Research Article

Regularity of Semigroups of Transformations Whose Characters Form the Semigroup of a $\Delta$-Structure

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In this paper, we make use of the notion of the character of a transformation on a fixed set $X$, provided by Purisang and Rakbud in 2016, and the notion of a $\Delta$-structure on $X$, provided by Magill Jr. and Subbiah in 1974, to define a sub-semigroup of the full-transformation semigroup $T(X)$. We also define a sub-semigroup of that semigroup. The regularity of those two semigroups is also studied.

1. Introduction and Preliminaries

For any semigroup $\mathcal{S}$, we call an element $a$ of $\mathcal{S}$ a regular element of $\mathcal{S}$ if there exists an element $b$ of $\mathcal{S}$ such that $aba = a$. A semigroup $\mathcal{S}$ is said to be regular if every element of $\mathcal{S}$ is regular. The semigroup of transformations on a nonempty set $X$, denoted by $T(X)$, is a well-known regular semigroup. In [1], Nenthein et al. studied the regularity of the following two sub-semigroups of $T(X)$:

$$
T(X, Y) = \{a \in T(X): \text{Ran}(a) \subseteq Y\},
$$

$$
\overline{T}(X, Y) = \{a \in T(X): \alpha(Y) \subseteq Y\},
$$

where $Y$ is a fixed nonempty subset of $X$ and $\text{Ran}(a)$ denotes the range of $a$ for all $a \in T(X)$. The authors obtained that the two semigroups are regular if and only if $|Y| = 1$ or $Y = X$. The regularity of some other sub-semigroups of $T(X)$ has been studied by many people (see [2–6] for some recent works).

In this paper, by a partition of a nonempty set $X$, we mean a family $\mathcal{F} = \{Y_i: i \in I\}$ of nonempty subsets of $X$ such that $X = \bigcup_{i \in I} Y_i$ and $Y_i \cap Y_j = \emptyset$ for all $i \neq j$. For a given partition $\mathcal{F} = \{Y_i: i \in I\}$ of a nonempty set $X$, let

$$
T_\mathcal{F}(X) = \{a \in T(X): \forall i \exists j \in I, a(Y_i) \subseteq Y_j\}.
$$

Note that $T_\mathcal{F}(X)$ is exactly the semigroup of all transformations on $X$ preserving the equivalence relation induced by partition $\mathcal{F}$. There have been several works involving transformations preserving an equivalence relation (see [3–5, 7–12] for some references).

Throughout this paper, let $X$ be a nonempty set, and let $\mathcal{F} = \{Y_i: i \in I\}$ be a partition of $X$, which are arbitrarily fixed. From the definition of a partition, Purisang and Rakbud [11] defined a function $\chi^{(a)}: I \rightarrow I$ associated with each $a \in T_\mathcal{F}(X)$ called the character of $a$, by

$$
\forall i, j \in I,
$$

$$
\chi^{(a)}(i) = j \iff a(Y_i) \subseteq Y_j.
$$

They also defined and studied the regularity of the following sub-semigroup of $T(X)$:

$$
T_{\mathcal{F}^{(j)}}(X) = \{a \in T(X): \forall i \exists j \in J, a(Y_i) \subseteq Y_j\}
$$

$$
= \{a \in T(X): \chi^{(a)}(i) \in T(I, J)\},
$$

where $J$ is an arbitrarily fixed nonempty subset of the index set $I$. We summarize some of their results as follows.

**Lemma 1** (see [11], Lemma 2.3). For every $\alpha, \beta \in T_{\mathcal{F}^{(j)}}(X)$, $\chi^{(\alpha \beta)} = \chi^{(\alpha)} \chi^{(\beta)}$.
By making use of the notion of the character, the following two equivalence relations \( \chi \) and \( \bar{\chi} \) on \( T^{(I)}_{\varphi}(X) \) were defined:

\[
\forall \alpha, \beta \in T^{(I)}_{\varphi}(X), \quad (\alpha, \beta) \in \chi \iff \chi^{(\alpha)} = \chi^{(\beta)},
\]

\[
\forall \alpha, \beta \in T^{(I)}_{\varphi}(X), \quad (\alpha, \beta) \in \bar{\chi} \iff \bar{\chi}^{(\alpha)} = \bar{\chi}^{(\beta)}.
\]

(5)

Note that, by Lemma 1, they both are congruence relations. The authors studied the regularity of the quotient semigroups \( T^{(I)}_{\varphi}(X)/\chi \) and \( T^{(I)}_{\varphi}(X)/\bar{\chi} \). The following are what they obtained.

**Theorem 1** (see [11], Theorem 2.4). The following statements hold:

1. \( T^{(I)}_{\varphi}(X)/\chi \equiv T(I, I) \) by the isomorphism \( [\alpha] \to \chi^{(\alpha)} \);
2. \( T^{(I)}_{\varphi}(X)/\bar{\chi} \equiv T(I, I) \) by the isomorphism \( [\alpha] \to \bar{\chi}^{(\alpha)} \),

where for each \( \alpha \in T^{(I)}_{\varphi}(X) \), \( [\alpha] \) and \( \bar{[\alpha]} \) denote the equivalence classes of \( \alpha \) under the equivalence relations \( \chi \) and \( \bar{\chi} \) respectively.

**Corollary 1** (see [11], Corollary 2.5). The following statements hold.

1. The three statements below are all equivalent:
   a. The quotient semigroup \( T^{(I)}_{\varphi}(X)/\chi \) is regular;
   b. The semigroup \( T(I, I) \) is regular;
   c. \( I = 1 \) or \( |I| = 1 \).
2. The quotient semigroup \( T^{(I)}_{\varphi}(X)/\bar{\chi} \) is regular.
3. The quotient semigroup \( T^{(I)}_{\varphi}(X)/\bar{\chi} \) is regular.

The regularity of the semigroup \( T^{(I)}_{\varphi}(X) \) was obtained as follows.

**Theorem 2** (see [11], Theorem 2.6). The semigroup \( T^{(I)}_{\varphi}(X) \) is regular if and only if \( |T^{(I)}_{\varphi}(X)| = 1 \) or \( T^{(I)}_{\varphi}(X) = T(I, X) \).

It is easy to see that for each \( \alpha \in T_{\varphi}(X) \), the equivalence class \( [\alpha] \) of \( \alpha \) under the equivalence relation \( \chi \) is a subsemigroup of \( T_{\varphi}(X) \) if and only if \( \chi^{(\alpha)} \) is an idempotent element of \( T(I) \). In [11], the authors studied the regularity of the semigroup \( [\alpha] \) when \( \alpha \) is an idempotent element of \( T(I) \). They also defined some further sub-semigroups of \( T_{\varphi}(X) \) by making use of the notion of the character as follows: let \( I_{\varphi}(X) \), \( S_{\varphi}(X) \), and \( B_{\varphi}(X) \) be the sets of all elements of \( T_{\varphi}(X) \) whose characters are injective, surjective, and bijective, respectively. Note that, by Lemma 1, the sets \( I_{\varphi}(X) \), \( S_{\varphi}(X) \), and \( B_{\varphi}(X) \) are sub-semigroups of \( T_{\varphi}(X) \). The regularity of each of these three semigroups was also studied.

It was observed by Rakbud [12] that the semigroups \( T^{(I)}_{\varphi}(X) \), \([\alpha] \) when \( \chi^{(\alpha)} \) is idempotent, \( I_{\varphi}(X) \), \( S_{\varphi}(X) \), and \( B_{\varphi}(X) \) can simultaneously be generalized by making use of the notion of the character as follows: for every sub-semigroup \( \delta \) of \( T(I) \), let

\[
T^{(\delta)}_{\varphi}(X) = \{ \alpha \in T_{\varphi}(X) : \chi^{(\alpha)} \in \delta \}.
\]

(6)

Note that, for each \( \gamma \in T(I) \), the function \( \alpha : X \to X \) defined on each \( \gamma \) by \( \alpha(Y) = \{ z \} \), where \( z \) is a fixed element of \( Y \), for all \( i \in I \), is an element of \( T_{\varphi}(X) \) whose character is exactly \( \gamma \). Hence, \( T^{(\delta)}_{\varphi}(X) \neq \emptyset \), and by Lemma 1, it is a sub-semigroup of \( T_{\varphi}(X) \).

Let \( \delta \) be a sub-semigroup of \( T(I) \). Then, by considering the congruence relation \( \chi \) on \( T_{\varphi}(X) \) restricted to \( T^{(\delta)}_{\varphi}(X) \), we have the quotient semigroup \( T^{(\delta)}_{\varphi}(X)/\chi \). Obviously, \( T^{(\delta)}_{\varphi}(X)/\chi = \{ [\alpha] : \alpha \in T^{(\delta)}_{\varphi}(X) \} \) and \( T^{(\delta)}_{\varphi}(X)/\chi \) is a subsemigroup of \( T_{\varphi}(X)/\chi \). Analogously to Theorem 1, the following result was established.

**Theorem 3** (see [12], Theorem 1.14). \( T^{(\delta)}_{\varphi}(X)/\chi \equiv \delta \) by the isomorphism defined by \( [\alpha] \to \chi^{(\alpha)} \).

Immediately from Theorem 3, the following corollary was obtained.

**Corollary 2** (see [12], Corollary 1.15). The quotient semigroup \( T^{(\delta)}_{\varphi}(X)/\chi \) is regular if and only if the semigroup \( \delta \) is regular.

Besides the above results, in [12], the author also used the notion of the character to define the notion of a weakly regular transformation and study the regularity of a semigroup of weakly regular transformations in that sense. However, the regularity of \( T^{(\delta)}_{\varphi}(X) \) has not been studied in general yet. This will be in our attention here when \( \delta \) has a certain property. We are mentioning that \( \delta \) is “the semigroup of a \( \Delta \)-structure” on \( I \).

We now refer to the definition of a \( \Delta \)-structure on a set and some other related ones from [13] by Magill and Subbiah. Let \( \mathcal{A} \) be a family of nonempty subsets of \( X \) such that \( X \in \mathcal{A} \). And, let \( \mathcal{M} = \{ \text{Hom}(A, B) : A, B \in \mathcal{A} \} \), where \( \text{Hom}(A, B) \) is a nonempty set of functions from \( A \) into \( B \) for all \( A, B \in \mathcal{A} \), with the following properties:

1. (\( \Delta 1 \)) End \( (X) = \text{Hom}(X, X) \) is a monoid;
2. (\( \Delta 2 \)) \( \text{ran}(\alpha) \in \mathcal{A} \) for all \( \alpha \in \text{End}(X) \);
3. (\( \Delta 3 \)) For all \( B \in \mathcal{A} \), \( \alpha \in \text{End}(X) \) and \( \beta \in \text{Hom}(\text{ran}(\alpha), B) \), \( \beta \alpha \in \text{End}(X) \);
4. (\( \Delta 4 \)) For all \( \alpha, \beta \in \text{End}(X) \) and \( A, B \in \mathcal{A} \) with \( \alpha(B) \subseteq A \) and \( \beta(A) \subseteq B \), if \( \text{id}_A \beta \alpha = \text{id}_A \alpha B = \text{id}_A \), then \( \beta \alpha \subseteq \text{Hom}(A, B) \) and \( A \beta \alpha \subseteq \text{Hom}(B, A) \), where \( \text{id}_A \) denotes the identity function on \( Z \) for any nonempty set \( Z \).

The pair \( (\mathcal{A}, \mathcal{M}) \) is called a \( \Delta \)-structure on \( X \), and the monoid \( \text{End}(X) \) is called the semigroup of the \( \Delta \)-structure \( (\mathcal{A}, \mathcal{M}) \). A subset \( A \) of \( X \) is called a \( \Delta \)-retract of \( X \) if there is an idempotent element \( \rho \) of \( \text{End}(X) \) such that \( A = \text{ran}(\rho) \).

For any \( A, B \in \mathcal{A} \), an element \( \lambda \) of \( \text{Hom}(A, B) \) is called a \( \Delta \)-isomorphism (from \( A \) onto \( B \)) if there is a \( \sigma \in \text{Hom}(B, A) \) such that \( \sigma \lambda = \text{id}_A \lambda = \lambda \sigma = \text{id}_B \). It is clear for any \( A, B \in \mathcal{A} \) that an element \( \lambda \) of \( \text{Hom}(A, B) \) is \( \Delta \)-isomorphism if and only if \( \lambda \) is bijective and \( \lambda^{-1} \in \text{Hom}(B, A) \).

In [13], the authors gave some characterizations of regular elements of the semigroup \( \text{End}(X) \) as follows.
Theorem 4 (see [13], Theorem 2.4). Let $\alpha \in \text{End}(X)$. Then, the following statements are equivalent:

1. $\alpha$ is regular;
2. $\text{Ran}(\alpha)$ is a $\Delta$-retract of $X$, and there is a $\Delta$-retract $A$ of $X$ such that $\alpha|_A$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$;
3. $\text{Ran}(\alpha)$ is a $\Delta$-retract of $X$, and there is $A \in \mathcal{A}$ such that $\alpha|_A$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$.

It is clear that $T(X) = \text{End}(X)$ when $X$ is equipped with the $\Delta$-structure $(\mathcal{A}, \mathcal{M})$, where $\mathcal{A}$ is the family of all nonempty subsets of $X$, and $\text{Hom}(A, B)$ is the set of all functions from $A$ into $B$ for all $A, B \in \mathcal{A}$. In this setting, the $\Delta$-retracts of $X$ are exactly the nonempty subsets of $X$, and the $\Delta$-isomorphisms are exactly the bijective functions. More interesting semigroups of $\Delta$-structures on $X$ were given in [13] as follows:

(i) The semigroup $S(X)$ of all continuous maps on $X$ is exactly $\text{End}(X)$ when $X$ is a topological space equipped with the $\Delta$-structure $(\mathcal{A}, \mathcal{M})$, where $\mathcal{A}$ is the family of all nonempty subsets of $X$ and $\text{Hom}(A, B)$ is the set of all continuous maps from $A$ into $B$ for all $A, B \in \mathcal{A}$. In this setting, the $\Delta$-retracts of $X$ are exactly the $\Delta$-retracts of the topological space $X$, and the $\Delta$-isomorphisms are exactly the homeomorphisms.

(ii) The semigroup $L(X)$ of all linear transformations on $X$ coincides with $\text{End}(X)$ when $X$ is a vector space equipped with the $\Delta$-structure $(\mathcal{A}, \mathcal{M})$, where $\mathcal{A}$ is the family of all subspaces of $X$ and $\text{Hom}(A, B)$ is the set of all linear transformations from $A$ into $B$ for all $A, B \in \mathcal{A}$. In this setting, the $\Delta$-retracts of $X$ are exactly the $\Delta$-retracts of the vector space $X$, and the $\Delta$-isomorphisms are exactly the isomorphisms.

(iii) The semigroup $\Gamma(X)$ of all closed maps on $X$ is exactly $\text{End}(X)$ when $X$ is a $T_1$-space equipped with the $\Delta$-structure $(\mathcal{A}, \mathcal{M})$, where $\mathcal{A}$ is the family of all nonempty closed subsets of $X$ and $\text{Hom}(A, B)$ is the set of all closed maps from $A$ into $B$ for all $A, B \in \mathcal{A}$. In this setting, the $\Delta$-retracts of $X$ are exactly the nonempty closed subsets of the topological space $X$, and the $\Delta$-isomorphisms are exactly the homeomorphisms.

The regularity of the semigroups $S(X), L(X), \text{and } \Gamma(X)$ was also deduced via Theorem 4. In addition, we note here that the semigroup $T(\mathcal{S}(X))$ of all transformations on $X$ preserving an equivalence relation $\mathcal{E}$ on $X$ can be considered as the semigroup of all continuous maps on $X$, where $X$ is equipped with the topology having the family of all equivalence classes as a base. This was proved by Huisheng [8] (see Theorem 2.8).

The main aim of this paper is to study the regularity of the semigroup $T(\mathcal{S}(X))$, introduced in [12], when $\mathcal{S}$ is a $\Delta$-structure on the index set $I$ of the partition $\mathcal{S}$. We also define, in this situation, a sub-semigroup of $T(\mathcal{S}(X))$ whose regularity coincides with that of the semigroup $\mathcal{S}$.

2. The Semigroup and Its Regularity

For any nonempty sets $Z$ and $W$ and partitions $\mathcal{P} = \{Z_i; i \in I\}$ and $\mathcal{Q} = \{W_j; i \in K\}$ of $Z$ and $W$, respectively, let $T_{\mathcal{P},\mathcal{Q}}(Z, W)$ be the set of all functions $\lambda: Z \rightarrow W$ satisfying the condition that for all $i \in H$, there is $j \in K$ such that $\lambda(Z_i) \subseteq W_j$. And, for each $\lambda \in T_{\mathcal{P},\mathcal{Q}}(Z, W)$, let $\chi_{\mathcal{P},\mathcal{Q}}(\lambda): H \rightarrow K$ be defined by

$$\chi_{\mathcal{P},\mathcal{Q}}(\lambda)(i) = \lambda(Z_i) \subseteq W_j.$$  \hspace{1cm} (7)

It is easy to verify that the map $\lambda \mapsto \chi_{\mathcal{P},\mathcal{Q}}(\lambda)$ from the set $T_{\mathcal{P},\mathcal{Q}}(Z, W)$ into the set of all functions from $H$ into $K$ is surjective. If we have three nonempty sets $Z, W$, and $U$ with partitions $\mathcal{S} = \{Z_i; i \in H\}$, $\mathcal{Q} = \{W_j; i \in K\}$, and $\mathcal{R} = \{U_i; i \in M\}$ of $Z, W$, and $U$, respectively, we see for any $\lambda \in T_{\mathcal{P},\mathcal{Q}}(Z, W)$ and $\rho \in T_{\mathcal{Q},\mathcal{R}}(W, U)$ that $\rho \lambda \in T_{\mathcal{P},\mathcal{R}}(Z, U)$ and $\chi_{\mathcal{P},\mathcal{R}}(\rho \lambda) = \chi_{\mathcal{P},\mathcal{Q}}(\lambda) \circ \chi_{\mathcal{Q},\mathcal{R}}(\rho)$.

For all $\alpha, \beta \in T_{\mathcal{S}}(X)$, let

$$T_{\mathcal{S}}(\alpha, \beta) = T_{\mathcal{P},\mathcal{Q}}(\text{Ran}(\alpha), \text{Ran}(\beta)),$$  \hspace{1cm} (8)

where

$$\mathcal{P} = \{\text{Ran}(\alpha) \cap Y_i; i \in \text{Ran}(\chi^{(\alpha)\mathcal{S}})\},$$

$$\mathcal{Q} = \{\text{Ran}(\beta) \cap Y_i; i \in \text{Ran}(\chi^{(\beta)\mathcal{S}})\}.$$  \hspace{1cm} (9)

And, for each $\lambda \in T_{\mathcal{S}}(\alpha, \beta)$, let $\chi_{\mathcal{S}}^{(\alpha)\mathcal{S}} = \chi_{\mathcal{S}}^{(\beta)\mathcal{S}}$. Let $(\mathcal{S}, \mathcal{E})$ be a $\Delta$-structure on the index set $I$ of the partition $\mathcal{S}$, and let

$$\text{End}_{\mathcal{S}}(X) = T_{\mathcal{S}}^{(\text{End}(I))}(X) = \{\alpha \in T_{\mathcal{S}}(X); \chi^{(\alpha)} \in \text{End}(I)\}.$$  \hspace{1cm} (10)

By the property $(\Delta 1)$ of the $\Delta$-structure $(\mathcal{S}, \mathcal{E})$ on $I$, we have that $\chi_{\mathcal{S}}^{(\alpha)\mathcal{S}} = \text{id}_I \in \text{End}(I)$, which yields that $\text{id}_I \in \text{End}_{\mathcal{S}}(X)$. Hence, $\text{End}_{\mathcal{S}}(X)$ is a submonoid of $T_{\mathcal{S}}(X)$.

Theorem 5. There is a $\Delta$-structure on $X$ such that $\text{End}(X) = \text{End}_{\mathcal{S}}(X)$.

Proof. Let

$$\mathcal{A} = \{\text{Ran}(\alpha); \alpha \in \text{End}_{\mathcal{S}}(X)\}.\hspace{1cm} (11)$$

Since $\text{End}_{\mathcal{S}}(X)$ is a monoid, we have $X \in \mathcal{A}$. For any $\alpha, \beta \in \text{End}_{\mathcal{S}}(X)$, let

$$\text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta)) = \{\lambda \in T_{\mathcal{S}}(\alpha, \beta); \chi_{\mathcal{S}}^{(\alpha)\mathcal{S}} \in \text{Hom}(\text{Ran}(\chi^{(\alpha)\mathcal{S}}), \text{Ran}(\chi^{(\beta)\mathcal{S}}))\}.$$  \hspace{1cm} (12)

Then, by the surjectivity of the map $\lambda \mapsto \chi_{\mathcal{S}}^{(\alpha)\mathcal{S}}$ from the set $T_{\mathcal{S}}(\alpha, \beta)$ into the set of all functions from $\text{Ran}(\chi^{(\alpha)\mathcal{S}})$ into $\text{Ran}(\chi^{(\beta)\mathcal{S}})$, we have that $\text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta)) \neq \emptyset$ for all $\alpha, \beta \in \text{End}_{\mathcal{S}}(X)$. Note that for all $\alpha, \beta, \lambda, \rho \in \text{End}_{\mathcal{S}}(X)$, if $\text{Ran}(\alpha) = \text{Ran}(\lambda)$ and $\text{Ran}(\beta) = \text{Ran}(\rho)$, then $\text{Ran}(\chi^{(\alpha)}) = \text{Ran}(\chi^{(\lambda)}) = \text{Ran}(\chi^{(\rho)})$.
Ran(\(\chi^{(1)}\)) and Ran(\(\chi^{(3)}\)) = Ran(\(\chi^{(2)}\)), which yields that Hom(Ran (\(\alpha\)), Ran (\(\beta\))) = Hom(Ran (\(\lambda\)), Ran (\(\rho\))). Let

\[ \mathcal{M} = \{\text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta)) : \alpha, \beta \in \text{End}_X(X)\}. \]  

Then, from the above explanation, the family \(\mathcal{M}\) is well defined. We will show that \((\mathcal{A}, \mathcal{M})\) is a \(\Delta\)-structure on \(X\). It is easy to see that

\[ \text{Hom}(X, X) = \text{Hom}(\text{Ran}(\text{id}_X), \text{Ran}(\text{id}_X)) \]
\[ = \{\lambda \in \text{End}_X(X) : \chi^{(1)} \in \text{End}(I)\} \]  

\[ = \text{End}_X(X). \]

Thus, the property \((\Delta_1)\) is satisfied. From the definition of the family \(\mathcal{A}\), we immediately have that the property \((\Delta_2)\) holds. Next, we will show that the property \((\Delta_3)\) is satisfied. Let \(\alpha, \beta \in \text{End}_X(X)\), and let \(\lambda \in \text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta))\). We want to show that \(\lambda \alpha \in \text{End}_X(X)\). Since \(\lambda \in \text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta))\), we have \(\chi^{(3)} \in \text{End}(\text{Ran}(\alpha), \text{Ran}(\beta))\). Since \(\alpha \in \text{End}_X(X)\), we have that \(\chi^{(1)} \in \text{End}(I)\). Thus, by the property \((\Delta_3)\) of the \(\Delta\)-structure \((\mathcal{A}, \mathcal{M})\) on \(I\), we get \(\chi^{(1)} \in \text{End}(I)\). It is clear that \(\lambda \alpha \in \text{End}_X(X)\), and that \(\chi^{(1)} \in \text{End}(I)\). Hence, by the membership of \(\chi^{(1)} \in \text{End}(I)\), we get that \(\alpha \in \text{End}_X(X)\) as desired. Finally, we will show that the property \((\Delta_4)\) holds. Let \(\alpha, \beta, \psi, \phi \in \text{End}_X(X)\) be such that \(\alpha(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\) and \(\beta(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\). Then, \(\chi^{(1)} \subseteq \text{End}(\text{Ran}(\phi))\). Similarly, by the inclusion \(\beta(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\), we obtain \(\chi^{(1)} \subseteq \text{End}(\text{Ran}(\phi))\). From this, we get that \(\alpha \mid \text{Ran}(\psi) \in \text{End}(\text{Ran}(\phi))\).

From now on, we will consider the \(\Delta\)-structure \((\mathcal{A}, \mathcal{M})\) on \(X\) defined in the proof of Theorem 5, and the semigroup \(\text{End}_X(X)\) of this \(\Delta\)-structure will be in our attention. By Theorem 4, the regularity of elements of the semigroup \(\text{End}_X(X)\) can roughly be characterized. To get more precise characterizations, according to Theorem 4, the notions of a \(\Delta\)-retract and a \(\Delta\)-isomorphism in the \(\Delta\)-structure \((\mathcal{A}, \mathcal{M})\) on \(X\) should particularly be studied. For that purpose, the following elementary theorem, stating some characterizations of idempotent elements in a transformation semigroup, is needed.

**Theorem 6.** Let \(Z\) be a nonempty set and \(\alpha \in T(Z)\). Then, the following statements are equivalent:

1. \(\alpha\) is idempotent;
2. \(\alpha \mid \text{Ran}(\alpha) = \text{id}_X(\text{Ran}(\alpha))\);
3. There is a partition \(\{Z_j : j \in E\}\) of \(Z\) and a subset \(\{z_j : j \in E\}\) of \(Z\) such that \(z_j \in z_j\) for all \(j \in E\).

In this situation, the partition \(\{Z_j : j \in E\}\) of \(Z\) is unique determined by \(\alpha\).

For each nonempty subset \(A\) of \(X\), we see that there is a unique subset of \(I\), denoted by \(I_A\), such that the family \(\{\alpha \in \text{End}_X(X) : \alpha(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\}\). For each nonempty subset \(A\) of \(X\), we see that there is a unique subset of \(I\), denoted by \(I_A\), such that the family \(\{\alpha \in \text{End}_X(X) : \alpha(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\}\). For each nonempty subset \(A\) of \(X\), we see that there is a unique subset of \(I\), denoted by \(I_A\), such that the family \(\{\alpha \in \text{End}_X(X) : \alpha(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\}\). For each nonempty subset \(A\) of \(X\), we see that there is a unique subset of \(I\), denoted by \(I_A\), such that the family \(\{\alpha \in \text{End}_X(X) : \alpha(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\}\). For each nonempty subset \(A\) of \(X\), we see that there is a unique subset of \(I\), denoted by \(I_A\), such that the family \(\{\alpha \in \text{End}_X(X) : \alpha(\text{Ran}(\psi)) \subseteq \text{Ran}(\phi)\}\).
of defining $a$ that $a \in T_{\varphi}(X)$ with $\chi^{(a)} = \gamma$. It is also clear that $a$ is idempotent with $\text{Ran}(a) = A$. □

From Proposition 1, the following corollary is easily obtained.

**Corollary 3.** Let $A \subseteq X$. Then, $A$ is a $\Delta$-retract of $X$ if and only if $J_A$ is a $\Delta$-retract of $I$.

**Proof.** Suppose that $A$ is a $\Delta$-retract of $X$. Then, there is an idempotent element $\alpha$ of $\text{End}_{\varphi}(X)$ such that $A = \text{Ran}(\alpha)$, which yields that $J_A = \text{Ran}(\chi^{(\alpha)})$. Since $\alpha$ is idempotent, we have by Lemma 1 that $\chi^{(\alpha)}$ is an idempotent element of $\text{End}(I)$. Hence, $J_A$ is a $\Delta$-retract of $I$. Conversely, suppose that $J_A$ is a $\Delta$-retract of $I$. Then, there is an idempotent element $\gamma$ of $\text{End}(I)$ such that $J_A = \text{Ran}(\gamma)$. Thus, by Proposition 1, the set $A$ is a $\Delta$-retract of $X$. □

**Proposition 2.** Let $\alpha, \beta \in \text{End}_{\varphi}(X)$, and let $\delta \in \text{Hom}(\text{Ran}(\alpha), \text{Ran}(\beta))$. If $\delta$ is a $\Delta$-isomorphism, then $\chi_{\alpha, \beta}^{(\delta)}$ is a $\Delta$-isomorphism.

**Proof.** Let $\psi \in \text{Hom}(\text{Ran}(\beta), \text{Ran}(\alpha))$ such that $\psi \beta = \id_{\text{Ran}(\beta)}$ and $\psi \lambda = \id_{\text{Ran}(\alpha)}$. By the membership of $\psi$ in $\text{Hom}(\text{Ran}(\beta), \text{Ran}(\alpha))$, we have $\chi_{\beta, \alpha}^{(\psi)} \in \text{Hom}(\chi^{(\delta)}, \text{Ran}(\chi^{(\alpha)}))$. Since $\lambda \psi = \id_{\text{Ran}(\beta)}$, we get $\chi_{\alpha, \beta}^{(\lambda \psi)} = \chi_{\beta, \alpha}^{(\psi)} = \id_{\text{Ran}(\chi^{(\beta)})}$. Similarly, since $\psi \lambda = \id_{\text{Ran}(\alpha)}$, we get $\chi_{\beta, \alpha}^{(\psi \lambda)} = \id_{\text{Ran}(\chi^{(\beta)})}$. Thus, $\chi_{\alpha, \beta}^{(\psi \lambda)}$ is a $\Delta$-isomorphism. □

In the following theorem, we provide a characterization of the regularity of elements of $\text{End}_{\varphi}(X)$ in terms of the $\Delta$-retract and the $\Delta$-isomorphism of $I$.

**Theorem 7.** Let $\alpha \in \text{End}_{\varphi}(X)$. Then, $\alpha$ is regular if and only if $\text{Ran}(\chi^{(\alpha)})$ is a $\Delta$-retract of $I$, and there is $A \subseteq X$ such that each of the following statements holds true:

(i) $J_A$ is a $\Delta$-retract of $I$;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a $\Delta$-isomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{\bigcap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

**Proof.** Suppose that $\text{Ran}(\chi^{(\alpha)})$ is a $\Delta$-retract of $I$, and that there exists a subset $A$ of $X$ such that each of the following statements holds true:

(i) $J_A$ is a $\Delta$-retract of $I$;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a $\Delta$-isomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(iii) $\alpha|_{\bigcap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

Since $\text{Ran}(\chi^{(\alpha)})$ is a $\Delta$-retract of $I$, we have by Corollary 3 that $\text{Ran}(\alpha)$ is a $\Delta$-retract of $X$. And, since $J_A$ is a $\Delta$-retract of $I$, there is an idempotent element $\gamma$ of $\text{End}(I)$ such that $J_A = \text{Ran}(\gamma)$. Thus, by Proposition 1, there is an idempotent element $\beta$ of $T_{\varphi}(X)$ such that $\chi^{(\beta)} = \gamma$ and $A = \text{Ran}(\beta)$. This yields that $A$ is a $\Delta$-retract of $X$. By condition (ii), we have that $\chi^{(\alpha)}|_{J_A}$ is a bijective function from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$. Thus, by condition (iii), we get that $\alpha|_{J_A}$, which is an element of $T_{\varphi}(\beta, \alpha)$, is bijective. By condition (ii) (ii) again, we have $\chi_{\text{Ran}}^{(\alpha)}(\beta) = \chi^{(\alpha)}|_{J_A} \in \text{Hom}(J_A, \text{Ran}(\chi^{(\alpha)}))$ and $\chi_{\text{Ran}}^{(\alpha)}(\beta)^{-1} = (\chi^{(\alpha)}|_{J_A})^{-1} \in \text{Hom}(\text{Ran}(\chi^{(\alpha)}), J_A)$. It follows that $\alpha|_{J_A}$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$. Therefore, by Theorem 4, we obtain that $\alpha$ is regular. Conversely, suppose that $\alpha$ is regular. Then, by Theorem 4, we get that $\text{Ran}(\alpha)$ is a $\Delta$-retract of $X$, and that there is a $\Delta$-retract $A$ of $X$ such that $\alpha|_{J_A}$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$. Since $\text{Ran}(\alpha)$ and $A$ are $\Delta$-retracts of $X$, we have by Corollary 3 that $\text{Ran}(\chi^{(\alpha)})$ and $J_A$ are $\Delta$-retracts of $I$, respectively. Since $\alpha|_{J_A}$ is a $\Delta$-isomorphism from $A$ onto $\text{Ran}(\alpha)$, we have that (iii) holds. And, by Proposition 2, we get that (ii) holds. □

If $\text{End}(I) = T(I)$, then $\text{End}_{\varphi}(X)$ becomes $T_{\varphi}(X)$. Hence, by Theorem 7, the following corollary is immediately obtained.

**Corollary 4.** Let $\alpha \in T_{\varphi}(X)$. Then, $\alpha$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $\chi^{(\alpha)}|_{J_A}$ is a bijective function from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(ii) $\alpha|_{\bigcap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

Note that by considering $T_{\varphi}(X)$ as the semigroup of all continuous maps on $X$, where $X$ is equipped with the topology having the family of all equivalence classes as a base, the regularity of $\alpha \in T_{\varphi}(X)$ can be deduced from Theorem 4 as well. This was provided by Huisheng [9]. The author obtained for any $\alpha \in T_{\varphi}(X)$ that $\alpha$ is regular if and only if for each $i \in I$, there is $\gamma \in I$ such that $Y_i \cap \text{Ran}(\alpha) = \alpha(Y_i)$. Here, we get another characterization of the regularity of elements of $T_{\varphi}(X)$ in terms of the character.

The following three corollaries are immediately obtained from Theorem 7 as well.

**Corollary 5.** Suppose that $I$ is a topological space, and let $\text{End}(I) = S(I)$. Let $\alpha \in \text{End}_{\varphi}(X)$. Then, $\alpha$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $\chi^{(\alpha)}|_{J_A}$ is a homeomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;

(ii) $\alpha|_{\bigcap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\text{Ran}(\alpha) \cap Y_{\chi^{(\alpha)}(i)}$ for all $i \in J_A$.

**Corollary 6.** Suppose that $I$ is a $T_1$-space, and let $\text{End}(I) = \Gamma(I)$. Let $\alpha \in \text{End}_{\varphi}(X)$. Then, $\alpha$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $J_A$ is a closed subset of $I$;

(ii) $\chi^{(\alpha)}|_{J_A}$ is a homeomorphism from $J_A$ onto $\text{Ran}(\chi^{(\alpha)})$;
(iii) $a|_{A \cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\operatorname{Ran}(a) \cap \chi_{X^{(a)(i)}}$ for all $i \in J_A$.

**Corollary 7.** Suppose that $I$ is a vector space, and let $\operatorname{End}(I) = L(I)$. Let $a \in \operatorname{End}_\beta(X)$. Then, $a$ is regular if and only if there is $A \subseteq X$ such that each of the following statements holds true:

(i) $J_A$ is a subspace of $I$;
(ii) $\chi^{(a)}|_{J_A}$ is an isomorphism from $J_A$ onto $\operatorname{Ran}(\chi^{(a)})$;
(iii) $a|_{A \cap Y_i}$ is a bijective function from $A \cap Y_i$ onto $\operatorname{Ran}(a) \cap \chi_{X^{(a)(i)}}$ for all $i \in J_A$.

We end this section with a discussion on the regularity of the quotient semigroup $\operatorname{End}_\beta(X)/\chi$, where $\chi$ is the congruence relation on $\operatorname{End}_\beta(X)$ defined by

$$\forall a, b \in \operatorname{End}_\beta(X),$$

$$a \approx b \iff \chi^{(a)} = \chi^{(b)}. \quad (15)$$

By virtue of Corollary 2, we immediately get that the semigroup $\operatorname{End}_\beta(X)/\chi$ is regular if and only if the semigroup $\operatorname{End}(I)$ is regular.

**Theorem 8.** Let $a \in E^{(1)}_\beta(X)$. Then, the following statements are equivalent:

1. $a$ is a regular element of $E^{(1)}_\beta(X)$;
2. $a$ is a regular element of $E^{(2)}_\beta(X)$;
3. $\chi^{(a)}$ is a regular element of $E^{(1)}_\beta(I)$.

**Proof.** $((3) \Rightarrow (2))$. Suppose that $\chi^{(a)}$ is a regular element of $\operatorname{End}(I)$. Then, by the regularity of $\chi^{(a)}$ in $\operatorname{End}(I)$, we get by Theorem 4 that $\chi^{(a)} \in \Delta$-retract of $I$, and that there is an idempotent $\gamma$ such that $\chi^{(a)} = \chi^{(a)} \gamma$. Hence, there is a bijective function from $\operatorname{Ran}(\chi^{(a)})$ onto $\operatorname{Ran}(\chi^{(a)} \gamma)$.

**Theorem 8.** Let $a \in E^{(1)}_\beta(X)$. Then, the following statements are equivalent:

1. $a$ is a regular element of $E^{(1)}_\beta(X)$;
2. $a$ is a regular element of $E^{(2)}_\beta(X)$;
3. $\chi^{(a)}$ is a regular element of $E^{(1)}_\beta(I)$.

**Proof.** $((3) \Rightarrow (2))$. Suppose that $\chi^{(a)}$ is a regular element of $\operatorname{End}(I)$. Then, by the regularity of $\chi^{(a)}$ in $\operatorname{End}(I)$, we get by Theorem 4 that $\chi^{(a)} \in \Delta$-retract of $I$, and that there is an idempotent $\gamma$ such that $\chi^{(a)} = \chi^{(a)} \gamma$. Hence, there is a bijective function from $\operatorname{Ran}(\chi^{(a)})$ onto $\operatorname{Ran}(\chi^{(a)} \gamma)$.
idempotent, it follows that $a\beta a(x) = \sigma(a(x)) = \sigma(z) = \alpha(x)$. Therefore, $a\beta a = \alpha$, and hence $\alpha$ is a regular element of $E^I_\varnothing(X)$.

((1) $\Rightarrow$ (3)). It follows directly from Lemma 1. □

**Corollary 9.** The semigroup $E^I_\varnothing(X)$ is regular if and only if $\text{End}(I)$ is regular.

4. Conclusions

The semigroup $T^{(\mathcal{D})}(X)$, where $\mathcal{D}$ is a sub-semigroup of $T(I)$, was first defined by Rakbud [12] in 2018 via the notion of the character introduced by Purisang and Rakbud [11] in 2016. Here, we focus on studying the regularity of the semigroup $T^{(\mathcal{D})}(X)$ when $\mathcal{D}$ is the semigroup of a $\Delta$-structure on $I$, which is defined as $\mathcal{D} = \text{End}(I)$. In our study, we obtain that $T^{(\mathcal{D})}(X)$, which is denoted by $\text{End}_{\varnothing}(X)$, is the semigroup of a $\Delta$-structure on $X$. From this, the regularity of elements of $\text{End}_{\varnothing}(X)$ can generally be explained via Theorem 4 established by Magill and Subbiah [13] in 1974. We also obtain a characterization of regular elements of $\text{End}_{\varnothing}(X)$ in terms of the $\Delta$-structure on $I$ (see Theorem 7). From this result, we deduce the regularity of $\text{End}_{\varnothing}(X)$ when $\text{End}(I)$ is one of the following semigroups: the transformation semigroup $T(I)$, the semigroup $S(I)$ of continuous maps on $I$ when $I$ is a topological space, the semigroup $\Gamma(I)$ of closed maps on $I$ when $I$ is a $T_1$-space, and the semigroup $L(I)$ of linear transformations on $I$ when $I$ is a vector space (see Corollaries 4–7). Apart from the regularity of $\text{End}_{\varnothing}(X)$, we provide a sub-semigroup of $\text{End}_{\varnothing}(X)$, namely, the semigroup $E^I_\varnothing(X)$, whose regularity coincides with that of $\text{End}(I)$. In [13], Magill and Subbiah also generally gave some characterizations of Green’s relations for regular elements of the semigroup of a $\Delta$-structure. Since our semigroup $\text{End}_{\varnothing}(X)$ is the semigroup of a $\Delta$-structure on $X$, some rough characterizations of Green’s relations for regular elements of $\text{End}_{\varnothing}(X)$ can immediately be deduced from the results of Magill Jr. and Subbiah.

We end this paper with some interesting questions:

1. Can Green’s relations for regular elements of $\text{End}_{\varnothing}(X)$ be characterized more deeply in terms of the $\Delta$-structure on $I$?

2. Can other notions such as the ideal, the rank, the left regularity, and the right regularity in the semigroup $\text{End}_{\varnothing}(X)$ be explained in terms of those in the semigroup $\text{End}(I)$?

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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