\colon E(G) \longrightarrow \{1, 2, \ldots, k_e\}$ and $f_v\colon V(G) \longrightarrow \{0, 2, \ldots, 2k_v\}$ such that $f(x) = f_v(x)$ if $x \in V(G)$ and $f(x) = f_e(x)$ if $x \in E(G)$, where $k = \max\{k_e, 2k_v\}$. The total k-labeling f is called a vertex irregular reflexive k-labeling of the graph G if every two different vertices x and x', and it holds $\text{wt}(x) \neq \text{wt}(x')$, where $\text{wt}(x) = f_v(x) + \Sigma_{xy\in E(G)} f_e(xy)$. Furthermore, since we require k-minimum for the graph G which has a vertex irregular reflexive labeling, the set of a vertex weight should be consecutive, otherwise it will not give a minimum rvs. Thus, the set of a vertex weight is $\text{Wt}(x) = \{\delta, \delta+1, \delta+2, \ldots, 1(2k_v) + \Delta k_e\}$. Since the minimum $k = \max\{k_e, 2k_v\}$ is the reflexive vertex strength, the maximum possible vertex weight of graph G is at most $k(1+\Delta)$. It implies $2k_v + \Delta k_e \geq \delta + (p-1).1 \leftrightarrow k + \Delta k \geq \delta + (p-1) \leftrightarrow k \geq \delta + p - 1/\Delta + 1$. Since $\text{rvs}(G)$ should be integer and we need a sharpest lower bound, it implies

$$\text{rvs}(G) \geq \left\lceil \frac{\delta+p-1}{\Delta+1} \right\rceil. \quad (2)$$

It completes the proof.

Theorem 1. *Let G be a graph of order n and contains l pendant vertex. If $l \geq n-l$, then*

$$\text{rvs}(G) = \begin{cases} \left\lceil \dfrac{l}{2} \right\rceil + 1, & \text{for } l \text{ even and } \dfrac{l}{2} \text{ odd}, \\ \left\lceil \dfrac{l}{2} \right\rceil, & \text{otherwise}. \end{cases} \quad (3)$$

Proof. Given that a graph G of order n is with l pendant vertices. The labeling of graph G is with respect to two components, namely, the pendant vertices and otherwise vertices. Thus, we will split our proof into two cases.

Case 1. Let V_l be a set of pendant vertices and the number of V_l is l. A pendant vertex consists of two elements, i.e., a vertex and an edge. The vertex weight on each pendant vertex must be different. Suppose we choose those vertex weights are $1, 2, \ldots, l$. Those vertex weights are obtained by summing the vertex and edge labels. To prove the above $\text{rvs}(G)$, let us suppose the maximum vertex weight of l. Let

$$\left\lceil \frac{l}{2} \right\rceil = \begin{cases} \dfrac{l+1}{2}, & \text{for } l \text{ odd}, \\ \dfrac{l}{2}, & \text{for } l \text{ even}. \end{cases} \quad (4)$$

Define an injection f as the labels. Since the weight l is contributed by one vertex and one edge labels, it will give four possibilities.

(1) If $l + 1/2$ is odd number, then $f(v) = (l + 1/2 - 1)$ and $f(e) = l + 1/2$, such that the vertex weight is $(l + 1/2 - 1) + (l + 1/2) = l + 1 - 1 = l$
(2) If $l + 1/2$ is even number, then $f(v) = (l + 1/2)$ and $f(e) = l + 1/2$, such that the vertex weight is $(l + 1/2) + (l + 1/2) = l + 1$
(3) If $l/2$ is odd number, then $f(v) = (l/2 - 1)$ and $f(e) = l/2$, such that the vertex weight is $(l/2 - 1) + (l/2) = l - 1$
(4) If $l/2$ is even number, then $f(v) = (l/2)$ and $f(e) = l/2$, such that the vertex weight is $(l/2) + (l/2) = l$

The vertex weight of point (1), (2), and (4) are, respectively, l and $l + 1$. It will give all weights are different, whereas, point (3) has a vertex weight of $l - 1$. Since the number of pendants is l, we will have at least two vertices which have the same weight. Therefore, for l is even and $l/2$ is odd, we need to add 1 for the largest vertex or edge labels. Thus, we will have a different weight for every pendant. Thus, the labels of vertex and edge of the pendant are the following.

From Table 1, it is easy to see that all vertex weights are different.

$$k = \begin{cases} \left\lceil \frac{l}{2} \right\rceil + 1, & \text{for } l \text{ even and } \frac{l}{2} \text{ odd,} \\ \left\lceil \frac{l}{2} \right\rceil, & \text{otherwise.} \end{cases} \quad (5)$$

Case 2. The vertex set apart from pendant vertices $V(G) - V_l$ must have a degree of at least two. The cardinality of $V(G) - V_l$ is less than or equal to the cardinality of V_l. It implies that the vertex or edge labels of pendant vertices can be re-used on labels of $V(G) - V_l$. Thus, the vertex weight of $V(G) - V_l$ will be different with the vertices of V_l since it has more combination, namely, $2k + 1, 2k + 2, \ldots, n$.

Based on Case 1 and Case 2, the reflexive vertex strength of graph G is

$$\text{rvs}(G) = \begin{cases} \left\lceil \frac{l}{2} \right\rceil + 1, & \text{for } l \text{ even and } \frac{l}{2} \text{ odd,} \\ \left\lceil \frac{l}{2} \right\rceil, & \text{otherwise.} \end{cases} \quad (6)$$

It concludes the proof.

Corollary 1. *Let S_n be a sunlet graph, and for every $n \geq 3$,*

$$\text{rvs}(S_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases} \quad (7)$$

Proof. Moreover, to determine the label of vertices $V(S_n) = \{u_i, v_i; 1 \leq i \leq n\}$ and edge set $E(S_n) = \{u_i v_i, 1 \leq i \leq n\} \cup \{u_i u_{i+1}, 1 \leq i \leq n-1\} \cup \{u_1 u_n\}$, we will use (Algorithm 1)

$$k = \left\{ \begin{array}{ll} \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{array} \right\} \quad (8)$$

It concludes the proof.

For an illustration, see Figure 1.

Theorem 2. Let H_n *be a helm graph, and for every* $n \geq 3$,

$$\text{rvs}(H_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases} \quad (9)$$

Proof. Let H_n be a helm graph with vertex set $V(H_n) = \{A, u_i, v_i; 1 \leq i \leq n\}$, $|V(H_n)| = 2n + 1$ and edge set $E(H_n) = \{Au_i, u_i v_i, 1 \leq i \leq n\} \cup \{u_i u_{i+1}, 1 \leq i \leq n-1\} \cup \{u_1 u_n\}$, $|E(H_n)| = 3n$. Helm graph has n pendant vertices and one central vertex of degree n. Since the central vertex has degree of much greater than the other vertices, it must have a different vertex weight than the others. Based on Theorem 1, we have the following lower bound:

$$\text{rvs}(H_n) \geq \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \frac{n}{2} \text{ odd,} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{otherwise.} \end{cases} \quad (10)$$

Furthermore, we will show the upper bound of vertex irregular reflexive k-labeling by defining the injection f and g in the following:

$$\begin{aligned} f(A) &= 0, \\ f(u_i) &= f(v_i), \\ f(v_i) &= \begin{cases} 0, & \text{if } 1 \leq i \leq k, \\ 2\left\lceil \frac{i-k}{2} \right\rceil, & \text{if } k+1 \leq i \leq n, \end{cases} \\ g(Au_i) &= g(u_i u_{i+1}) = g(u_1 u_n) = k, \\ g(u_i v_i) &= \begin{cases} i, & \text{if } 1 \leq i \leq k, \\ k-1, & \text{if } k+1 \leq i \leq n, \ (i \text{ odd for } k \text{ even}) \text{ and } (i \text{ even for } k \text{ odd}), \\ k, & \text{if } k+1 \leq i \leq n, \ (i \text{ even for } k \text{ even}) \text{ and } (i \text{ odd for } k \text{ odd}), \end{cases} \end{aligned} \quad (11)$$

Table 1: Labeling of vertex and edge on pendant vertices.

v_i	v_1	v_2	v_3	$\dots$	v_k	v_{k+1}	v_{k+2}	v_{k+3}	v_{k+4}	$\dots$	v_{l-1}	v_l
$f(v)$	0	0	0	$\dots$	0	2	2	4	4	$\dots$	k	k
$f(e)$	1	2	3	$\dots$	k	$k-1$	k	$k-1$	k	$\dots$	$k-1$	k
$w(v_i)$	1	2	3	$\dots$	k	$k+1$	$k+2$	$k+3$	$k+4$	$\dots$	$2k-1$	$2k$

(1) Define $v \in V(G)$, $e \in E(G)$ and injecton f for labeling of the graph elements
(2) Assign the labels of vertices and edges of pendants v_i according to Theorem 1.
(3) Observe that the vertex weight on each pendant will be $1, 2, \dots, k, k+1, \dots, n = 2k$ or $1, 2, \dots, k, k+1, \dots, n = 2k-1$.
(8) The vertex weights of point (4),(7) are l, but 5, 6 are respectively is $l+1$ and $l-1$. Since the number of pendant vertices is l, it will exist two type of vertices which have the same weights. Do the following.
(9) When l is even and $l/2$ is odd, add the label of each vertex or edge by 1.
(10) Apart from pendants, assign the label of vertices u_i as the label of pendant vertices v_i, but the labels of edges $u_i u_{i+1}$, $1 \le i \le n-1$ with k, thus the vertex weights are $2k+1, 2k+2, \dots, 3k, 3k+1, \dots 4k$ or $2k+1, 2k+2, \dots, 3k, 3k+1, \dots 4k-1$.
(11) Observe, all pendant vertices and otherwise show a different vertex weight.
(11) STOP.

Algorithm 1: The vertices' and edges' labels.

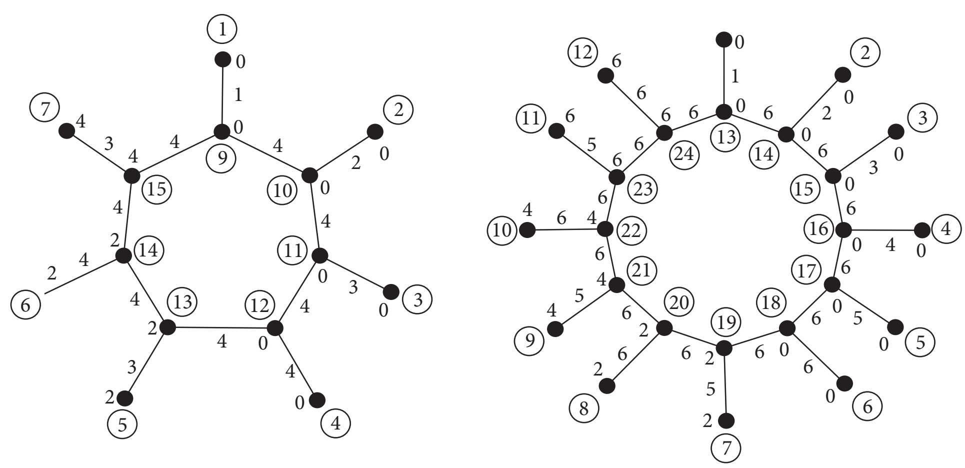

Figure 1: The illustration of labeling on S_7 and S_{12}.

where

$$k = \begin{cases} \left\lceil \dfrac{n}{2} \right\rceil + 1, & \text{if } n \text{ even and } \dfrac{n}{2} \text{ odd}, \\ \left\lceil \dfrac{n}{2} \right\rceil, & \text{otherwise}. \end{cases} \tag{12}$$

Based on the above injection, the overall vertex weight sets are

$$\begin{aligned} w(v_i) &= i, \\ w(v_i) &= i + 3k, \\ w(A) &= nk. \end{aligned} \tag{13}$$

It is easy to see that the above elements of set are all different. It concludes the proof.

Theorem 3. *SS_n be a subdivided star graph, and for every $n \ge 3$,*

$$\operatorname{rvs}(SS_n) = \left\lceil \frac{2n}{3} \right\rceil. \tag{14}$$

Proof. Let SS_n be a subdivided star graph with vertex set $V(SS_n) = \{A, x_i, y_i; 1 \le i \le n\}$, $|V(SS_n)| = 2n + 1$ and edge set $E(SS_n) = \{Ay_i, x_i y_i, 1 \le i \le n\}$, $|E(H_n)| = 2n$. The maximum degree of SS_n is n. The graph SS_n has one central vertex of degree n. Since the central vertex has degree of much greater than the other vertices, it must have a different vertex weight than the others. Based on Lemma 1, we have the following lower bound:

$$\operatorname{rvs}(G) \ge \left\lceil \frac{p + \delta - 1}{\Delta + 1} \right\rceil = \left\lceil \frac{2n + 1 - 1}{2 + 1} \right\rceil = \left\lceil \frac{2n}{3} \right\rceil. \tag{15}$$

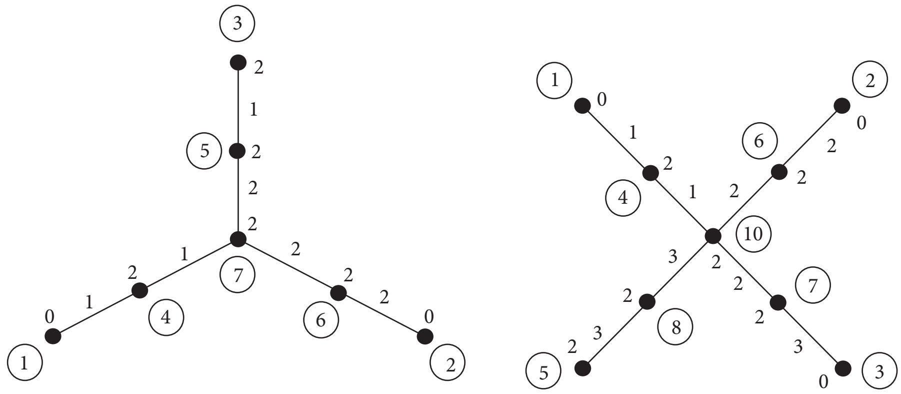

Figure 2: The illustration of labeling on SS_3 and.SS_4.

For the illustration of the vertex irregular reflexive, k-labeling of SS_3 and SS_4 can be depicted in Figure 2.

Furthermore, we will show the upper bound of vertex irregular reflexive k-labeling by defining the injection f and g. For $n \geq 5$, we have the following:

$$f(A) = 0,$$

$$f(x_i) = \begin{cases} 0, & \text{if } 1 \leq i \leq k, \\ 2\left\lceil \dfrac{i-k}{2} \right\rceil, & \text{if } k+1 \leq i \leq n. \end{cases} \quad (16)$$

For $n \equiv 3, 4 \pmod 6$, we have the following function for y_i and Ay_i:

$$f(y_i) = \begin{cases} 2\left\lceil \dfrac{n-k}{2} \right\rceil, & \text{if } 1 \leq i \leq k, \\ 2\left\lceil \dfrac{n-k}{2} \right\rceil + 2\left\lceil \dfrac{i-k}{2} \right\rceil, & \text{if } k+1 \leq i \leq n-1, \\ 2\left\lceil \dfrac{n-k}{2} \right\rceil + 2\left\lceil \dfrac{n-1-k}{2} \right\rceil, & \text{if } i = n, \end{cases} \quad (17)$$

$$g(Ay_i) = \begin{cases} k-1, & \text{if } 1 \leq i \leq n-2, \\ k, & \text{if } n-1 \leq i \leq n. \end{cases}$$

For otherwise n, we have

$$f(y_i) = \begin{cases} 2\left\lceil \dfrac{n-k}{2} \right\rceil, & \text{if } 1 \le i \le k, \\ 2\left\lceil \dfrac{n-k}{2} \right\rceil + 2\left\lceil \dfrac{i-k}{2} \right\rceil, & \text{if } k+1 \le i \le n \end{cases}$$

$$g(Ay_i) = \begin{cases} k-1, & \text{if } 1 \le i \le n \text{ for } n \equiv 5 (\text{mod}6), \\ k, & \text{if } 1 \le i \le n \text{ for otherwise}, \end{cases} \quad (18)$$

$$g(x_i y_i) = \begin{cases} i, & \text{if } 1 \le i \le k, \\ k-1, & \text{if } k+1 \le i \le n, \ (i \text{ odd for } k \text{ even}) \text{and} (i \text{ even for } k \text{ odd}), \\ k, & \text{if } k+1 \le i \le n, \ (i \text{ even for } k \text{ even}) \text{and} (i \text{ odd for } k \text{ odd}), \end{cases}$$

where

$$k = \left\lceil \frac{2n}{3} \right\rceil. \quad (19)$$

Based on the above injection, the overall vertex weight sets of the subdivided star SS_n are

$$w(x_i) = i,$$
$$w(y_i) = \begin{cases} n+i, & \text{if } 1 \le i \le n-2 \text{ for } n \equiv 3,4 (\text{mod}6), \\ 2n-1, & \text{if } i = n \text{ for } n \equiv 3,4 (\text{mod}6), \\ 2n, & \text{if } i = n-1 \text{ for } n \equiv 3,4 (\text{mod}6), \\ n+i, & \text{if } 1 \le i \le n \text{ for otherwise}, \end{cases}$$
$$w(A) = \begin{cases} n(k-1)+2, & \text{if } n \equiv 3,4 (\text{mod}6), \\ n(k-1), & \text{if } n \equiv 5 (\text{mod}6), \\ n(k), & \text{otherwise}. \end{cases} \quad (20)$$

It is easy to see that the above elements of the set are all different. It concludes the proof.

Theorem 4. Let $\text{Br}_{n,m}$ *be a broom graph, and for every* $n \ge 2, n+1 \le m$,

$$\text{rvs}(\text{Br}_{n,m}) = \begin{cases} \left\lceil \dfrac{n+m}{3} \right\rceil + 1, & \text{if } n+m \equiv 3 (\text{mod}6), \\ \left\lceil \dfrac{n+m}{3} \right\rceil, & \text{otherwise}. \end{cases} \quad (21)$$

Proof. Let $\text{Br}_{n,m}$ be a broom graph with vertex set $V(\text{Br}_{n,m}) = \{A, v_i, u_j; 1 \le i \le n, 1 \le j \le m\}$, $|V(Br_{n,m})| = n+m+1$, and edge set $E(\text{Br}_{n,m}) = \{\text{Av}_i, 1 \le i \le n\} \cup \{\text{Au}_m\} \cup \{u_j u_{j+1}, 1 \le j \le m-1\}$, $|E(H_n)| = n+m$.

The Broom graph $\text{Br}_{n,m}$ has n pendant vertices and one central vertex of degree n. Since the central vertex has a degree much greater than the other vertices, it must have a different vertex weight than the others. Based on Lemma 1, we have the following lower bound:

$$\text{rvs}(\text{Br}_{n,m}) \ge \left\lceil \frac{p+\delta-1}{\Delta+1} \right\rceil = \left\lceil \frac{n+m}{3} \right\rceil, \quad (22)$$

for $n+m \equiv 3 (\text{mod}6)$, $n+m = 3k$ and k is odd. Since the vertices u_j apart from vertex A have degree of at most 2, the labels of u_j are $(n+m/3-1)$, and the label of edges, which are incident to u_j, are $(n+m/3)$. Thus, the vertex weight u_j is $3(n+m/3)-1 = n+m-1$. Furthermore, since the number of vertices of Broom graph is $n+m$, there must be at least two vertices with the same vertex weight. Thus, we need to add 1 on the sharpest lower bound:

$$\text{rvs}(\text{Br}_{n,m}) \ge \left\lceil \frac{n+m}{3} \right\rceil + 1, \quad \text{if } (n+m) \equiv 3 (\text{mod}6). \quad (23)$$

Furthermore, we will show that k is an upper bound of the reflexive vertex strength of Broom graph $\text{Br}_{n,m}$. Let

$$k = \begin{cases} \left\lceil \dfrac{n+m}{3} \right\rceil + 1, & \text{if } n+m \equiv 3 ((\text{mod}6), \\ \left\lceil \dfrac{n+m}{3} \right\rceil, & \text{otherwise}. \end{cases} \quad (24)$$

Define an injection f and g of the vertex irregular reflexive k-labeling of Broom graph $\text{rvs}(\text{Br}_{n,m})$ as follows:

(1) Given that the vertex weight $V(G) = \{u_j, 2 \le j \le m\}$ by $w(u_j) = n + j, 2 \le j \le m$.
(2) Assign the labels of vertices u_j by $f(u_j) = f(U - j - 1$ and assign the labels of edges which are incident to u_j by $1, 2, 3, \ldots, k$ such that it meets with given vertex weight.
(3) When on point (ii), the label of edges is more than k, relabel the vertices with $f(u_j) = f(u_{j-1}) + 2$ as well as relabel the edges which are incident to u_j by $1, 2, 3, \ldots, k$ such that it meets with given vertex weight $w(u_j) = n + j, 2 \le j \le m$.
(4) STOP.

Algorithm 2: The vertices' and edges' labels.

$$\begin{aligned}
f(A) &= \begin{cases} k-1, & \text{if } k \text{ odd}, \\ k, & \text{if } k \text{ even}, \end{cases} \\
f(v_i) &= \begin{cases} 0, & \text{if } 1 \le i \le k, \\ 2\left\lceil \dfrac{i-k}{2} \right\rceil, & \text{if } k+1 \le i \le n+1, \end{cases} \\
f(u_1) &= f(v_{n+1}), \\
g(Av_i) &= \begin{cases} i, & \text{if } 1 \le i \le k, \\ k-1, & \text{if } k+1 \le i \le n+1,\ (i \text{ odd for } k \text{ even}) \text{and} (i \text{ even for } k \text{ odd}), \\ k, & \text{if } k+1 \le i \le n+1,\ (i \text{ even for } k \text{ even}) \text{and} (i \text{ odd for } k \text{ odd}), \end{cases} \\
g(u_1u_2) &= g(Av_{n+1}).
\end{aligned} \tag{25}$$

Based on the above injection, the overall vertex weight sets of $Br_{n,m}$ for $v_i; 1 \le i \le n+1$ is

$$w(v_i) = i, \quad 1 \le i \le n+1. \tag{26}$$

Moreover, to determine label of vertices $V(G) = \{u_j, 2 \le j \le m\}$ and $E(G) = \{Au_m\} \cup \{u_j u_{j+1}, 1 \le j \le m-1\}$, we will use Algorithm 2.

It is easy to see that the above elements of set $w(v_i)$ and $w(u_j)$ are all different. It concludes the proof.

3. Concluding Remark

In this paper, we have studied the construction of the reflexive vertex k-labeling of any graph with pendant vertex. We have determined a sharp lower bound of the reflexive vertex strength of any graph G in Lemma 1, as well as obtained the exact value the reflexive vertex strength of any graph G in Theorem 1. By this lemma and theorem, we finally determined the reflexive vertex k-labeling of some families of graph with a pendant vertex. However, we need to find an upper bound of the reflexive vertex strength of any graph and study the reflexive vertex k-labeling of other families of graph or some graph operations. Therefore, we propose the following open problems:

(1) Determine an upper bound of reflexive vertex strength of any graph to find the gap between lower bound and upper bound, and continue to determine the exact values for reflexive vertex strength of any other special graphs
(2) Determine the construction of the reflexive vertex k-labeling of any regular graph, planar graph, or some graph operations

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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