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Research Article

Regularity of Semigroups of Transformations Whose Characters Form the Semigroup of a Δ-Structure

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In this paper, we make use of the notion of the character of a transformation on a fixed set X, provided by Purisang and Rakbud in 2016, and the notion of a Δ-structure on X, provided by Magill Jr. and Subbiah in 1974, to define a sub-semigroup of the full-transformation semigroup $T(X)$. We also define a sub-semigroup of that semigroup. The regularity of those two semigroups is also studied.

1. Introduction and Preliminaries

For any semigroup $\mathcal{S}$, we call an element a of $\mathcal{S}$ a *regular element* of $\mathcal{S}$ if there exists an element b of $\mathcal{S}$ such that $aba = a$. A semigroup $\mathcal{S}$ is said to be *regular* if every element of $\mathcal{S}$ is regular. The semigroup of transformations on a nonempty set X, denoted by $T(X)$, is a well-known regular semigroup. In [1], Nenthein et al. studied the regularity of the following two sub-semigroups of $T(X)$:

$$\begin{aligned} T(X,Y) &:= \{\alpha \in T(X)\colon \operatorname{Ran}(\alpha)\subseteq Y\},\\ \overline{T}(X,Y) &:= \{\alpha \in T(X)\colon \alpha(Y)\subseteq Y\}, \end{aligned} \tag{1}$$

where Y is a fixed nonempty subset of X and $\operatorname{Ran}(\alpha)$ denotes the range of α for all $\alpha \in T(X)$. The authors obtained that the two semigroups are regular if and only $|Y| = 1$ or $Y = X$. The regularity of some other sub-semigroups of $T(X)$ has been studied by many people (see [2–6] for some recent works).

In this paper, by a *partition* of a nonempty set X, we mean a family $\mathcal{F} = \{Y_i\colon\ i \in I\}$ of nonempty subsets of X such that $X = \cup_{i\in I} Y_i$ and $Y_i \cap Y_j = \varnothing$ for all $i, j \in I$ with $i \neq j$. For a given partition $\mathcal{F} = \{Y_i\colon\ i \in I\}$ of a nonempty set X, let

$$T_{\mathcal{F}}(X) = \left\{\alpha \in T(X)\colon \forall i \in I \exists j \in I, \alpha\left(Y_i\right)\subseteq Y_j\right\}. \tag{2}$$

Note that $T_{\mathcal{F}}(X)$ is exactly the semigroup of all transformations on X preserving the equivalence relation induced by partition $\mathcal{F}$. There have been several works involving transformations preserving an equivalence relation (see [3–5, 7–12] for some references).

Throughout this paper, let X be a nonempty set, and let $\mathcal{F} = \{Y_i\colon\ i \in I\}$ be a partition of X, which are arbitrarily fixed. From the definition of a partition, Purisang and Rakbud [11] defined a function $\chi^{(\alpha)}\colon I \longrightarrow I$ associated with each $\alpha \in T_{\mathcal{F}}(X)$ called the *character* of α, by

$$\begin{aligned} &\forall i, j \in I,\\ &\chi^{(\alpha)} i = j \Longleftrightarrow \alpha\left(Y_i\right)\subseteq Y_j. \end{aligned} \tag{3}$$

They also defined and studied the regularity of the following sub-semigroup of $T(X)$:

$$\begin{aligned} T_{\mathcal{F}}^{(J)}(X) &:= \left\{\alpha \in T(X)\colon \forall i \in I \exists j \in J, \alpha\left(Y_i\right)\subseteq Y_j\right\}\\ &= \left\{\alpha \in T(X)\colon \chi^{(\alpha)} \in T(I,J)\right\}, \end{aligned} \tag{4}$$

where J is an arbitrarily fixed nonempty subset of the index set I. We summarize some of their results as follows.

Lemma 1 (see [11], Lemma 2.3). *For every $\alpha, \beta \in T_{\mathcal{F}}^{(J)}(X)$, $\chi^{(\alpha\beta)} = \chi^{(\alpha)}\chi^{(\beta)}$.*

By making use of the notion of the character, the following two equivalence relations χ and $\tilde{\chi}$ on $T_{\mathcal{F}}^{(J)}(X)$ were defined:

$$\begin{aligned} &\forall \alpha,\beta \in T_{\mathcal{F}}^{(J)}(X), \quad (\alpha,\beta)\in\chi \Longleftrightarrow \chi^{(\alpha)} = \chi^{(\beta)}, \\ &\forall \alpha,\beta \in T_{\mathcal{F}}^{(J)}(X), \quad (\alpha,\beta)\in\tilde{\chi} \Longleftrightarrow \chi^{(\alpha)}|_J = \chi^{(\beta)}|_J. \end{aligned} \tag{5}$$

Note that, by Lemma 1, they both are congruence relations. The authors studied the regularity of the quotient semigroups $T_{\mathcal{F}}^{(J)}(X)/\chi$ and $T_{\mathcal{F}}^{(J)}(X)/\tilde{\chi}$. The following are what they obtained.

Theorem 1 (see [11], Theorem 2.4). *The following statements hold:*

(1) $T_{\mathcal{F}}^{(J)}(X)/\chi \cong T(I,J)$ by the isomorphism $[\alpha]\mapsto\chi^{(\alpha)}$;
(2) $T_{\mathcal{F}}^{(J)}(X)/\tilde{\chi} \cong T(J)$ by the isomorphism $\widetilde{[\alpha]}\mapsto\chi^{(\alpha)}|_J$,

where for each $\alpha\in T_{\mathcal{F}}^{(J)}(X)$, $[\alpha]$ and $\widetilde{[\alpha]}$ denote the equivalence classes of α under the equivalence relations χ and $\tilde{\chi}$, respectively.

Corollary 1 (see [11], Corollary 2.5). *The following statements hold.*

(1) The three statements below are all equivalent:

(a) The quotient semigroup $T_{\mathcal{F}}^{(J)}(X)/\chi$ is regular;
(b) The semigroup $T(I,J)$ is regular;
(c) $J=I$ or $|J|=1$.

(2) The quotient semigroup $T_{\mathcal{F}}(X)/\chi$ is regular.

(3) The quotient semigroup $T_{\mathcal{F}}^{(J)}(X)/\tilde{\chi}$ is regular.

The regularity of the semigroup $T_{\mathcal{F}}^{(J)}(X)$ was obtained as follows.

Theorem 2 (see [11], Theorem 2.6). *The semigroup $T_{\mathcal{F}}^{(J)}(X)$ is regular if and only if $|T_{\mathcal{F}}^{(J)}(X)|=1$ or $T_{\mathcal{F}}^{(J)}(X)=T(X)$.*

It is easy to see that for each $\alpha\in T_{\mathcal{F}}(X)$, the equivalence class $[\alpha]$ of α under the equivalence relation χ is a sub-semigroup of $T_{\mathcal{F}}(X)$ if and only if $\chi^{(\alpha)}$ is an idempotent element of $T(I)$. In [11], the authors studied the regularity of the semigroup $[\alpha]$ when α is an idempotent element of $T(I)$. They also defined some further sub-semigroups of $T_{\mathcal{F}}(X)$ by making use the notion of the character as follows: let $I_{\mathcal{F}}(X)$, $S_{\mathcal{F}}(X)$, and $B_{\mathcal{F}}(X)$ be the sets of all elements of $T_{\mathcal{F}}(X)$ whose characters are injective, surjective, and bijective, respectively. Note that, by Lemma 1, the sets $I_{\mathcal{F}}(X)$, $S_{\mathcal{F}}(X)$, and $B_{\mathcal{F}}(X)$ are sub-semigroups of $T(X)$. The regularity of each of these three semigroups was also studied.

It was observed by Rakbud [12] that the semigroups $T_{\mathcal{F}}^{(J)}(X)$, $[\alpha]$ when $\chi^{(\alpha)}$ is idempotent, $I_{\mathcal{F}}(X)$, $S_{\mathcal{F}}(X)$, and $B_{\mathcal{F}}(X)$ can simultaneously be generalized by making use of the notion of the character as follows: for every sub-semigroup $\mathcal{S}$ of $T(I)$, let

$$T_{\mathcal{F}}^{(\mathcal{S})}(X) = \left\{\alpha\in T_{\mathcal{F}}(X) : \chi^{(\alpha)}\in\mathcal{S}\right\}. \tag{6}$$

Note that, for each $\gamma\in T(I)$, the function $\alpha: X\longrightarrow X$ defined on each Y_i by $\alpha(Y_i)=\{z_i\}$, where z_i is a fixed element of $Y_{\gamma(i)}$ for all $i\in I$, is an element of $T_{\mathcal{F}}(X)$ whose character is exactly γ. Hence, $T_{\mathcal{F}}^{(\mathcal{S})}(X)\neq\varnothing$, and by Lemma 1, it is a sub-semigroup of $T_{\mathcal{F}}(X)$.

Let $\mathcal{S}$ be a sub-semigroup of $T(I)$. Then, by considering the congruence relation χ on $T_{\mathcal{F}}(X)$ restricted to $T_{\mathcal{F}}^{(\mathcal{S})}(X)$, we have the quotient semigroup $T_{\mathcal{F}}^{(\mathcal{S})}(X)/\chi$. Obviously, $T_{\mathcal{F}}^{(\mathcal{S})}(X)/\chi = \left\{[\alpha] : \alpha\in T_{\mathcal{F}}^{(\mathcal{S})}(X)\right\}$ and $T_{\mathcal{F}}^{(\mathcal{S})}(X)/\chi$ is a sub-semigroup of $T_{\mathcal{F}}(X)/\chi$. Analogously to Theorem 1, the following result was established.

Theorem 3 (see [12], Theorem 1.14). *$T_{\mathcal{F}}^{(\mathcal{S})}(X)/\chi\cong\mathcal{S}$ by the isomorphism defined by $[\alpha]\mapsto\chi^{(\alpha)}$.*

Immediately from Theorem 3, the following corollary was obtained.

Corollary 2 (see [12], Corollary 1.15). *The quotient semigroup $T_{\mathcal{F}}^{(\mathcal{S})}(X)/\chi$ is regular if and only if the semigroup $\mathcal{S}$ is regular.*

Besides the above results, in [12], the author also used the notion of the character to define the notion of a weakly regular transformation and study the regularity of a semigroup of weakly regular transformations in that sense. However, the regularity of $T_{\mathcal{F}}^{(\mathcal{S})}(X)$ has not been studied in general yet. This will be in our attention here when $\mathcal{S}$ has a certain property. We are mentioning that $\mathcal{S}$ is "the semigroup of a Δ-structure"on I.

We now refer to the definition of a Δ-structure on a set and some other related ones from [13] by Magill and Subbiah. Let $\mathcal{A}$ be a family of nonempty subsets of X such that $X\in\mathcal{A}$. And, let $\mathcal{M}=\{\mathrm{Hom}(A,B): A,B\in\mathcal{A}\}$, where $\mathrm{Hom}(A,B)$ is a nonempty set of functions from A into B for all $A,B\in\mathcal{A}$, with the following properties:

($\Delta 1$) $\mathrm{End}(X) := \mathrm{Hom}(X,X)$ is a monoid;
($\Delta 2$) $\mathrm{Ran}(\alpha)\in\mathcal{A}$ for all $\alpha\in\mathrm{End}(X)$;
($\Delta 3$) For all $B\in\mathcal{A}$, $\alpha\in\mathrm{End}(X)$ and $\beta\in\mathrm{Hom}(\mathrm{Ran}(\alpha),B)$, $\beta\alpha\in\mathrm{End}(X)$;
($\Delta 4$) For all $\alpha,\beta\in\mathrm{End}(X)$ and $A,B\in\mathcal{A}$ with $\alpha(B)\subseteq A$ and $\beta(A)\subseteq B$, if $(\alpha\beta)|_A=\mathrm{id}_A$ and $(\beta\alpha)|_B=\mathrm{id}_B$, then $\beta|_A\in\mathrm{Hom}(A,B)$ and $\alpha|_B\in\mathrm{Hom}(B,A)$, where id_Z denotes the identity function on Z for any nonempty set Z.

The pair $(\mathcal{A},\mathcal{M})$ is called a Δ-*structure* on X, and the monoid $\mathrm{End}(X)$ is called the *semigroup of the* Δ-*structure* $(\mathcal{A},\mathcal{M})$. A subset A of X is called a Δ-*retract* of X if there is an idempotent element ρ of $\mathrm{End}(X)$ such that $A=\mathrm{Ran}(\rho)$. For any $A,B\in\mathcal{A}$, an element λ of $\mathrm{Hom}(A,B)$ is called a Δ-*isomorphism* (from A onto B) if there is $\sigma\in\mathrm{Hom}(B,A)$ such that $\sigma\lambda=\mathrm{id}_A$ and $\lambda\sigma=\mathrm{id}_B$. It is clear for any $A,B\in\mathcal{A}$ that an element λ of $\mathrm{Hom}(A,B)$ is Δ-isomorphism if and only if λ is bijective and $\lambda^{-1}\in\mathrm{Hom}(B,A)$.

In [13], the authors gave some characterizations of regular elements of the semigroup $\mathrm{End}(X)$ as follows.

Theorem 4 (see [13], Theorem 2.4). *Let $\alpha \in End(X)$. Then, the following statements are equivalent:*

(1) α is regular;

(2) $Ran(\alpha)$ is a Δ-retract of X, and there is a Δ-retract A of X such that $\alpha|_A$ is a Δ-isomorphism from A onto $Ran(\alpha)$;

(3) $Ran(\alpha)$ is a Δ-retract of X, and there is $A \in \mathscr{A}$ such that $\alpha|_A$ is a Δ-isomorphism from A onto $Ran(\alpha)$.

It is clear that $T(X) = \mathrm{End}(X)$ when X is equipped with the Δ-structure $(\mathscr{A}, \mathscr{M})$, where $\mathscr{A}$ is the family of all nonempty subsets of X, and $\mathrm{Hom}(A, B)$ is the set of all functions from A into B for all $A, B \in \mathscr{A}$. In this setting, the Δ-retracts of X are exactly the nonempty subsets of X, and the Δ-isomorphisms are exactly the bijective functions. More interesting semigroups of Δ-structures on X were given in [13] as follows:

(i) The semigroup $S(X)$ of all continuous maps on X is exactly $\mathrm{End}(X)$ when X is a topological space equipped with the Δ-structure $(\mathscr{A}, \mathscr{M})$, where $\mathscr{A}$ is the family of all nonempty subsets of X and $\mathrm{Hom}(A, B)$ is the set of all continuous maps from A into B for all $A, B \in \mathscr{A}$. In this setting, the Δ-retracts of X are exactly the Δ-retracts of the topological space X, and the Δ-isomorphisms are exactly the homeomorphisms.

(ii) The semigroup $L(X)$ of all linear transformations on X coincides with $\mathrm{End}(X)$ when X is a vector space equipped with the Δ-structure $(\mathscr{A}, \mathscr{M})$, where $\mathscr{A}$ is the family of all subspaces of X and $\mathrm{Hom}(A, B)$ is the set of all linear transformations from A into B for all $A, B \in \mathscr{A}$. In this setting, the Δ-retracts of X are exactly the subspaces of the vector space X, and the Δ-isomorphisms are exactly the vector space isomorphisms.

(iii) The semigroup $\Gamma(X)$ of all closed maps on X is exactly $\mathrm{End}(X)$ when X is a T_1-space equipped with the Δ-structure $(\mathscr{A}, \mathscr{M})$, where $\mathscr{A}$ is the family of all nonempty closed subsets of X and $\mathrm{Hom}(A, B)$ is the set of all closed maps from A into B for all $A, B \in \mathscr{A}$. In this setting, the Δ-retracts of X are exactly the nonempty closed subsets of the topological space X, and the Δ-isomorphisms are exactly the homeomorphisms.

The regularity of the semigroups $S(X)$, $L(X)$, and $\Gamma(X)$ was also deduced via Theorem 4. In addition, we note here that the semigroup $T_{\mathscr{E}}(X)$ of all transformations on X preserving an equivalence relation $\mathscr{E}$ on X can be considered as the semigroup of all continuous maps on X, where X is equipped with the topology having the family of all equivalence classes as a base. This was proved by Huisheng [8] (see Theorem 2.8).

The main aim of this paper is to study the regularity of the semigroup $T_{\mathscr{F}}^{(\mathscr{S})}(X)$, introduced in [12], when $\mathscr{S}$ is the semigroup of a Δ-structure on the index set I of the partition $\mathscr{F}$. We also define, in this situation, a sub-semigroup of $T_{\mathscr{F}}^{(\mathscr{S})}(X)$ whose regularity coincides with that of the semigroup $\mathscr{S}$.

2. The Semigroup and Its Regularity

For any nonempty sets Z and W and partitions $\mathscr{P} = \{Z_i\colon i \in H\}$ and $\mathscr{Q} = \{W_i\colon i \in K\}$ of Z and W, respectively, let $T_{\mathscr{P},\mathscr{Q}}(Z, W)$ be the set of all functions $\lambda\colon Z \longrightarrow W$ satisfying the condition that for all $i \in H$, there is $j \in K$ such that $\lambda(Z_i) \subseteq W_j$. And, for each $\lambda \in T_{\mathscr{P},\mathscr{Q}}(Z, W)$, let $\chi_{\mathscr{P},\mathscr{Q}}^{(\lambda)}\colon H \longrightarrow K$ be defined by

$$\begin{gathered}\forall (i, j) \in H \times K,\\ \chi_{\mathscr{P},\mathscr{Q}}^{(\lambda)}(i) = j \Longleftrightarrow \lambda(Z_i) \subseteq W_j.\end{gathered} \tag{7}$$

It is easy to verify that the map $\lambda \mapsto \chi_{\mathscr{P},\mathscr{Q}}^{(\lambda)}$ from the set $T_{\mathscr{P},\mathscr{Q}}(Z, W)$ into the set of all functions from H into K is surjective. If we have three nonempty sets Z, W, and U with partitions $\mathscr{P} = \{Z_i\colon i \in H\}$, $\mathscr{Q} = \{W_i\colon i \in K\}$, and $\mathscr{R} = \{U_i\colon i \in M\}$ of Z, W, and U, respectively, we see for any $\lambda \in T_{\mathscr{P},\mathscr{Q}}(Z, W)$ and $\rho \in T_{\mathscr{Q},\mathscr{R}}(W, U)$ that $\rho\lambda \in T_{\mathscr{P},\mathscr{R}}(Z, U)$ and $\chi_{\mathscr{P},\mathscr{R}}^{(\rho\lambda)} = \chi_{\mathscr{Q},\mathscr{R}}^{(\rho)}\chi_{\mathscr{P},\mathscr{Q}}^{(\lambda)}$.

For all $\alpha, \beta \in T_{\mathscr{F}}(X)$, let

$$T_{\mathscr{F}}(\alpha, \beta) = T_{\mathscr{P},\mathscr{Q}}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta)), \tag{8}$$

where

$$\begin{aligned}\mathscr{P} &= \left\{\mathrm{Ran}(\alpha) \cap Y_i\colon\ i \in \mathrm{Ran}\left(\chi^{(\alpha)}\right)\right\},\\ \mathscr{Q} &= \left\{\mathrm{Ran}(\beta) \cap Y_i\colon\ i \in \mathrm{Ran}\left(\chi^{(\beta)}\right)\right\}.\end{aligned} \tag{9}$$

And, for each $\lambda \in T_{\mathscr{F}}(\alpha, \beta)$, let $\chi_{\alpha,\beta}^{(\lambda)} = \chi_{\mathscr{P},\mathscr{Q}}^{(\lambda)}$.

Let $(\mathscr{J}, \mathscr{C})$ be a Δ-structure on the index set I of the partition $\mathscr{F}$, and let

$$\mathrm{End}_{\mathscr{F}}(X) = T_{\mathscr{F}}^{(\mathrm{End}(I))}(X) := \left\{\alpha \in T_{\mathscr{F}}(X)\colon \chi^{(\alpha)} \in \mathrm{End}(I)\right\}. \tag{10}$$

By the property $(\Delta 1)$ of the Δ-structure $(\mathscr{J}, \mathscr{C})$ on I, we have that $\chi^{(\mathrm{id}_X)} = \mathrm{id}_I \in \mathrm{End}(I)$, which yields that $\mathrm{id}_X \in \mathrm{End}_{\mathscr{F}}(X)$. Hence, $\mathrm{End}_{\mathscr{F}}(X)$ is a submonoid of $T_{\mathscr{F}}(X)$.

Theorem 5. *There is a Δ-structure on X such that $End(X) = End_{\mathscr{F}}(X)$.*

Proof. Let

$$\mathscr{A} = \{\mathrm{Ran}(\alpha)\colon\quad \alpha \in \mathrm{End}_{\mathscr{F}}(X)\}. \tag{11}$$

Since $\mathrm{End}_{\mathscr{F}}(X)$ is a monoid, we have $X \in \mathscr{A}$. For any $\alpha, \beta \in \mathrm{End}_{\mathscr{F}}(X)$, let

$$\mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta)) = \Big\{\lambda \in T_{\mathscr{F}}(\alpha, \beta)\colon \chi_{\alpha,\beta}^{(\lambda)} \in \mathrm{Hom}\left(\mathrm{Ran}\left(\chi^{(\alpha)}\right), \mathrm{Ran}\left(\chi^{(\beta)}\right)\right)\Big\}. \tag{12}$$

Then, by the surjectivity of the map $\lambda \mapsto \chi_{\alpha,\beta}^{(\lambda)}$ from the set $T_{\mathscr{F}}(\alpha, \beta)$ into the set of all functions from $\mathrm{Ran}(\chi^{(\alpha)})$ into $\mathrm{Ran}(\chi^{(\beta)})$, we have that $\mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta)) \neq \varnothing$ for all $\alpha, \beta \in \mathrm{End}_{\mathscr{F}}(X)$. Note that for all $\alpha, \beta, \lambda, \rho \in \mathrm{End}_{\mathscr{F}}(X)$, if $\mathrm{Ran}(\alpha) = \mathrm{Ran}(\lambda)$ and $\mathrm{Ran}(\beta) = \mathrm{Ran}(\rho)$, then $\mathrm{Ran}(\chi^{(\alpha)}) =$

$\mathrm{Ran}(\chi^{(\lambda)})$ and $\mathrm{Ran}(\chi^{(\beta)}) = \mathrm{Ran}(\chi^{(\rho)})$, which yields that $\mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta)) = \mathrm{Hom}(\mathrm{Ran}(\lambda), \mathrm{Ran}(\rho))$. Let

$$\mathscr{M} = \{\mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta)): \quad \alpha, \beta \in \mathrm{End}_{\mathscr{F}}(X)\}. \qquad (13)$$

Then, from the above explanation, the family $\mathscr{M}$ is well defined. We will show that $(\mathscr{A}, \mathscr{M})$ is a Δ-structure on X. It is easy to see that

$$\begin{aligned}\mathrm{Hom}(X, X) &= \mathrm{Hom}(\mathrm{Ran}(\mathrm{id}_X), \mathrm{Ran}(\mathrm{id}_X)) \\ &= \{\lambda \in T_{\mathscr{F}}(X): \chi^{(\lambda)} \in \mathrm{End}(I)\} \\ &= \mathrm{End}_{\mathscr{F}}(X).\end{aligned} \qquad (14)$$

Thus, the property (Δ1) is satisfied. From the definition of the family $\mathscr{A}$, we immediately have that the property (Δ2) holds. Next, we will show that the property (Δ3) is satisfied. Let $\alpha, \beta \in \mathrm{End}_{\mathscr{F}}(X)$, and let $\lambda \in \mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta))$. We want to show that $\lambda\alpha \in \mathrm{End}_{\mathscr{F}}(X)$. Since $\lambda \in \mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta))$, we have $\chi^{(\lambda)}_{\alpha,\beta} \in \mathrm{Hom}(\mathrm{Ran}(\chi^{(\alpha)}), \mathrm{Ran}(\chi^{(\beta)}))$. Since $\alpha \in \mathrm{End}_{\mathscr{F}}(X)$, we have that $\chi^{(\alpha)} \in \mathrm{End}(I)$. Thus, by the property (Δ3) of the Δ-structure $(\mathscr{J}, \mathscr{C})$ on I, we get $\chi^{(\lambda)}_{\alpha,\beta}\chi^{(\alpha)} \in \mathrm{End}(I)$. It is clear that $\lambda\alpha \in T_{\mathscr{F}}(X)$, and that $\chi^{(\lambda\alpha)} = \chi^{(\lambda)}_{\alpha,\beta}\chi^{(\alpha)}$. Hence, by the membership of $\chi^{(\lambda)}_{\alpha,\beta}\chi^{(\alpha)}$ in $\mathrm{End}(I)$, we obtain that $\lambda\alpha \in \mathrm{End}_{\mathscr{F}}(X)$ as desired. Finally, we will show that the property (Δ4) holds. Let $\alpha, \beta, \psi, \phi \in \mathrm{End}_{\mathscr{F}}(X)$ be such that $\alpha(\mathrm{Ran}(\phi)) \subseteq \mathrm{Ran}(\psi)$ and $\beta(\mathrm{Ran}(\psi)) \subseteq \mathrm{Ran}(\phi)$. Then, $\chi^{(\alpha)}, \chi^{(\beta)}, \chi^{(\psi)}, \chi^{(\phi)} \in \mathrm{End}(I)$. Thus, by the property (Δ2) of the Δ-structure $(\mathscr{J}, \mathscr{C})$ on I, we get that $\mathrm{Ran}(\chi^{(\psi)})$ and $\mathrm{Ran}(\chi^{(\phi)})$ belong to $\mathscr{J}$. Since $\mathrm{Ran}(\alpha\phi) = \alpha(\mathrm{Ran}(\phi)) \subseteq \mathrm{Ran}(\psi)$, we have that $\mathrm{Ran}(\chi^{(\alpha\phi)}) \subseteq \mathrm{Ran}(\chi^{(\psi)})$. To see this explicitly, let $j \in \mathrm{Ran}(\chi^{(\alpha\phi)})$. Then, there is $i \in I$ such that $\chi^{(\alpha\phi)}(i) = j$, which yields that $\alpha\phi(Y_i) \subseteq Y_j$. Fix $x \in Y_i$. Since $\mathrm{Ran}(\alpha\phi) \subseteq \mathrm{Ran}(\psi)$, we get that $\alpha\phi(x) \in \mathrm{Ran}(\psi)$. Therefore, there is $z \in X$ such that $\alpha\phi(x) = \psi(z)$. Let $k \in I$ be such that $z \in Y_k$, and let $\nu = \chi^{(\psi)}(k)$. Then, $\psi(Y_k) \subseteq Y_\nu$; so $\alpha\phi(x) \in Y_\nu$, which implies that $Y_j = Y_\nu$. Hence, $j = \nu \in \mathrm{Ran}(\chi^{(\psi)})$. Similarly, by the inclusion $\beta(\mathrm{Ran}(\psi)) \subseteq \mathrm{Ran}(\phi)$, we obtain $\mathrm{Ran}(\chi^{(\beta\psi)}) \subseteq \mathrm{Ran}(\chi^{(\phi)})$. We now have two inclusions $\mathrm{Ran}(\chi^{(\alpha\phi)}) \subseteq \mathrm{Ran}(\chi^{(\psi)})$ and $\mathrm{Ran}(\chi^{(\beta\psi)}) \subseteq \mathrm{Ran}(\chi^{(\phi)})$. From these, we get by Lemma 1 that $\chi^{(\alpha)}(\mathrm{Ran}(\chi^{(\phi)})) \subseteq \mathrm{Ran}(\chi^{(\psi)})$ and $\chi^{(\beta)}(\mathrm{Ran}(\chi^{(\psi)})) \subseteq \mathrm{Ran}(\chi^{(\phi)})$. By the assumptions that $\alpha(\mathrm{Ran}(\phi)) \subseteq \mathrm{Ran}(\psi)$ and that $\beta(\mathrm{Ran}(\psi)) \subseteq \mathrm{Ran}(\phi)$, we have $\alpha|_{\mathrm{Ran}(\phi)}: \mathrm{Ran}(\phi) \longrightarrow \mathrm{Ran}(\psi)$ and $\beta|_{\mathrm{Ran}(\psi)}: \mathrm{Ran}(\psi) \longrightarrow \mathrm{Ran}(\phi)$, respectively. And, from the membership of α in $T_{\mathscr{F}}(X)$, we get that for all $i \in \mathrm{Ran}(\chi^{(\phi)})$, there exists $j \in \mathrm{Ran}(\chi^{(\psi)})$ such that $\alpha|_{\mathrm{Ran}(\phi)}(Y_i \cap \mathrm{Ran}(\phi)) \subseteq Y_j$. Thus, $\alpha|_{\mathrm{Ran}(\phi)} \in T_{\mathscr{F}}(\phi, \psi)$. By the inclusion $\chi^{(\alpha)}(\mathrm{Ran}(\chi^{(\phi)})) \subseteq \mathrm{Ran}(\chi^{(\psi)})$, we have $\chi^{(\alpha)}|_{\mathrm{Ran}(\chi^{(\phi)})}: \mathrm{Ran}(\chi^{(\phi)}) \longrightarrow \mathrm{Ran}(\chi^{(\psi)})$, which yields that $\chi^{(\alpha|_{\mathrm{Ran}(\phi)})}_{\phi,\psi} = \chi^{(\alpha)}|_{\mathrm{Ran}(\chi^{(\phi)})}$. Similarly, we have that $\beta|_{\mathrm{Ran}(\psi)} \in T_{\mathscr{F}}(\psi, \phi)$, and that $\chi^{(\beta|_{\mathrm{Ran}(\psi)})}_{\psi,\phi} = \chi^{(\beta)}|_{\mathrm{Ran}(\chi^{(\psi)})}$. Suppose that $(\beta\alpha)|_{\mathrm{Ran}(\phi)} = \mathrm{id}_{\mathrm{Ran}(\phi)}$, and that $(\alpha\beta)|_{\mathrm{Ran}(\psi)} = \mathrm{id}_{\mathrm{Ran}(\psi)}$. We want to show that $\alpha|_{\mathrm{Ran}(\phi)} \in \mathrm{Hom}(\mathrm{Ran}(\phi), \mathrm{Ran}(\psi))$, and that $\beta|_{\mathrm{Ran}(\psi)} \in \mathrm{Hom}(\mathrm{Ran}(\psi), \mathrm{Ran}(\phi))$. By the inclusions $\chi^{(\alpha)}(\mathrm{Ran}(\chi^{(\phi)})) \subseteq \mathrm{Ran}(\chi^{(\psi)})$ and $\chi^{(\beta)}(\mathrm{Ran}(\chi^{(\psi)})) \subseteq \mathrm{Ran}(\chi^{(\phi)})$, we have, respectively, that $(\chi^{(\beta)}\chi^{(\alpha)})|_{\mathrm{Ran}(\chi^{(\phi)})}: \mathrm{Ran}(\chi^{(\phi)}) \longrightarrow \mathrm{Ran}(\chi^{(\phi)})$ and $(\chi^{(\alpha)}\chi^{(\beta)})|_{\mathrm{Ran}(\chi^{(\psi)})}: \mathrm{Ran}(\chi^{(\psi)}) \longrightarrow \mathrm{Ran}(\chi^{(\psi)})$. And, since $(\beta\alpha)|_{\mathrm{Ran}(\phi)} = \mathrm{id}_{\mathrm{Ran}(\phi)}$ and $(\alpha\beta)|_{\mathrm{Ran}(\psi)} = \mathrm{id}_{\mathrm{Ran}(\psi)}$, it follows that $(\chi^{(\beta)}\chi^{(\alpha)})|_{\mathrm{Ran}(\chi^{(\phi)})} = \mathrm{id}_{\mathrm{Ran}(\chi^{(\phi)})}$ and $(\chi^{(\alpha)}\chi^{(\beta)})|_{\mathrm{Ran}(\chi^{(\psi)})} = \mathrm{id}_{\mathrm{Ran}(\chi^{(\psi)})}$, respectively. Therefore, by the property (Δ4) of the Δ-structure $(\mathscr{J}, \mathscr{C})$ on I, we obtain that $\chi^{(\alpha|_{\mathrm{Ran}(\phi)})}_{\phi,\psi} = \chi^{(\alpha)}|_{\mathrm{Ran}(\chi^{(\phi)})} \in \mathrm{Hom}(\mathrm{Ran}(\chi^{(\phi)}), \mathrm{Ran}(\chi^{(\psi)}))$ and $\chi^{(\beta|_{\mathrm{Ran}(\psi)})}_{\psi,\phi} = \chi^{(\beta)}|_{\mathrm{Ran}(\chi^{(\psi)})} \in \mathrm{Hom}(\mathrm{Ran}(\chi^{(\psi)}), \mathrm{Ran}(\chi^{(\phi)}))$. These yield, respectively, that $\alpha|_{\mathrm{Ran}(\phi)} \in \mathrm{Hom}(\mathrm{Ran}(\phi), \mathrm{Ran}(\psi))$ and $\beta|_{\mathrm{Ran}(\psi)} \in \mathrm{Hom}(\mathrm{Ran}(\psi), \mathrm{Ran}(\phi))$. □

From now on, we will consider the Δ-structure $(\mathscr{A}, \mathscr{M})$ on X defined in the proof of Theorem 5, and the semigroup $\mathrm{End}_{\mathscr{F}}(X)$ of this Δ-structure will be in our attention. By Theorem 4, the regularity of elements of the semigroup $\mathrm{End}_{\mathscr{F}}(X)$ can roughly be characterized. To get more precise characterizations, according to Theorem 4, the notions of a Δ-retract and a Δ-isomorphism in the Δ-structure $(\mathscr{A}, \mathscr{M})$ on X should particularly be studied. For that purpose, the following elementary theorem, stating some characterizations of idempotent elements in a transformation semigroup, is needed.

Theorem 6. *Let Z be a nonempty set and $\alpha \in T(Z)$. Then, the following statements are equivalent:*

(1) α is idempotent;

(2) $\alpha|_{Ran(\alpha)} = id_{Ran(\alpha)}$;

(3) There is a partition $\{Z_j: j \in E\}$ of Z and a subset $\{z_j: j \in E\}$ of Z such that $z_j \in Z_j$ for all $j \in E$ and $\alpha(Z_j) = \{z_j\}$ for all $j \in E$.

In this situation, the partition $\{Z_j: j \in E\}$ and the subset $\{Z_j: j \in E\}$ of Z are unique determined by α.

For each nonempty subset A of X, we see that there is a unique subset of I, denoted by J_A, such that the family $\{A \cap Y_i: i \in J_A\}$ is a partition of A. In particular, we have that J_A is exactly $\mathrm{Ran}(\chi^{(\alpha)})$ if $A = \mathrm{Ran}(\alpha)$ for some $\alpha \in T_{\mathscr{F}}(X)$.

Proposition 1. *Let $\gamma \in T(I)$, and let $A \subseteq X$. If γ is idempotent and $J_A = Ran(\gamma)$, then there is an idempotent element α of $T_{\mathscr{F}}(X)$ such that $\chi^{(\alpha)} = \gamma$ and $A = Ran(\alpha)$.*

Proof. Suppose that γ is idempotent, and that $J_A = \mathrm{Ran}(\gamma)$. Then, by Theorem 6, there is a partition $\{I_j: j \in E\}$ of I and a subset $\{i_j: j \in E\}$ of I such that $i_j \in I_j$ and $\gamma(I_j) = \{i_j\}$ for all $j \in E$. From this, we have that $J_A = \mathrm{Ran}(\gamma) = \{i_j: j \in E\}$. For each $j \in E$, let $\alpha_j \in T_{\mathscr{F}_j}(\cup_{i \in I_j} Y_i)$, where $\mathscr{F}_j = \{Y_i: i \in I_j\}$, be idempotent such that $\chi^{(\alpha_j)}(i) = i_j$ for all $i \in I_j$ and $\mathrm{Ran}(\alpha_j) = A \cap Y_{i_j}$. And, finally, let $\alpha: X \longrightarrow X$ be defined by $\alpha|_{\cup_{i \in I_j} Y_i} = \alpha_j$ for all $j \in E$. Since $\{\cup_{i \in I_j} Y_i: j \in E\}$ is a partition of X, it follows that α is well defined. It is clear by the way

of defining α that $\alpha \in T_{\mathcal{F}}(X)$ with $\chi^{(\alpha)} = \gamma$. It is also clear that α is idempotent with $\mathrm{Ran}(\alpha) = A$. □

From Proposition 1, the following corollary is easily obtained.

Corollary 3. *Let $A \subseteq X$. Then, A is a Δ-retract of X if and only if J_A is a Δ-retract of I.*

Proof. Suppose that A is a Δ-retract of X. Then, there is an idempotent element α of $\mathrm{End}_{\mathcal{F}}(X)$ such that $A = \mathrm{Ran}(\alpha)$, which yields that $J_A = \mathrm{Ran}(\chi^{(\alpha)})$. Since α is idempotent, we have by Lemma 1 that $\chi^{(\alpha)}$ is an idempotent element of $\mathrm{End}(I)$. Hence, J_A is a Δ-retract of I. Conversely, suppose that J_A is a Δ-retract of I. Then, there is an idempotent element γ of $\mathrm{End}(I)$ such that $J_A = \mathrm{Ran}(\gamma)$. Thus, by Proposition 1, the set A is a Δ-retract of X. □

Proposition 2. *Let $\alpha, \beta \in \mathrm{End}_{\mathcal{F}}(X)$, and let $\lambda \in \mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta))$. If λ is a Δ-isomorphism, then $\chi^{(\lambda)}_{\alpha,\beta}$ is a Δ-isomorphism.*

Proof. Let $\lambda \in \mathrm{Hom}(\mathrm{Ran}(\alpha), \mathrm{Ran}(\beta))$. Then, $\chi^{(\lambda)}_{\alpha,\beta} \in \mathrm{Hom}(\mathrm{Ran}(\chi^{(\alpha)}), \mathrm{Ran}(\chi^{(\beta)}))$. Suppose that λ is a Δ-isomorphism. Then, there exists $\psi \in \mathrm{Hom}(\mathrm{Ran}(\beta), \mathrm{Ran}(\alpha))$ such that $\lambda\psi = \mathrm{id}_{\mathrm{Ran}(\beta)}$ and $\psi\lambda = \mathrm{id}_{\mathrm{Ran}(\alpha)}$. By the membership of ψ in $\mathrm{Hom}(\mathrm{Ran}(\beta), \mathrm{Ran}(\alpha))$, we have $\chi^{(\psi)}_{(\beta,\alpha)} \in \mathrm{Hom}(\mathrm{Ran}(\chi^{(\beta)}), \mathrm{Ran}(\chi^{(\alpha)}))$. Since $\lambda\psi = \mathrm{id}_{\mathrm{Ran}(\beta)}$, we get that $\chi^{(\lambda)}_{\alpha,\beta}\chi^{(\psi)}_{\beta,\alpha} = \chi^{(\lambda\psi)}_{\beta,\beta} = \chi^{(\mathrm{id}_{\mathrm{Ran}(\beta)})}_{\beta,\beta} = \mathrm{id}_{\mathrm{Ran}(\chi^{(\beta)})}$. Similarly, since $\psi\lambda = \mathrm{id}_{\mathrm{Ran}(\alpha)}$, we get $\chi^{(\psi)}_{\beta,\alpha}\chi^{(\lambda)}_{\alpha,\beta} = \mathrm{id}_{\mathrm{Ran}(\chi^{(\alpha)})}$. Thus, $\chi^{(\lambda)}_{\alpha,\beta}$ is a Δ-isomorphism. □

In the following theorem, we provide a characterization of the regularity of elements of $\mathrm{End}_{\mathcal{F}}(X)$ in terms of the Δ-retract and the Δ-isomorphism of I.

Theorem 7. *Let $\alpha \in \mathrm{End}_{\mathcal{F}}(X)$. Then, α is regular if and only if $\mathrm{Ran}(\chi^{(\alpha)})$ is a Δ-retract of I, and there is $A \subseteq X$ such that each of the following statements holds true:*

(i) J_A is Δ-retract of I;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a Δ-isomorphism from J_A onto $\mathrm{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{A\cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\mathrm{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

Proof. Suppose that $\mathrm{Ran}(\chi^{(\alpha)})$ is a Δ-retract of I, and that there exists a subset A of X such that each of the following statements holds true:

(i) J_A is Δ-retract of I;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a Δ-isomorphism from J_A onto $\mathrm{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{A\cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\mathrm{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

Since $\mathrm{Ran}(\chi^{(\alpha)})$ is a Δ-retract of I, we have by Corollary 3 that $\mathrm{Ran}(\alpha)$ is a Δ-retract of X. And, since J_A is a Δ-retract of I, there is an idempotent element γ of $\mathrm{End}(I)$ such that $J_A = \mathrm{Ran}(\gamma)$. Thus, by Proposition 1, there is an idempotent element β of $T_{\mathcal{F}}(X)$ such that $\chi^{(\beta)} = \gamma$ and $A = \mathrm{Ran}(\beta)$. This yields that A is a Δ-retract of X. By condition (ii), we have that $\chi^{(\alpha)}|_{J_A}$ is a bijective function from J_A onto $\mathrm{Ran}(\chi^{(\alpha)})$. Thus, by condition (*iii*), we get that $\alpha|_A$, which is an element of $T_{\mathcal{F}}(\beta, \alpha)$, is bijective. By condition (ii) once again, we have $\chi^{(\alpha|_A)}_{\beta,\alpha} = \chi^{(\alpha)}|_{J_A} \in \mathrm{Hom}(J_A, \mathrm{Ran}(\chi^{(\alpha)}))$ and $\chi^{((\alpha|_A)^{-1})}_{\alpha,\beta} = (\chi^{(\alpha)}|_{J_A})^{-1} \in \mathrm{Hom}(\mathrm{Ran}(\chi^{(\alpha)}), J_A)$. It follows that $\alpha|_A$ is a Δ-isomorphism from A onto $\mathrm{Ran}(\alpha)$. Therefore, by Theorem 4, we obtain that α is regular. Conversely, suppose that α is regular. Then, by Theorem 4, we get that $\mathrm{Ran}(\alpha)$ is a Δ-retract of X, and that there is a Δ-retract A of X such that $\alpha|_A$ is a Δ-isomorphism from A onto $\mathrm{Ran}(\alpha)$. Since $\mathrm{Ran}(\alpha)$ and A are Δ-retracts of X, we have by Corollary 3 that $\mathrm{Ran}(\chi^{(\alpha)})$ and J_A are Δ-retracts of I, respectively. Since $\alpha|_A$ is a Δ-isomorphism from A onto $\mathrm{Ran}(\alpha)$, we have that (iii) holds. And, by Proposition 2, we get that (ii) holds. □

If $\mathrm{End}(I) = T(I)$, then $\mathrm{End}_{\mathcal{F}}(X)$ becomes $T_{\mathcal{F}}(X)$. Hence, by Theorem 7, the following corollary is immediately obtained.

Corollary 4. *Let $\alpha \in T_{\mathcal{F}}(X)$. Then, α is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:*

(i) $\chi^{(\alpha)}|_{J_A}$ is a bijective function from J_A onto $\mathrm{Ran}(\chi^{(\alpha)})$;

(ii) $\alpha|_{A\cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\mathrm{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

Note that by considering $T_{\mathcal{F}}(X)$ as the semigroup of all continuous maps on X, where X is equipped with the topology having the family of all equivalence classes as a base, the regularity of $\alpha \in T_{\mathcal{F}}(X)$ can be deduced from Theorem 4 as well. This was provided by Huisheng [9]. The author obtained for any $\alpha \in T_{\mathcal{F}}(X)$ that α is regular if and only if for each $i \in I$, there is $j \in I$ such that $Y_i \cap \mathrm{Ran}(\alpha) \subseteq \alpha(Y_j)$. Here, we get another characterization of the regularity of elements of $T_{\mathcal{F}}(X)$ in terms of the character.

The following three corollaries are immediately obtained from Theorem 7 as well.

Corollary 5. *Suppose that I is a topological space, and let $\mathrm{End}(I) = S(I)$. Let $\alpha \in \mathrm{End}_{\mathcal{F}}(X)$. Then, α is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:*

(i) $\chi^{(\alpha)}|_{J_A}$ is a homeomorphism from J_A onto $\mathrm{Ran}(\chi^{(\alpha)})$;

(ii) $\alpha|_{A\cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\mathrm{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

Corollary 6. *Suppose that I is a T_1-space, and let $\mathrm{End}(I) = \Gamma(I)$. Let $\alpha \in \mathrm{End}_{\mathcal{F}}(X)$. Then, α is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:*

(i) J_A is a closed subset of I;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a homeomorphism from J_A onto $\mathrm{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{A\cap Y_i}$ *is a bijective function from* $A\cap Y_i$ *onto* $\mathrm{Ran}(\alpha)\cap Y_{\chi^{(\alpha)}(i)}$ *for all* $i\in J_A$.

Corollary 7. *Suppose that* I *is a vector space, and let* $\mathrm{End}(I)=L(I)$. *Let* $\alpha\in \mathrm{End}_{\mathcal{F}}(X)$. *Then,* α *is regular if and only if there is* $A\subseteq X$ *such that each of the following statements holds true:*

(i) J_A *is a subspace of* I;

(ii) $\chi^{(\alpha)}|_{J_A}$ *is an isomorphism from* J_A *onto* $\mathrm{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{A\cap Y_i}$ *is a bijective function from* $A\cap Y_i$ *onto* $\mathrm{Ran}(\alpha)\cap Y_{\chi^{(\alpha)}(i)}$ *for all* $i\in J_A$.

We end this section with a discussion on the regularity of the quotient semigroup $\mathrm{End}_{\mathcal{F}}(X)/\chi$, where χ is the congruence relation on $\mathrm{End}_{\mathcal{F}}(X)$ defined by

$$\begin{gathered}\forall\alpha,\beta\in \mathrm{End}_{\mathcal{F}}(X),\\ \alpha\chi\beta\Longleftrightarrow \chi^{(\alpha)}=\chi^{(\beta)}.\end{gathered}\tag{15}$$

By virtue of Corollary 2, we immediately get that the semigroup $\mathrm{End}_{\mathcal{F}}(X)/\chi$ is regular if and only if the semigroup $\mathrm{End}(I)$ is regular.

3. A Subsemigroup of $\mathrm{End}_{\mathcal{F}}(X)$ and Its Regularity

In this section, we define a sub-semigroup of $\mathrm{End}_{\mathcal{F}}(X)$ and study the regularity of that semigroup. Let κ be a cardinal number, and let

$$M_{\mathcal{F}}^{(\kappa)}(X)=\left\{\alpha\in T_{\mathcal{F}}(X)\colon \forall i\in \mathrm{Ran}\left(\chi^{(\alpha)}\right), |Y_i\cap \mathrm{Ran}(\alpha)|\le\kappa\right\}.\tag{16}$$

Lemma 2. *Let* $\sigma,\rho\in M_{\mathcal{F}}^{(\kappa)}(X)$ *and* $\gamma\in T_{\mathcal{F}}(\sigma,\rho)$. *Then,* $\gamma\sigma\in M_{\mathcal{F}}^{(\kappa)}(X)$.

Proof. Let $\beta=\gamma\sigma$. We now want to show that $\beta\in M_{\mathcal{F}}^{(\kappa)}(X)$. It is clear that $\beta\in T_{\mathcal{F}}(X)$. Next, let $i\in \mathrm{Ran}(\chi^{(\beta)})$. Since $\chi^{(\beta)}=\chi_{\sigma,\rho}^{(\gamma)}\chi^{(\sigma)}$, we have $\mathrm{Ran}(\chi^{(\beta)})=\chi_{\sigma,\rho}^{(\gamma)}(\mathrm{Ran}(\chi^{(\sigma)}))\subseteq \mathrm{Ran}(\chi_{\sigma,\rho}^{(\gamma)})\subseteq \mathrm{Ran}(\chi^{(\rho)})$. Thus, $i\in \mathrm{Ran}(\chi^{(\rho)})$), and there is $j\in \mathrm{Ran}(\chi^{(\sigma)})$ such that $\chi_{\sigma,\rho}^{(\gamma)}(j)=i$, which yields that $\gamma(Y_j\cap \mathrm{Ran}(\sigma))\subseteq Y_i\cap \mathrm{Ran}(\rho)$. Since $j\in \mathrm{Ran}(\chi^{(\sigma)})$, there is $k\in I$ such that $\chi^{(\sigma)}(k)=j$, which implies that $\sigma(Y_k)\subseteq Y_j$. Thus, $\beta(Y_k)=\gamma(\sigma(Y_k))=\gamma(\sigma(Y_k)\cap \mathrm{Ran}(\sigma))\subseteq\gamma(Y_j\cap \mathrm{Ran}(\sigma))\subseteq Y_i\cap \mathrm{Ran}(\rho)$. From this, we obtain that $Y_i\cap \mathrm{Ran}(\beta)\ne\varnothing$. Since $i\in \mathrm{Ran}(\chi^{(\rho)})$, we have that $|Y_i\cap \mathrm{Ran}(\rho)|\le\kappa$. And, since $\mathrm{Ran}(\beta)\subseteq \mathrm{Ran}(\gamma)\subseteq \mathrm{Ran}(\rho)$, it follows that $0<|Y_i\cap \mathrm{Ran}(\beta)|\le|Y_i\cap \mathrm{Ran}(\rho)|\le\kappa$. Thus, $\beta\in M_{\mathcal{F}}^{(\kappa)}(X)$. □

Corollary 8. *The set* $M_{\mathcal{F}}^{(\kappa)}(X)$ *is a sub-semigroup of* $T_{\mathcal{F}}(X)$.

Proof. Let $\alpha,\beta\in M_{\mathcal{F}}^{(\kappa)}(X)$. Then, $\alpha|_{\mathrm{Ran}(\beta)}\in T_{\mathcal{F}}(\beta,\alpha)$. Thus, by Lemma 2, we get that $\alpha\beta=\alpha|_{\mathrm{Ran}(\beta)}\beta\in M_{\mathcal{F}}^{(\kappa)}(X)$. □

Let

$$E_{\mathcal{F}}^{(\kappa)}(X)=\mathrm{End}_{\mathcal{F}}(X)\cap M_{\mathcal{F}}^{(\kappa)}(X).\tag{17}$$

Note that for each $\gamma\in T(I)$, there is $\alpha\in M_{\mathcal{F}}^{(\kappa)}(X)$ such that $\chi^{(\alpha)}=\gamma$. Such a function α can easily be defined as follows: for each $j\in \mathrm{Ran}(\gamma)$, let z_j be a fixed element of Y_j and let $\alpha\colon X\longrightarrow X$ be defined by $\alpha(\cup_{i\in\gamma^{-1}(j)}Y_i)=\{z_j\}$ for all $j\in \mathrm{Ran}(\gamma)$. Thus, by the note, we have that $E_{\mathcal{F}}^{(\kappa)}(X)\ne\varnothing$. Furthermore, it is a sub-semigroup of $\mathrm{End}_{\mathcal{F}}(X)$.

Theorem 8. *Let* $\alpha\in E_{\mathcal{F}}^{(1)}(X)$. *Then, the following statements are equivalent:*

(1) α *is a regular element of* $E_{\mathcal{F}}^{(1)}(X)$;

(2) α *is a regular element of* $\mathrm{End}_{\mathcal{F}}(X)$;

(3) $\chi^{(\alpha)}$ *is a regular element of* $\mathrm{End}(I)$.

Proof. ((3)⇒(2)). Suppose that $\chi^{(\alpha)}$ is a regular element of $\mathrm{End}(I)$. Then, by the regularity of $\chi^{(\alpha)}$ in $\mathrm{End}(I)$, we get by Theorem 4 that $\mathrm{Ran}(\chi^{(\alpha)})$ is a Δ-retract of I, and that there is an idempotent element γ of $\mathrm{End}(I)$ such that $\chi^{(\alpha)}|_{\mathrm{Ran}(\gamma)}$ is a Δ-isomorphism from $\mathrm{Ran}(\gamma)$ onto $\mathrm{Ran}(\chi^{(\alpha)})$. Since γ is an idempotent element of $\mathrm{End}(I)$, we have by Theorem 6 that there is a partition $\{I_j\colon j\in E\}$ of I and a subset $\{i_j\colon j\in E\}$ of I with $i_j\in I_j$ for all $j\in E$ such that $\gamma(I_j)=\{i_j\}$ for all $j\in E$. So, $\mathrm{Ran}(\gamma)=\{i_j\colon j\in E\}$. Since $\chi^{(\alpha)}|_{\mathrm{Ran}(\gamma)}$ is a Δ-isomorphism from $\mathrm{Ran}(\gamma)$ onto $\mathrm{Ran}(\chi^{(\alpha)})$, we have that $\chi^{(\alpha)}|_{\mathrm{Ran}(\gamma)}\in \mathrm{Hom}(\mathrm{Ran}(\gamma),\mathrm{Ran}(\chi^{(\alpha)}))$ is bijective, and that $(\chi^{(\alpha)}|_{\mathrm{Ran}(\gamma)})^{-1}\in \mathrm{Hom}(\mathrm{Ran}(\chi^{(\alpha)}),\mathrm{Ran}(\gamma))$. So, $\mathrm{Ran}(\chi^{(\alpha)})=\{\chi^{(\alpha)}(i_j)\colon j\in E\}$ with $|\mathrm{Ran}(\chi^{(\alpha)})|=|E|$. Since $\alpha\in M_{\mathcal{F}}^{(1)}(X)$, we have that $|\mathrm{Ran}(\alpha)\cap Y_i|=1$ for all $i\in \mathrm{Ran}(\chi^{(\alpha)})$. Let $\mathrm{Ran}(\alpha)\cap Y_{\chi^{(\alpha)}(i_j)}=\{a_j\}$ for all $j\in E$. Then, $\mathrm{Ran}(\alpha)=\{a_j\colon j\in E\}$. For each $j\in E$, fix $b_j\in\alpha^{-1}(a_j)\cap Y_{i_j}$. Let $\beta\colon X\longrightarrow X$ be defined by $\beta(\cup_{i\in I_j}Y_i)=\{b_j\}$ for all $j\in E$. Since the family $\{\cup_{i\in I_j}Y_i\colon j\in E\}$ is a partition of X, we get that β is well defined. It is clear that β is an idempotent element of $T_{\mathcal{F}}(X)$, and that $\chi^{(\beta)}=\gamma\in \mathrm{End}(I)$. Thus, β is an idempotent element of $\mathrm{End}_{\mathcal{F}}(X)$ yields that $\mathrm{Ran}(\beta)$ is a Δ-retract of X. Moreover, we have $\alpha|_{\mathrm{Ran}(\beta)\cap Y_i}$ is a bijective function from $\mathrm{Ran}(\beta)\cap Y_i$ onto $\mathrm{Ran}(\alpha)\cap Y_{\chi^{(\alpha)}(i)}$ for all $i\in \mathrm{Ran}(\chi^{(\beta)})$. Therefore, by Theorem 7, we obtain that α is regular.

((2)⇒(1)). Suppose that α is a regular element of $\mathrm{End}_{\mathcal{F}}(X)$. Then, by Theorem 4 and Proposition 1, there exists an idempotent element σ of $\mathrm{End}_{\mathcal{F}}(X)$ such that $\chi^{(\sigma)}=\chi^{(\alpha)}$ and $\mathrm{Ran}(\alpha)=\mathrm{Ran}(\sigma)$. Since $\alpha\in E_{\mathcal{F}}^{(1)}(X)$, it follows that $\sigma\in E_{\mathcal{F}}^{(1)}(X)$. By Theorem 7, there exists an idempotent element ρ of $\mathrm{End}_{\mathcal{F}}(X)$ such that

(a) $\chi^{(\alpha)}|_{\mathrm{Ran}(\chi^{(\rho)})}$ is a $\triangle$-isomorphism from $\mathrm{Ran}(\chi^{(\rho)})$ onto $\mathrm{Ran}(\chi^{(\sigma)})$;

(b) $\alpha|_{\mathrm{Ran}(\rho)\cap Y_i}$ is a bijective function from $\mathrm{Ran}(\rho)\cap Y_i$ onto $\mathrm{Ran}(\sigma)\cap Y_{\chi^{(\sigma)}(i)}$ for all $i\in \mathrm{Ran}(\chi^{(\rho)})$.

From (a) and (b), we have that $(\alpha|_{\mathrm{Ran}(\rho)})^{-1}\in \mathrm{Hom}(\mathrm{Ran}(\sigma),\mathrm{Ran}(\rho))$ and $\rho\in E_{\mathcal{F}}^{(1)}(X)$, respectively. Let $\beta=(\alpha|_{\mathrm{Ran}(\rho)})^{-1}\sigma$. Then, by the property ($\Delta 3$), we have that $\beta\in \mathrm{End}_{\mathcal{F}}(X)$. And, by Lemma 2, we immediately obtain that $\beta\in M_{\mathcal{F}}^{(1)}(X)$. Thus, $\beta\in E_{\mathcal{F}}^{(1)}(X)$. Finally, we will show that $\alpha\beta\alpha=\alpha$. Let $x\in X$. Then, $\alpha(x)\in \mathrm{Ran}(\sigma)$. Hence, there is $z\in X$ such that $\alpha(x)=\sigma(z)$. And, since σ is

idempotent, it follows that $\alpha\beta\alpha(x) = \sigma(\alpha(x)) = \sigma(\sigma(z)) = \sigma(z) = \alpha(x)$. Therefore, $\alpha\beta\alpha = \alpha$, and hence α is a regular element of $E_{\mathcal{F}}^{(1)}(X)$.

((1)⇒(3)). It follows directly from Lemma 1. □

Corollary 9. *The semigroup $E_{\mathcal{F}}^{(1)}(X)$ is regular if and only if $\mathrm{End}(I)$ is regular.*

4. Conclusions

The semigroup $T_{\mathcal{F}}^{(\mathcal{S})}(X)$, where $\mathcal{S}$ is a sub-semigroup of $T(I)$, was first defined by Rakbud [12] in 2018 via the notion of the character introduced by Purisang and Rakbud [11] in 2016. Here, we focus on studying the regularity of the semigroup $T_{\mathcal{F}}^{(\mathcal{S})}(X)$ when $\mathcal{S}$ is the semigroup of a Δ-structure on I, which is written as $\mathcal{S} = \mathrm{End}(I)$. In our study, we obtain that $T_{\mathcal{F}}^{(\mathcal{S})}(X)$, which is denoted by $\mathrm{End}_{\mathcal{F}}(X)$, is the semigroup of a Δ-structure on X. From this, the regularity of elements of $\mathrm{End}_{\mathcal{F}}(X)$ can generally be explained via Theorem 4 established by Magill and Subbiah [13] in 1974. We also obtain a characterization of regular elements of $\mathrm{End}_{\mathcal{F}}(X)$ in terms of the Δ-structure on I (see Theorem 7). From this result, we deduce the regularity of $\mathrm{End}_{\mathcal{F}}(X)$ when $\mathrm{End}(I)$ is one of the following semigroups: the transformation semigroup $T(I)$, the semigroup $S(I)$ of continuous maps on I when I is a topological space, the semigroup $\Gamma(I)$ of closed maps on I when I is a T_1-space, and the semigroup $L(I)$ of linear transformations on I when I is a vector space (see Corollaries 4–7). Apart from the regularity of $\mathrm{End}_{\mathcal{F}}(X)$, we provide a sub-semigroup of $\mathrm{End}_{\mathcal{F}}(X)$, namely, the semigroup $E_{\mathcal{F}}^{(1)}(X)$, whose regularity coincides with that of $\mathrm{End}(I)$. In [13], Magill and Subbiah also generally gave some characterizations of Green's relations for regular elements of the semigroup of a Δ-structure. Since our semigroup $\mathrm{End}_{\mathcal{F}}(X)$ is the semigroup of a Δ-structure on X, some rough characterizations of Green's relations for regular elements of $\mathrm{End}_{\mathcal{F}}(X)$ can immediately be deduced from the results of Magill Jr. and Subbiah.

We end this paper with some interesting questions:

(1) Can Green's relations for regular elements of $\mathrm{End}_{\mathcal{F}}(X)$ be characterized more deeply in terms of the Δ-structure on I?

(2) Can other notions such as the ideal, the rank, the left regularity, and the right regularity in the semigroup $\mathrm{End}_{\mathcal{F}}(X)$ be explained in terms of those in the semigroup $\mathrm{End}(I)$?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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