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Research Article

Admissible Almost Type $\mathcal{Z}$-Contractions and Fixed Point Results

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In this paper, we introduce a new concept of α-admissible almost type $\mathcal{Z}$-contraction and prove some fixed point results for this new class of contractions in the context of complete metric spaces. The presented results generalize and unify several existing results in the literature.

1. Introduction and Preliminaries

Fixed point theory is one of the fundamental research areas in nonlinear functional analysis, and it plays a powerful role in the resolution of several mathematical problems with potential applicability in various fields such economics and game theory (see, e.g., [1, 2]) and computer sciences (see, e.g., [3, 4]).

The well-known Banach contraction principle (BCP) is the most impressive theoretical development in the evolution of the metric fixed point theory, and this principle has been extended, improved, and generalized in different approaches (see [5–18]). In particular, in 2015, Khojasteh et al. [5] coined the concept of simulation functions and defined a new class of nonlinear contractions, namely, $\mathcal{Z}$-contractions, which generalize the Banach contraction principle and unify several known types of contractions. For other results related to this interesting approach, refer [8, 17–21].

In [6], Samet et al. introduced the notion of α-ψ-contractive-type mappings by defining the concept of α-admissibility and using Bianchini–Grandolfi gauge functions, and the authors inspected for the existence and the uniqueness of fixed points for such mappings. Later on, Karapinar and Samet [7] generalized the results derived in [6] by proposing the concept of generalized α-ψ-contractive type. In 2016, Karapinar [8] originated the concept of α-admissible $\mathcal{Z}$-contraction by combining the ideas in [5, 6] to obtain some interesting fixed point results in the context of complete metric spaces. In fact, Karapinar [8] proved, among other results, that several existing fixed point theorems can be expressed easily from the main results of [8].

In the present paper, we introduce a new concept of α-admissible almost type $\mathcal{Z}$-contraction with respect to a simulation function ζ, and we prove some results about existence and uniqueness of fixed points for such mappings; the presented results unify several well-known types of contraction and generalize several existing results in the literature.

Let Ψ be the family of nondecreasing functions $\psi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying the following condition.

(**B**) There exist $n_0 \in \mathbb{N}$ and $\alpha \in (0,1)$ and a convergent series of nonnegative terms $\sum_{n=1}^{\infty} v_n$ such that

$$\psi^{n+1}(t) \le \alpha\psi^n(t) + v_n, \quad \text{for } n \ge n_0 \text{ and all at } \in \mathbb{R}^+. \tag{1}$$

The class of such functions is called as Bianchini–Grandolfi gauge functions [10, 22] or c-comparison functions [11].

Lemma 1 (see [11]). *If $\psi \in \Psi$, then the following holds:*

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \longrightarrow \infty$ for all $t \in \mathbb{R}^+$.

(ii) $\psi(t) < t$, for any $t \in \mathbb{R}^+$.

(iii) ψ is continuous at 0.

(iv) The series $\sum_{k=1}^{\infty} \psi_k(t)$ converges for all $t \in \mathbb{R}^+$.

In 2012, Karapinar and Samet [7] introduced a new class of contractive mappings via the following concept of α-admissibility presented in [6].

Definition 1 (see [6]). Let $T: X \longrightarrow X$ be a self-mapping and $\alpha: X \times X \longrightarrow [0, \infty)$ be a function. T is said to be α-admissible if

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1, \quad \text{for all } x, y \in X. \tag{2}$$

Example 1 (see [6]). Let $X = [0, \infty)$, and we define the mappings $T: X \longrightarrow X$ and $\alpha: X \times X \longrightarrow [0, \infty)$ as follows. $Tx = \sqrt{x}$, for all $x \in X$, and

$$\alpha(x, y) = \begin{cases} e^{x-y}, & \text{if } x \geq y, \\ 0, & \text{if } x < y. \end{cases} \tag{3}$$

Then, T is α-admissible.

Definition 2 (see [7]. Let (X, d) be a metric space and $T: X \longrightarrow X$ be a given mapping. We say that T is a generalized α-ψ-contractive mapping if there exist two functions $\alpha: X \times X \longrightarrow [0, \infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y) d(Tx, Ty) \leq \psi(M(x, y)), \quad \text{for all } x, y \in X, \tag{4}$$

where $M(x, y) = \max\{d((x, y), \quad d(x, Tx) + d(y, Ty)/2), (d(x, Ty) + d(y, Tx)/2)\}$.

In 2014, Popescu [12] suggested the concept of triangular α-orbital admissible as an improvement of the triangular α-admissible notion proposed in [9].

Definition 3 (see [12]). Let $T: X \longrightarrow X$ be a mapping and $\alpha: X \times X \longrightarrow [0, \infty)$ be a function. Then, T is said to be α-orbital admissible if

$$\alpha(x, Tx) \geq 1 \longrightarrow \alpha\left(Tx, T^2 x\right) \geq 1. \tag{5}$$

Moreover, T is called a triangular α-orbital admissible if it satisfies the following conditions:

(i) **T**1: T is α-orbital admissible.

(ii) **T**2: $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1 \Longrightarrow \alpha(x, Ty) \geq 1$.

On the contrary, Khojasteh et al. [5] defined a new family of contractions by using the following notion of simulation functions.

Definition 4 (see [5]). The function $\zeta: [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ is said to be a simulation function, if it satisfies the following conditions:

(i) ζ_1: $\zeta(0, 0) = 0$.

(ii) ζ_2: $\zeta(t, s) < s - t$ for all $t, s > 0$.

(iii) ζ_3: if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \longrightarrow \infty} t_n = \lim_{n \longrightarrow \infty} s_n > 0$, then $\lim_{n \longrightarrow \infty} \sup \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Example 2 (see [5]). Let $\zeta_i: [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$, $i = 1, 2, 3$ be defined as follows:

(1) $\zeta_1(t, s) = \widetilde{\phi}(s) - \phi(t)$ for all $t, s \in [0, \infty)$, where $\widetilde{\phi}, \phi: [0, \infty) \longrightarrow [0, \infty)$ are two continuous functions such that $\widetilde{\phi}(t) = \phi(t) = 0$ if and only if $t = 0$ and $\widetilde{\phi}(t) < t \leq \phi(t)$ for all $t > 0$.

(2) $\zeta_2(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, \infty)$, where $\varphi: [0, \infty) \longrightarrow [0, \infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

(3) $\zeta_3(t, s) = s - (f(t, s)/g(t, s))t$ for all $t, s \in [0, \infty)$, where $f, g: [0, \infty) \longrightarrow (0, \infty)$ are continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.

(4) ζ_i for $i = 1, 2, 3$ are simulation functions. For other interesting examples of simulation functions, readers are referred to [5, 19, 20].

Definition 5 (see [5]). Let (X, d) be a metric space, $T: X \longrightarrow X$ a self-mapping, and $\zeta \in \mathcal{Z}$. We say that T is a $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied:

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \quad \text{for all } x, y \in X. \tag{6}$$

Now, we state the result proved in [5] as follows.

Theorem 1 (see [5]). *Let (X, d) be a complete metric space and $T: X \longrightarrow X$ be a $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then, T has unique fixed point in X, and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of T.*

2. Main Results

First, we present the following concept.

Definition 6. Let (X, d) be a metric space and $\zeta \in \mathcal{Z}$. We say that $T: X \longrightarrow X$ is an α-admissible almost $\mathcal{Z}$-contraction if there exists $\alpha: X \times X \longrightarrow [0, \infty)$ and a constant $L \geq 0$ such that

$$\zeta(\alpha(x, y) d(Tx, Ty), M(x, y) + LN(x, y)) \geq 0, \quad \text{for all } x, y \in X, \tag{7}$$

where

$$N(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}. \tag{8}$$

Remark 1. If T is an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, then

$$\alpha(x,y)d(Tx,Ty) < M(x,y) + LN(x,y), \quad \text{for all } x,y \in X. \tag{9}$$

Our first result is the following theorem.

Theorem 2. *Let (X,d) be a complete metric space, and let $T: X \longrightarrow X$ be an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ and satisfying the following conditions:*

(i) T is triangular α-orbital admissible.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

(iii) T is continuous.

Then, there exists $z \in X$ such that $Tz = z$.

Proof. Using condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, and let $\{x_n\}$ be the iterative sequence in X defined by

$$x_{n+1} = Tx_n, \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

If $x_{m+1} = Tx_m$ for some $m \in \mathbb{N}$, then x_m is a fixed point of T. Therefore, to continue our proof, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Since T is an α-admissible mapping, we have

$$\begin{aligned} &\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1, \quad \text{implies that} \\ &\alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1. \end{aligned} \tag{11}$$

By induction, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Applying the condition (7) with $x = x_n$ and $y = x_{n-1}$ and using (12), we get

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}), M(x_n, x_{n-1}) + LN(x_n, x_{n-1})) \\ &= \zeta(\alpha(x_n, x_{n-1})d(x_{n+1}, x_n), M(x_n, x_{n-1}) + LN(x_n, x_{n-1})) \\ &< M(x_n, x_{n-1}) + LN(x_n, x_{n-1}) - \alpha(x_n, x_{n-1})d(x_{n+1}, x_n), \end{aligned} \tag{13}$$

where

$$\begin{aligned} N(x_n, x_{n-1}) &= \min\{d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_{n-1})\} \\ &= \min\{d(x_n, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_n, x_n), d(x_{n-1}, x_{n+1})\} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{2}, \frac{d(x_n, x_{n-1}) + d(x_{n-1}, Tx_n)}{2}\right\} \\ &= \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}, \frac{d(x_{n-1}, x_{n+1})}{2}\right\} \\ &\leq \max\left\{d(x_n, x_{n-1}), \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{2}\right\} \\ &\leq \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\}. \end{aligned} \tag{15}$$

By (13) and taking in account (12), (14), and (15), we derive that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) \\ &< \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} \end{aligned} \tag{16}$$

for all $n \geq 1$. Now, if $\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ for some $n \geq 1$, then from the above inequality, we get

$$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) < d(x_n, x_{n+1}), \tag{17}$$

which is a contradiction. Therefore,

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n-1}), \quad \text{for all } n \geq 1. \tag{18}$$

Hence,

$$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) < d(x_n, x_{n-1}), \quad \text{for all } n \geq 1. \tag{19}$$

Consequently, we deduce that the sequence $\{d(x_n, x_{n-1})\}$ is a decreasing positive real number. Thus, there exists $r \geq 0$ such that $\lim_{n\longrightarrow\infty} d(x_n, x_{n-1}) = r \geq 0$. We claim that

$$\lim_{n\longrightarrow\infty} d(x_n, x_{n-1}) = 0. \tag{20}$$

On the contrary, assume that $r > 0$. It follows from the inequality (19) that

$$\lim_{n\longrightarrow\infty} \alpha(x_n, x_{n-1})d(x_n, x_{n+1}) = r. \tag{21}$$

Now, if we take the sequences $\{\delta_n = \alpha(x_n, x_{n-1}) d(x_n, x_{n+1})\}$ and $\{\tau_n = d(x_n, x_{n-1})\}$ and considering (21), then $\lim_{n\longrightarrow\infty}\delta_n = \lim_{n\longrightarrow\infty}\tau_n = r$; therefore, by (ζ_3), we get that

$$0 \leq \lim_{n\longrightarrow\infty} \sup \zeta(\alpha(x_n, x_{n-1})d(x_n, x_{n+1}), d(x_n, x_{n-1}) < 0 \tag{22}$$

is a contradiction, we deduce that $r = 0$, and equation (20) holds.

Next, we show that $\{x_n\}$ is Cauchy sequence in X, reasoning by the method of reductio ad absurdum. Suppose

to the contrary that $\{x_n\}$ is not a Cauchy sequence. So, there exists $\varepsilon > 0$; for every $N \in \mathbb{N}$, there exist $n, m \in \mathbb{N}$ such that $N < m < n$ and $d(x_m, x_n) > \varepsilon$. In account of (20), there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \varepsilon, \quad \text{for all } n > n_0. \tag{23}$$

We can find two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $m_k > n_k \geq n_0$ and

$$d(x_{m_k}, x_{n_k}) > \varepsilon, \quad \text{for all } k, \tag{24}$$

where m_k is the smallest index satisfying (24). Then,

$$d(x_{m_{k-1}}, x_{n_k}) \leq \varepsilon, \quad \text{for all } k. \tag{25}$$

Now, using (24), (25), and the triangular inequality, we obtain

$$\begin{aligned} \varepsilon < d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \varepsilon. \end{aligned} \tag{26}$$

Letting $k \longrightarrow \infty$ and using equation (20), we derive that

$$\lim_{n\longrightarrow\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{27}$$

Again, using the triangular inequality, we get

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}), \quad \text{for all } k. \tag{28}$$

Also, we have

$$d(x_{m_k+1}, x_{n_k+1}) \leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}), \quad \text{for all } k. \tag{29}$$

By taking the limit as $k \longrightarrow \infty$ on both sides of (28), (29), and using (20), we deduce that

$$\lim_{n\longrightarrow\infty} d(x_{m_k+1}, x_{n_k+1}) = \varepsilon. \tag{30}$$

By the same reasoning as above, we get that

$$\lim_{n\longrightarrow\infty} d(x_{m_k}, x_{n_k+1}) = \lim_{n\longrightarrow\infty} d(x_{m_k+1}, x_{n_k}) = \varepsilon. \tag{31}$$

As T is triangular α-orbital admissible, we have

$$\alpha(x_{m_k}, x_{n_k}) \geq 1. \tag{32}$$

Moreover, since T is an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we obtain

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{m_k}, x_{n_k}) d(Tx_{m_k}, Tx_{n_k}), M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k})) \\ &= \zeta(\alpha(x_{m_k}, x_{n_k}) d(x_{m_k+1}, x_{n_k+1}), M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k})) \\ &< M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}) - \alpha(x_{m_k}, x_{n_k}) d(x_{m_k+1}, x_{n_k+1}). \end{aligned} \tag{33}$$

Hence,

$$\begin{aligned} 0 < d(x_{m_k+1}, x_{n_k+1}) &\leq \alpha(x_{m_k}, x_{n_k}) d(x_{m_k+1}, x_{n_k+1}) \\ &< M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}), \end{aligned} \tag{34}$$

for all $k \geq n_1$, where

$$\begin{aligned} M(x_{m_k}, x_{n_k}) &= \max\left\{ d(x_{m_k}, x_{n_k}), \frac{d(x_{m_k}, Tx_{m_k}) + d(x_{n_k}, Tx_{n_k})}{2}, \frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2} \right\} \\ &= \max\left\{ d(x_{m_k}, x_{n_k}), \frac{d(x_{m_k}, x_{m_k+1}) + d(x_{n_k}, x_{n_k+1})}{2}, \frac{d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})}{2} \right\}, \end{aligned} \tag{35}$$

$$\begin{aligned} N(x_{m_k}, x_{n_k}) &= \min\{d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), d(x_{m_k}, Tx_{n_k}), d(x_{n_k}, Tx_{m_k})\} \\ &= \min\{d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{n_k+1}), d(x_{n_k}, x_{m_k+1})\}. \end{aligned} \tag{36}$$

Taking the limit as $k \longrightarrow \infty$ in (35) and (36), using (20), (27), (30), and (31), we get

$$\lim_{k\longrightarrow\infty} M(x_{m_k}, x_{n_k}) = \varepsilon, \tag{37}$$

$$\lim_{k\longrightarrow\infty} N(x_{m_k}, x_{n_k}) = 0. \tag{38}$$

From (34), (37), and (38), we derive that $\mu_n = \alpha(x_{m_k}, x_{n_k}) d(x_{m_k+1}, x_{n_k+1}) \longrightarrow \varepsilon$ as $\nu_n = M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}) \longrightarrow \varepsilon$; therefore by (ζ_3), we get

$$\begin{aligned} &\lim_{k\longrightarrow\infty} \sup \zeta(\alpha(x_{m_k}, x_{n_k}) d(x_{m_k+1}, x_{n_k+1}), \\ &\quad M(x_{m_k}, x_{n_k}) + LN(x_{m_k}, x_{n_k}) < 0, \end{aligned} \tag{39}$$

which is a contradiction. It follows that $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d). Therefore, there exists $z \in X$ such that

$$\lim_{n\longrightarrow\infty} d(x_n, z) = 0. \tag{40}$$

Furthermore, by the continuity of T, we obtain that

$$\lim_{n\longrightarrow\infty} d(x_{n+1}, Tz) = \lim_{n\longrightarrow\infty} d(Tx_n, Tz) = 0. \tag{41}$$

Taking into account (40), (41), and the uniqueness of the limit, we deduce that z is a fixed point of T and $Tz = z$.

Theorem 3. *Let (X, d) be a complete metric space and $T: X \longrightarrow X$ be an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ satisfying the following conditions:*

(i) T is triangular α-orbital admissible.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then, there exists $z \in X$ such that $Tz = z$.

Proof. Following the lines of the proof of Theorem 2, we obtain that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$ is a Cauchy sequence in X. Since (X, d) is complete, there exists $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. By (12) and the condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$. Using (7), we obtain that

$$\begin{aligned} 0 &\leq \zeta\left(\alpha\left(x_{n(k)}, z\right) d\left(Tx_{n(k)}, Tz\right), M\left(x_{n(k)}, z\right) + LN\left(x_{n(k)}, z\right)\right) \\ &= \zeta\left(\alpha\left(x_{n(k)}, z\right) d\left(x_{n(k)+1}, Tz\right), M\left(x_{n(k)}, z\right) + LN\left(x_{n(k)}, z\right)\right) \\ &< M\left(x_{n(k)}, z\right) + LN\left(x_{n(k)}, z\right) - \alpha\left(x_{n(k)}, z\right) d\left(x_{n(k)+1}, Tz\right). \end{aligned} \tag{42}$$

Hence,

$$\begin{aligned} d\left(x_{n(k)+1}, Tz\right) &\leq \alpha\left(x_{n(k)}, z\right) d\left(x_{n(k)+1}, Tz\right) \\ &< M\left(x_{n(k)}, z\right) + LN\left(x_{n(k)}, z\right), \end{aligned} \tag{43}$$

where

$$\begin{aligned} M\left(x_{n(k)}, z\right) &= \max\left\{d\left(x_{n(k)}, z\right), \frac{d\left(x_{n(k)}, x_{n(k)+1}\right) + d(z, Tz)}{2}, \frac{d\left(x_{n(k)}, Tz\right) + d\left(z, x_{n(k)+1}\right)}{2}\right\}, \\ N\left(x_{n(k)}, z\right) &= \min\left\{d\left(x_{n(k)}, x_{n(k)+1}\right), d(z, Tz), d\left(x_{n(k)}, Tz\right), d\left(z, x_{n(k)+1}\right)\right\}. \end{aligned} \tag{44}$$

Letting $k \longrightarrow \infty$ in the above equalities, we get that

$$\lim_{k \longrightarrow \infty} M\left(x_{n(k)}, z\right) = \frac{d(z, Tz)}{2}, \tag{45}$$

$$\lim_{k \longrightarrow \infty} N\left(x_{n(k)}, z\right) = 0. \tag{46}$$

Suppose that $d(z, Tz) > 0$. From (43), we derive that

$$d\left(x_{n(k)+1}, Tz\right) < M\left(x_{n(k)}, z\right) + LN\left(x_{n(k)}, z\right). \tag{47}$$

Now, letting $k \longrightarrow \infty$ in the above inequality, taking into account (45) and (46), we obtain that

$$d(z, Tz) \leq \frac{d(z, Tz)}{2} \tag{48}$$

is a contradiction, and hence, $d(z, Tz) = 0$, that is, $z = Tz$.

To ensure the uniqueness of a fixed point of a α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, we shall consider the following condition: (**C**) For all $x, y \in \text{Fix}(T)$, we have $\alpha(x, y) \geq 1$, where Fix (T) denotes the set of fixed points of T.

Theorem 4. *Adding condition* (**C**) *to the hypotheses of Theorem 2 (resp., Theorem 3), we obtain the uniqueness of the fixed point of T.*

Proof. We argue by contradiction, suppose that there exist $z, z^* \in X$ such that $z = Tz$ and $z^* = Tz^*$ with $z \neq z^*$. From assumption (**C**), we have

$$\alpha(z, z^*) \geq 1. \tag{49}$$

Therefore, it follows from equation (7) and ζ_2 that

$$\begin{aligned} 0 &\leq \zeta\left(\alpha(z, z^*) d(Tz, Tz^*), M(z, z^*) + LN(z, z^*)\right) \\ &= \zeta\left(\alpha(z, z^*) d(z, z^*), M(z, z^*) + LN(z, z^*)\right) \\ &< M(z, z^*) + LN(z, z^*) - \alpha(z, z^*) d(z, z^*), \end{aligned} \tag{50}$$

where

$$M(z,z^*) = \max\left\{d(z,z^*), \frac{d(z,Tz)+d(z^*,Tz^*)}{2}, \frac{d(z,Tz^*)+d(z^*,Tz)}{2}\right\}$$
$$= \max\left\{d(z,z^*), \frac{d(z,z)+d(z^*,z^*)}{2}, \frac{d(z,z^*)+d(z^*,z)}{2}\right\} = d(z,z^*), \tag{51}$$

$$N(z,z^*) = \min\{d(z,Tz), d(z^*,Tz^*), d(z,Tz^*), d(z^*,Tz)\} = \min\{d(z,z), d(z^*,z^*), d(z,z^*), d(z^*,z)\} = 0. \tag{52}$$

From (50), together with (51) and (52), we deduce that

$$0 < d(z,z^*) - \alpha(z,z^*)d(z,z^*). \tag{53}$$

Using (49), it follows that

$$d(z,z^*) \le \alpha(z,z^*)d(z,z^*) < d(z,z^*), \tag{54}$$

which is a contradiction. Hence, $z = z^*$.

3. Consequences

In this section, we will show that several known fixed point results can be easily concluded from our obtained results.

Corollary 1 (see Karapinar and Samet [7]). *Let (X,d) be a complete metric space. Suppose that $T: X \longrightarrow X$ is a generalized α-ψ-contractive mapping and satisfies the following conditions:*

(i) T is α-admissible.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$.

(iii) Either T is continuous or

(iv) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \longrightarrow x \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \ge 1$ for all k.

Then, there exists $u \in X$ such that $Tu = u$.

Proof. Taking $L = 0$ and a simulation function $\zeta_{EB}: [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ defined by $\zeta_{EB}(t,s) = \psi(s) - t$ for all $s,t \in [0,\infty)$ where $\psi \in \Psi$, in Theorem 4, we obtain that

$$\alpha(x,y)d(Tx,Ty) \le \psi(M(x,y)), \quad \text{for all } x,y \in X. \tag{55}$$

The mapping T is an α-admissible almost $\mathcal{Z}$-contraction with respect to $\zeta_{EB} \in \mathcal{Z}$, and the conclusion follows.

Corollary 2 (see [7]). *Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi(M(x,y)), \quad \text{for all } x,y \in X. \tag{56}$$

Then, T has a unique fixed point.

Proof. It suffices to choose the mapping $\alpha: X \times X \longrightarrow [0,\infty)$ such that $\alpha(x,y) = 1$, for all $x,y \in X$ and $L = 0$ with $\zeta = \zeta_{EB}$ in Theorem 4.

Corollary 3 (see Berinde [23]). *Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exists a function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi(d(x,y)), \quad \text{for all } x,y \in X. \tag{57}$$

Then, T has a unique fixed point.

Corollary 4 (see Ćirić [13]). *Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0,1)$ such that*

$$d(Tx,Ty) \le k \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}, \tag{58}$$

for all $x,y \in X$. Then, T has a unique fixed point.

Proof. It is enough to choose the mapping $\alpha: X \times X \longrightarrow [0,\infty)$such that $\alpha(x,y) = 1$, for all $x,y \in X$ and $L = 0$ with $\zeta_C(t,s) = ks - t$ in Theorem 4.

Corollary 5 (see Hardy and Rogers [14]). *Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $\beta,\gamma,\eta \ge 0$ with $\beta + 2\gamma + 2\eta \in (0,1)$ such that*

$$d(Tx,Ty) \le \beta d(x,y) + \gamma[d(x,Tx) + d(y,Ty)] + \eta[d(x,Ty) + d(y,Tx)], \tag{59}$$

for all $x,y \in X$. Then, T has a unique fixed point.

Corollary 6 (see Banach contraction principle). *Let (X,d) be a complete metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0,1)$ such that*

$$d(Tx,Ty) \le k\,d(x,y), \quad \text{for all } x,y \in X. \tag{60}$$

Then, T has a unique fixed point.

Corollary 7 (see Kannan [15]). *Let (X, d) be a complete metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], \quad \text{for all } x, y \in X. \tag{61}$$

Then, T has a unique fixed point.

Corollary 8 (see Chatterjea [16]). *Let (X, d) be a complete metric space and $T: X \longrightarrow X$ be a given mapping. Suppose that there exists a constant $k \in (0, 1/2)$ such that*

$$d(Tx, Ty) \leq k[d(x, Ty) + d(y, Tx)], \quad \text{for all } x, y \in X. \tag{62}$$

Then, T has a unique fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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