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Research Article

Some Valid Generalizations of Boyd and Wong Inequality and (ψ, ϕ)-Weak Contraction in Partially Ordered b – Metric Spaces

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In this manuscript, we use (ψ, ϕ)-weak contraction to generalize coincidence point results which are established in the context of partially ordered b-metric spaces. The presented work explicitly generalized some recent results from the existing literature. Examples are also provided to show the authenticity of the established work.

1. Introduction

Banach contraction principle [1] is one of the most important result in analysis. It is widely considered as the source of fixed point theory. Banach principle has been modified by many researchers by changing either the contractive condition or the underlying spaces. Boyd and Wong [2] modified the contraction condition in metric spaces (X, d) with control function

$$d(ft, fb) \leq \psi(d(t, b)), \quad \text{for all } t, b \text{ in } X. \tag{1}$$

The ϕ-weak contraction condition was initiated by Alber and Guerre-Delabriere in the Hilbert spaces, see [3]. Rhoudes, in [4], showed that every ϕ-weak contractive mapping in metric spaces has a unique fixed point. The generalized ϕ-weak contractive condition in complete metric spaces was introduced by Zhang and Song in [5]. In [6], Dutta and Choudhury studied unique fixed point results with the help of (ψ, ϕ)-weak contraction in metric spaces. Doric, in [7], studied the result of [6] with generalized (ψ, ϕ)-weak contraction in metric spaces. The concept of fixed point results for two pairs of mappings was introduced by Jungck in [8], for commuted mappings. Jungck in [8, 9] states this new assumption, with compatibility and weak compatibility of mappings. Some further work for generalized (ψ, ϕ)-weak contraction in metric spaces can be found in [6, 10, 11]. Ran and Reurring, in [12], introduced partially ordered metric spaces. A b-metric space with partial ordering is called partially ordered b-metric space. Recently, (ψ, ϕ)-weak contraction in partially ordered b-metric spaces gain the attention of many researcher. In this direction, fixed point and coincidence point results are discussed by many authors. For more details of fixed point and coincidence point results and their applications, comparison of different contraction conditions, and related results in b-metric spaces, we refer the reader to [13–40] along with the references mentioned therein. To show that a sequence is Cauchy in metric-type spaces, Jovanovic et al. [41] proved Lemma 1. As b-metric spaces have discontinuous structure, therefore, in this manuscript, we give a generalization of Boyd and Wong [2] inequality and (ψ, ϕ)-weak contraction, to establish coincidence and fixed point results.

2. Preliminaries

Definition 1 (see [42]). Let $(X, \prec)$ be a partially ordered set, and assume that the map $d: X \times X \longrightarrow R^+$ satisfies the following conditions, For all $b, t, r \in X$ and $s \geq 1$:

$(b_1) d(b,t) = 0$ if and only if $b = t$

$(b_2) d(b,t) = d(t,b)$

$(b_3) d(b,r) \leq s(d(b,t) + d(t,r))$

Then, $(X, d, \prec)$ is called partially ordered b-metric space.

Lemma 1 (see [41]). *"Let (X,d) be b-metric space and $s \geq 1$. When a sequence $\{a_m\}$ satisfies the following condition,*

$$d(a_m, a_{m+1}) \leq K\, d(a_{m-1}, a_m), \tag{2}$$

for some, $0 < K < (1/s)$ and $m = 1, 2, \ldots$.

Then, $\{a_m\}$ is b-Cauchy sequence in (X,d).

Definition 2 (see [9]). The pair of mappings (f,g) is compatible in the metric space (X,d) if and only if

$$\lim_{m \longrightarrow \infty} d(fgn_m, gfn_m) = 0, \tag{3}$$

whenever $\{n_m\}$ in sequence such that

$$\lim_{m \longrightarrow \infty} fn_m = \lim_{m \longrightarrow \infty} gn_m = r, \text{ with } r \in X. \tag{4}$$

Definition 3 (see [8]). In the metric space (X,d), the pair of mappings $f, g: X \longrightarrow X$ is weakly compatible if (f,g) is commutative at the point of coincidence (i.e., $gfn = fgn$ whenever $fn = gn$).

Definition 4 (see [43]). The pair of mappings $f, g: X \longrightarrow X$ defined on a partially ordered set X is called

(a) Weakly increasing if $fn \prec g(fn)$ and $gn \prec f(gn)$, for all $n \in X$

(b) Partially weakly increasing if $\forall n \in X$, $fn \prec g(fn)$

Definition 5 (see [44, 45]). Let $f, g, h: X \longrightarrow X$ be three mappings on partially ordered set $(X, \prec)$ such that $fn \prec hn$ and $gn \prec hn$. Then, pair (f,g) is said to be

(i) Weakly increasing with respect to h if and only if for all $n \in X$, $fn \prec gw$, for all $w \in h^{-1}(fn)$ and $gn \prec fw$, and for all $w \in h^{-1}(gn)$

(ii) Partially weakly increasing w.r.t h if and only if $fn \prec gw$ for all $w \in h^{-1}(fn)$.

Theorem 1 (see [45]). *Let self-mappings f, g be continuous mappings in partially ordered complete metric space $(X, \prec, d)$ with $f(X) \subseteq g(X)$. Assume that the pair (f,g) is weakly compatible, whereas f is partially weakly increasing with respect to g such that gb and gt are comparable elements and satisfying the following condition:*

$$\psi(d(fb, ft)) \leq \psi(d(gb, gt)) - \phi(d(gb, gt)), \quad \text{for all } b, t \text{ in } X, \tag{5}$$

where $\phi, \psi: [0, \infty) \longrightarrow [0, \infty)$ are altering distance functions. Then, mappings f, g have coincidence point.

Theorem 2 (see [46]). *Let $f, g: X \longrightarrow X$ be mappings on a partially ordered complete b-metric space $(X, \prec, d)$. Assume that f, g is partially weakly increasing and satisfying the following condition:*

$$\psi(d(ft, gb)) \leq \psi(M(t,b)) - \phi(M(t,b)), \quad \forall t, b \in X, \tag{6}$$

where $\phi, \psi: [0, \infty) \longrightarrow [0, \infty)$ are altering distance and lower semicontinuous functions, respectively, with $\phi(0) = 0$ and

$$M(t,b) = \max\left\{d(t,b), d(t,ft), d(b,gb), \frac{d(t,gb) + d(b,ft)}{2s}\right\}, \tag{7}$$

where $s \geq 1$. Furthermore, assume that either f or g is continuous mapping or space X is regular. Then, f, g have a common fixed point.

Theorem 3 (see [19]). *"Let $(X, \prec, d)$ be partially ordered complete b-metric space and f, g, h, and $k: X \longrightarrow X$ be continuous mappings such that $f(X) \subseteq h(X)$ and $g(X) \subseteq k(X)$. Suppose for comparable elements kb and ht, the following condition hold*

$$\psi\left(s^2 d(fb, gt)\right) \leq \psi(M(b,t)) - \phi(M(b,t)), \quad \forall b, t \in X,$$

$$M(b,t) = \max\left\{d(kb,ht), d(kb,fb), d(ht,gt), \frac{d(kb,gt) + d(ht,fb)}{2s}\right\}, \tag{8}$$

with altering distance functions, $\phi, \psi: [0, \infty) \longrightarrow [0, \infty)$ and $s \geq 1$. Let the pairs (f,k) and (g,h) be compatible and the pairs (f,g) and (g,f) be partially weakly increasing w.r.t h and k, respectively. Then, f, g, h, and k have a coincidence point."

Theorem 4 (see [18]). *"Let f, g, h, $k: X \longrightarrow X$ be continuous mappings on a partially ordered complete b-metric space $(X, \prec, d)$ with $f(X) \subseteq h(X)$ and $g(X) \subseteq k(X)$. Assume that compatible pairs (f,h) and (g,k) and comparable elements kt and hb satisfy the following condition:*

$$\begin{aligned} &s^{\epsilon} d(ft, gb) \leq M(t,b), \\ &M(t,b) = \max\{d(kt,hb), d(kt,ft), d(hb,gb), \\ &\frac{d(kt,gb) + d(hb,ft)}{2s}\}, \end{aligned} \tag{9}$$

where $\epsilon > 1$, while $s \geq 1$ and the pairs of mappings (f,g) and (g,f) are partially weakly increasing w.r.t k and h, respectively. When ha and ka are comparable, then $fa = ga = ha = ka$."

Theorem 5 (see [20]). *Let the mappings f, g: $X \longrightarrow X$ be defined on a partially ordered complete b-metric space $(X, \prec, d)$ and f be g-weakly isotone increasing. Let ψ: $[0,\infty) \longrightarrow [0,\infty)$ be a function such that $\psi(n) < n$ for all $n > 0$ and $\psi(0) = 0$ satisfy the condition:*

$$s^4 d(ft, gb) \le M(t,b), \quad \forall t, b \in X,$$
$$M(t,b) = \max\{\psi(d(t,b)), \psi(d(t,ft)), \psi\left(d(b,gb)), \psi\left(\frac{d(t,gb)+d(b,ft)}{2s}\right)\right\}, \tag{10}$$

where $s \ge 1$. If the mappings f or g are continuous in X. Then, (f,g) has a common fixed point.

3. Main Results

We begin with the following result.

Theorem 6. *Let A, B, G, F: $X \longrightarrow X$ be continuous mappings on a partially ordered complete b-metric space, $(X, \prec, d)$ with $A(X) \subseteq G(X)$ and $B(X) \subseteq F(X)$. Assume that compatible pairs (A,F) and (B,G) and comparable elements Gb and Ft satisfy the following contractive condition:*

$$s^{\epsilon} d(At, Bb) \le \psi(M(t,b)), \quad \forall t, b \in X, \tag{11}$$

where $\epsilon > 1$ and ψ: $[0,\infty) \longrightarrow [0,\infty)$ satisfy $\psi(n) \le n$ for all $n > 0$, $\psi(0) = 0$, and

$$M(t,b) \in \{d(Ft,Gb), d(Gt,At), d(Fb,Bb), \frac{d(Ft,Bb)+d(Gb,At)}{2s}\}, \tag{12}$$

where $s \ge 1$ and the pairs (A,B) and (B,A) are partially weakly increasing w.r.t G, F, respectively. Then, (A,F) and (B,G) have a coincidence point. Furthermore, if b is a coincidence point for comparable elements Gb and Fb, then $Ab = Bb = Fb = Gb$.

Proof. Let $t_0 \in X$. Since $A(X) \subseteq F(X)$ and $B(X) \subseteq G(X)$, then there exist $t_1, t_2 \in X$ such that $At_0 = Ft_1$ and $Bt_1 = Gt_2$. The sequence $\{b_m\}$ is constructed as follows:

$$b_{2m+1} = At_{2m} = Ft_{2m+1},$$
$$b_{2m+2} = Bt_{2m+1} = Gt_{2m+2}, \quad \text{where } m = 0, 1, 2, \ldots. \tag{13}$$

The pairs of mappings (A,B) and (B,A) are partially weakly increasing w.r.t F and G, respectively. Therefore,

$$Ft_1 = At_0 \prec Bt_1 = Gt_2 \prec At_2 = Ft_3, \quad \text{for all } t_1 \in F^{-1}(At_0),\ t_2 \in G^{-1}(Bt_1). \tag{14}$$

By repeating this process, we deduce that

$$b_{2m+1} \prec b_{2m+2}, \quad \forall m \in N \cup \{0\}. \tag{15}$$

Assume that

$$b_{2m} \ne b_{2m+1}, \quad \forall m \in N. \tag{16}$$

We discuss three steps:

Step I: at this step, we have to prove that $d(b_{2m+1}, b_{2m}) \le d(b_{2m}, b_{2m-1})$.

Since $Ft_{2m} = b_{2m}$ and $Gt_{2m+1} = b_{2m+1}$ are comparable, therefore, from (11), we have

$$s^{\epsilon} d(b_{2m+1}, b_{2m}) = s^{\epsilon} d(At_{2m}, Bt_{2m-1}) \le \psi(M(t_{2m}, t_{2m-1})), \tag{17}$$

with

$$M(t_{2m}, t_{2m-1}) \in \left\{d(b_{2m}, b_{2m-1}), d(b_{2m}, b_{2m+1}), d(b_{2m-1}, b_{2m}), \frac{d(b_{2m}, b_{2m}) + d(b_{2m-1}, b_{2m+1})}{2s}\right\},$$
$$M(t_{2m}, t_{2m-1}) \in \left\{d(b_{2m}, b_{2m-1}), d(b_{2m}, b_{2m+1}), \frac{d(b_{2m+1}, b_{2m-1})}{2s}\right\}. \tag{18}$$

To show that $d(b_{2m,}, b_{2m+1}) \le d(b_{2m-1}, b_{2m})$, suppose that

$$d(b_{2m-1}, b_{2m}) < d(b_{2m}, b_{2m+1}). \tag{19}$$

Here, we discuss three possible cases of $M(t_{2m}, t_{2m-1})$.

Case (1): if $M(t_{2m}, t_{2m-1}) = d(b_{2m}, b_{2m-1})$, from (17), we have

$$s^{\epsilon} d(b_{2m}, b_{2m+1}) \le \psi(d(b_{2m}, b_{2m-1})). \tag{20}$$

Since $\psi(n) \le n$ for $n > 0$, therefore

$$s^{\epsilon} d(b_{2m}, b_{2m+1}) \le d(b_{2m}, b_{2m-1}), \tag{21}$$

which contradict assumption (19).

Case (2): if $M(t_{2m}, t_{2m-1}) = d(b_{2m}, b_{2m+1})$, since $\psi(n) \le n$ for $n > 0$, therefore, from (17), one can write

$$s^{\epsilon} d(b_{2m}, b_{2m+1}) \le \psi(d(b_{2m}, b_{2m+1})) \le d(b_{2m}, b_{2m+1}), \tag{22}$$

which is held unless $d(b_{2m}, b_{2m+1}) = 0$, which implies that $b_{2m} = b_{2m+1}$, but it contradicts (16).

Therefore,

$$d(b_{2m+1}, b_{2m}) \leq d(b_{2m}, b_{2m-1}). \tag{23}$$

Case (3): if $M(t_{2m}, t_{2m-1}) = (d(b_{2m-1}, b_{2m+1})/2s)$.
Using triangle inequality and assumption (19), we have

$$\begin{aligned} M(t_{2m}, t_{2m-1}) &\leq \frac{s(d(b_{2m-1}, b_{2m}) + d(b_{2m}, b_{2m+1}))}{2s} \\ &< d(b_{2m}, b_{2m+1}). \end{aligned} \tag{24}$$

Since $\psi(n) \leq n$ for all $n > 0$, thus, from (17), one has

$$\begin{aligned} s^{\epsilon} d(b_{2m}, b_{2m+1}) &\leq \psi(d(b_{2m}, b_{2m+1})) \\ &< d(b_{2m}, b_{2m+1}). \end{aligned} \tag{25}$$

This is again contradiction. Hence, for all cases, we conclude that

$$d(b_{2m+1}, b_{2m}) \leq d(b_{2m}, b_{2m-1}). \tag{26}$$

Step II: in this step, we have to prove that the sequence b_m is b-Cauchy by using Lemma 1, for all three cases.

Case (1): if $M(t_{2m}, t_{2m-1}) = d(b_{2m}, b_{2m-1})$.
From (17), one can write

$$s^{\epsilon} d(b_{2m}, b_{2m+1}) \leq \psi(d(b_{2m}, b_{2m-1})). \tag{27}$$

As $\psi(n) \leq n$ for $n > 0$, so

$$\begin{aligned} s^{\epsilon} d(b_{2m}, b_{2m+1}) &\leq d(b_{2m}, b_{2m-1}), \\ d(b_{2m}, b_{2m+1}) &\leq (1/s^{\epsilon}) d(b_{2m}, b_{2m-1}). \end{aligned} \tag{28}$$

Taking $K = 1/s^{\epsilon}$, we have

$$d(b_{2m}, b_{2m+1}) \leq K\, d(b_{2m}, b_{2m-1}), \quad \text{for } K \in \left[0, \frac{1}{s}\right). \tag{29}$$

Case (2): if $M(t_{2m}, t_{2m-1}) = d(b_{2m}, b_{2m+1})$, from (26), we have $d(b_{2m}, b_{2m+1}) \leq d(b_{2m}, b_{2m-1})$. From (17), one has

$$s^{\epsilon} d(b_{2m}, b_{2m+1}) \leq \psi(d(b_{2m}, b_{2m-1})). \tag{30}$$

Since $\psi(n) \leq n$, for $n > 0$, therefore

$$s^{\epsilon} d(b_{2m}, b_{2m+1}) \leq d(b_{2m}, b_{2m-1}). \tag{31}$$

Thus,

$$d(b_{2m}, b_{2m+1}) \leq \frac{1}{s^{\epsilon}} d(b_{2m}, b_{2m-1}). \tag{32}$$

Let $K = 1/s^{\epsilon}$. Then,

$$d(b_{2m}, b_{2m+1}) \leq K\, d(b_{2m}, b_{2m-1}), \quad \text{for } K \in \left[0, \frac{1}{s}\right). \tag{33}$$

Case (3): if $M(t_{2m}, t_{2m-1}) = (d(b_{2m-1}, b_{2m+1})/2s)$.
Using triangle inequality and (26), it is reduced to case (2). Therefore, (2) holds for three cases.

Similarly, if $Ft_{2m+1} = b_{2m+1}$ and $Gt_{2m+2} = b_{2m+2}$, one can easily show that

$$d(b_{2m+1}, b_{2m+2}) \leq K d(b_{2m}, b_{2m+1}). \tag{34}$$

Hence,

$$\begin{aligned} &d(b_{2m+1}, b_{2m+2}) \leq K d(b_{2m}, b_{2m+1}) \\ &\quad \text{with } b_{2m} \neq b_{2m+1} \forall m \in N. \end{aligned} \tag{35}$$

Define $d_k = d(b_k, b_{k+1})$. Suppose $d_{k_0} = 0$, for some k_0, then $b_{k_0} = b_{k_{0+1}}$.
In case $k_0 = 2m$, then $b_{2m} = b_{2m+1}$. We have

$$\begin{aligned} M(t_{2m}, t_{2m+1}) &\in \left\{0, d(b_{2m+1}, b_{2m+2}), 0, \frac{0 + d(b_{2m}, b_{2m+2})}{2s}\right\} \\ &= \left\{0, d(b_{2m+2}, b_{2m+1}), \frac{d(b_{2m+2}, b_{2m})}{2s}\right\}. \end{aligned} \tag{36}$$

If

$$M(t_{2m}, t_{2m+1}) = d(b_{2m+2}, b_{2m+1}). \tag{37}$$

Then, from (17),

$$s^{\epsilon} d(b_{2m+1}, b_{2m+2}) \leq \psi(d(b_{2m+2}, b_{2m+1})). \tag{38}$$

Since $\psi(n) \leq n$ for all $n > 0$ and $\psi(0) = 0$, therefore,

$$s^{\epsilon} d(b_{2m+1}, b_{2m+2}) \leq d(b_{2m+1}, b_{2m+2}). \tag{39}$$

This is only possible if $d(b_{2m+1}, b_{2m+2}) = 0$. Hence, $b_{2m+1} = b_{2m+2}$.
Thus, $b_{2m} = b_{2m+1} = b_{2m+2}$. Similarly, one can show for the remaining two cases.

By the same process, if $k_0 = 2m + 1$, then $b_{2m+1} = b_{2m+2}$, which gives $b_{2m+2} = b_{2m+3}$.

Therefore, $\{b_k\}$ is a constant sequence with $k \geq k_0$, and (2) is also valid for constant sequence.

Hence, $\{b_m\}$ is b-Cauchy sequence. The completeness of space X implies that there exist $b \in X$ such that $b_m \longrightarrow b$. Consequently, the subsequences will also converge to $b \in X$:

$$\begin{aligned} At_{2m} &\longrightarrow b, \\ Bt_{2m+1} &\longrightarrow b, \\ Gt_{2m+1} &\longrightarrow b, \\ Ft_{2m} &\longrightarrow b. \end{aligned} \tag{40}$$

Step III: now, we show that b is a coincidence point of A, F, since

$$\begin{aligned} &\lim_{m\longrightarrow\infty} d(At_{2m}, b) = \lim_{m\longrightarrow\infty} d(Gt_{2m+1}, b) = \lim_{m\longrightarrow\infty} d(b_{2m+1}, b) = 0, \\ &\lim_{m\longrightarrow\infty} d(Bt_{2m+1}, b) = \lim_{m\longrightarrow\infty} d(Ft_{2m+2}, b) = \lim_{m\longrightarrow\infty} d(b_{2m+2}, b) = 0. \end{aligned} \tag{41}$$

As from the compatibility of (A, F), we have

$$\lim_{m\longrightarrow\infty} d\left(FAt_{2m}, AFt_{2m}\right) = 0. \tag{42}$$

Moreover, $\lim_{m\longrightarrow\infty} d\left(At_{2m}, b\right) = 0$ and $\lim_{m\longrightarrow\infty} d\left(Ft_{2m}, b\right) = 0$.

As A and F are continuous mappings, therefore

$$\lim_{m\longrightarrow\infty} d\left(FAt_{2m}, Fb\right) = \lim_{m\longrightarrow\infty} \left(AFt_{2m}, Ab\right) = 0. \tag{43}$$

Using triangle inequality twice, one can write

$$\begin{aligned} d(Fb, Ab) &\le s\left(d\left(Fb, FAt_{2m}\right) + d\left(FAt_{2m}, Ab\right)\right), \\ d(Fb, Ab) &\le sd\left(Fb, FAt_{2m}\right) + s^2\left(d\left(FAt_{2m}, AFt_{2m}\right)\right. \\ &\quad \left. + d\left(AFt_{2m}, Ab\right)\right). \end{aligned} \tag{44}$$

Taking limit $m \longrightarrow \infty$ and using (41) and (42) in the above inequality, we obtain

$$\lim_{m\longrightarrow\infty} d(Fb, Ab) \le 0. \tag{45}$$

Thus, $d(Fb, Ab) = 0$. Thus, $Ab = Fb$. Therefore, b is coincidence at point of A, F.

Similarly, $Gb = Bb$.

From comparability of element Fb, Gb and (11), one can write

$$s^{\epsilon} d(Ab, Bb) \le \psi(M(b, b)), \tag{46}$$

where

$$\begin{aligned} M(b, b) &\in \left\{d(Fb, Gb), d(Fb, Ab), d(Gb, Bb),\right. \\ &\quad \left.\frac{d(Fb, Bb) + d(Gb, Ab)}{2s}\right\} \\ &= \left\{d(Ab, Bb), 0, 0, \frac{d(Ab, Bb)}{s}\right\}. \end{aligned} \tag{47}$$

If $M(b, b) = d(Ab, Bb)$, then

$$s^{\epsilon} d(Ab, Bb) \le \psi(d(Ab, Bb)). \tag{48}$$

This is possible if $Ab = Bb$. Hence, $Ab = Bb = Fb = Gb$.

On the same process, from the other two cases, we can show that $Ab = Bb = Fb = Gb$.

In the following, the result condition of continuity and compatibility is relaxed for mappings. □

Theorem 7. *Let A, B, G, F: $X \longrightarrow X$ be four mappings on regular partially ordered complete b-metric spaces $(X, \prec, d)$ such that $A(X) \subseteq F(X)$, $B(X) \subseteq G(X)$, and $F(X)$, $G(X)$ are b-closed subsets of X. Suppose the pairs (A, B) and (B, A) are partially weakly increasing with respect to F, G, respectively, whereas pairs (A, G), (B, F) are weak compatible and satisfy the following condition:*

$$s^{\epsilon} d(At, Bb) \le \psi(M(t, b)), \quad \forall t, b \in X, \tag{49}$$

where $\epsilon > 1$ and ψ: $[0, \infty) \longrightarrow [0, \infty)$ with $\psi(n) \le n$ for $n > 0$, where $\psi(0) = 0$, and

$$\begin{aligned} M(t, b) &\in \left\{d(Gt, Fb), d(Gt, At), d(Fb, Bb),\right. \\ &\quad \left.\frac{d(Gt, Bb) + d(Fb, At)}{2s}\right\}, \end{aligned} \tag{50}$$

with $s \ge 1$. Then, A, B, F, and G have a coincidence point. Furthermore, if Gb, Fb are comparable, then $Ab = Bb = Fb = Gb$.

Proof. Following the lines of the proof of Theorem 6, one can easily show that there exists a sequence $\{b_m\}$ which converges to some $b \in X$. Therefore,

$$\lim_{m\longrightarrow\infty} d\left(b_m, b\right) = 0. \tag{51}$$

If $\{b_{2m+1}\} \subseteq F(X)$ and $\{b_{2m+2}\} \subseteq G(X)$ but $F(X)$ and $G(X)$ are b-closed subsets of X, then there must be some $x, k \in X$, from which $b = Fx$ and $b = Gk$. Hence, from the construction of the sequence given in Theorem 6, we have

$$\begin{aligned} \lim_{m\longrightarrow\infty} Ft_{2m+1} &= \lim_{m\longrightarrow\infty} b_{2m+1} = b = Gk, \\ \lim_{m\longrightarrow\infty} Bt_{2m+1} &= \lim_{m\longrightarrow\infty} b_{2m+2} = b = Gk. \end{aligned} \tag{52}$$

Now, we prove that $Ab = Gb$.

From regularity of X, we have $Ft_{2m+1} \prec Gk$ and $Bt_{2m+1} \prec Gk$.

Thus, from (49), one can write

$$s^{\epsilon} d\left(Ak, Bt_{2m+1}\right) \le \psi\left(M\left(k, t_{2m+1}\right)\right), \tag{53}$$

where

$$\begin{aligned} M\left(k, t_{2m+1}\right) &\in \left\{d\left(Gk, Ft_{2m+1}\right), d(Gk, Ak), d\left(Ft_{2m+1}, Bt_{2m+1}\right), \frac{d\left(Gk, Bt_{2m+1}\right) + d\left(Ft_{2m+1}, Ak\right)}{2s}\right\} \\ &\le \left\{d(Gk, Gk), d(Gk, Ak), d(Gk, Gk), \frac{d(Gk, Gk) + d(Gk, Ak)}{2s}\right\} \\ &= \left\{0, d(Gk, Ak), 0, \frac{d(Gk, Ak)}{2s}\right\}. \end{aligned} \tag{54}$$

Since $\psi(n) \le n$ for all $n > 0$ and $\psi(0) = 0$ and using (54) in (53), we obtain

$$s^{\epsilon} d(Ak, Gk) \le \psi\left(\frac{d(Gk, Ak)}{2s}\right) \le d(Gk, Ak). \tag{55}$$

This is only possible if $d(Gk, Ak) = 0$. It implies that $Gk = Ak$. Hence, $b = Gk = Ak$. Now, from weak co$Ab = Gb$mpatibility of A and G, we can write $Ab = AGk = GAk = Gb$. Thus, . Similarly, we can show that $Fb = Bb$. Furthermore, following the last lines of the proof of Theorem 6, we have $Ab = Gb = Fb = Bb$.

By putting $G = F = I$(identity mapping) in Theorems (6 and 7), then the following corollary is obtained with

$$M(t, b) \in \left\{d(t, b), d(t, At), d(b, Bb), \frac{d(t, Bb) + d(b, At)}{2s}\right\}. \tag{56}$$

□

Corollary 1. *Let A, B: $X \longrightarrow X$ be two mappings on $(X, \prec, d)$. Let A, B be partially weakly increasing, whereas ψ: $[0, \infty) \longrightarrow [0, \infty)$ with $\psi(n) \le n$ for $n > 0$ and $\psi(0) = 0$, satisfy the following condition with comparable elements $t, b \in X$*

$$s^{\epsilon} d(At, Bb) \le \psi(M(t, b)). \tag{57}$$

Also,

(i) A or B is continuous

Or

(ii) X is regular

Then, b is a common fixed point of A, B.

Example 1. Let $X = [0, \infty)$. Define $d(t, b) = |t - b|^2$, for all t, b in X, then clearly d is a partially ordered b-metric on X with $s = 2$ and partially ordering $\prec$ is defined by

$$t \prec b \Leftrightarrow b \le t, \quad \text{for all } t, b \text{ in } X. \tag{58}$$

Assume that A, B are continuous mappings defined as follows:

$$A(t) = \ln\left(1 + \frac{t}{4}\right), \qquad B(t) = \ln\left(1 + \frac{t}{5}\right). \tag{59}$$

Since $1 + (t/5) \le e^{t/5}$. Therefore, $BA(t) = \ln(1 + (A(t)/5)) \le A(t)$. Therefore, $A(t) \prec BA(t)$. Thus, the pair (A, B) is partially weakly increasing. Now, we show that

$$\psi(s^{\epsilon} d(At, Bb)) \le \psi(M(t, b)), \quad \forall t, b \in X. \tag{60}$$

Define $\psi(m) = bv$, where $1 \le b \le 16$ and take $1 < \epsilon \le 4$. Here, we discuss the following cases.

Case (1): if $(t/4) < (b/5)$, then

$$1 \le \frac{1 + (b/5)}{1 + (t/4)} \le \frac{1 + (b/4)}{1 + (t/4)} \Longrightarrow 0 \le \ln\left(\frac{1 + (b/5)}{1 + (t/4)}\right) \le \ln\left(\frac{1 + (b/4)}{1 + (t/4)}\right). \tag{61}$$

Using mean value theorem for $\ln(1 + x)$, where $x \in [t/4, b/5]$, we obtain

$$\begin{aligned} 2^{\epsilon} d(At, Bb) &= 2^{\epsilon} |Bb - At|^2 \\ &= 2^{\epsilon} \left|\ln\left(1 + \frac{b}{5}\right) - \ln\left(1 + \frac{t}{4}\right)\right|^2 \\ &= 2^{\epsilon} \left|\ln\left(\frac{1 + (b/5)}{1 + (t/4)}\right)\right|^2 \\ &\le 2^{\epsilon} \left|\ln\left(\frac{1 + (b/4)}{1 + (t/4)}\right)\right|^2 \\ &\le 2^{\epsilon} \left|\ln\left(1 + \frac{b}{4}\right) - \ln\left(1 + \frac{t}{4}\right)\right|^2 \\ &\le 2^{\epsilon} \left|\frac{b}{4} - \frac{t}{4}\right|^2 \\ &\le 2^{\epsilon - 4} d(t, b) \\ &= \psi(M(t, b)). \end{aligned} \tag{62}$$

Case (2): if $(b/5) < (t/4)$, then $0 < ((t/4) - (b/5)) \le ((t/4) - (b/4))$. Therefore, $2^{\epsilon} d(At, Bb) = 2^{\epsilon} |At - Bb|^2 = 2^{\epsilon} |\ln(1 + (t/4)) - \ln(1 + (b/5))|^2 = 2^{\epsilon} |(t/4) - (b/5)|^2 \le 2^{\epsilon} |(t/4) - (b/4)|^2 = 2^{\epsilon - 4} d(t, b) = \psi(M(t, b))$.

Hence, condition of Corollary 1 holds, and 0 is coincidence point.

Remark 1. If we put $\psi(n) = n$ in Theorems 6 and 7, we can obtain Theorems (2.1, 2.2) of [18], respectively.

Remark 2. As A, B are partially weakly increasing, therefore A is B-weakly isotone increasing. Thus, by substitution $\epsilon = 4$, in Corollary 1, we get Theorem 2.1 of [20]. Moreover, if $A = B$ and $\epsilon = 4$ in Corollary 1, then Corollary 2.4 of [20] is obtained.

Remark 3. Clearly, the conditions of our Corollary 1 holds for Example 2.8 of [20] and the corresponding conclusion holds. By substituting $\epsilon = 4$ in Example 2, then condition of Theorem 2.1 of [20] does not hold but our condition (57) of Corollary 1 holds.

In the remaining part of this manuscript, we discuss coincidence point of two compatible pairs of mappings with generalized (ψ, ϕ)-weak contractive condition.

Throughout the rest of this paper, we consider the following, $\epsilon > 1$, and for all t, b in X, we define

$$M(t,b) \in \left\{ d(Gt,Fb), d(Gt,At), d(Fb,Bb), \frac{d(Gt,Bb)+d(Fb,At)}{2s} \right\}, \tag{63}$$

and ϕ, ψ: $[0,\infty) \longrightarrow [0,\infty)$ are altering distance functions and $(X, \prec, d)$ is partially ordered complete b-metric space.

Theorem 8. *Let A, B, G, F: $X \longrightarrow X$ be continuous self-mappings on $(X, \prec, d)$, where $A(X) \subseteq F(X)$ and $B(X) \subseteq G(X)$. Suppose that the pairs (A,F) and (B,G) are compatible, whereas the pairs (A,B) and (B,A) are partially weakly increasing with respect to F and G, respectively, and satisfy the following condition:*

$$\psi(s^{\epsilon} d(At,Bb)) \le \psi(M(t,b)) - \phi(M(t,b)), \quad \forall t,b \in X. \tag{64}$$

Then A, B, F, and G have a coincidence point. Moreover, if Gb, Fb are comparable then $Ab = Bb = Fb = Gb$.

Proof. Let $t_0 \in X$. Since $A(X) \subseteq F(X)$ and $B(X) \subseteq G(X)$, so there exist $t_1, t_2 \in X$ such that $At_0 = Ft_1$ and $Bt_1 = Gt_2$. Construct the sequence $\{b_m\}$ as follows:

$$b_{2m+1} = At_{2m} = Ft_{2m+1},\ b_{2m+2} = Bt_{2m+1} = Gt_{2m+2}, \quad \text{for } m = 0,1,2,\ldots. \tag{65}$$

Since the pairs (A,B) and (B,A) are partially weakly increasing with respect to F and G, therefore,

$$Ft_1 = At_0 \prec Bt_1 = Gt_2 \prec At_2 = Ft_3, \quad \text{where } t_1 \in F^{-1}(At_0), t_2 \in G^{-1}(Bt_1). \tag{66}$$

Repeating the above process, we can write

$$b_{2m+1} \prec b_{2m+2}, \quad \forall m \in N \cup \{0\}. \tag{67}$$

Assume that

$$b_{2m} \ne b_{2m+1}, \quad \forall m \in N. \tag{68}$$

We discuss the proof in three steps.

Step I: first, we prove that $d(b_{2m+1}, b_{2m}) \le d(b_{2m}, b_{2m-1})$. Since $Ft_{2m} = b_{2m}$ and $Gt_{2m+1} = b_{2m+1}$ are comparable, therefore (64) implies that

$$\begin{aligned} \psi(s^{\epsilon} d(b_{2m+1}, b_{2m})) &= \psi(s^{\epsilon} d(At_{2m}, Bt_{2m-1})) \\ &\le \psi(M(t_{2m}, t_{2m-1})) \\ &\quad - \phi(M(t_{2m}, t_{2m-1})), \end{aligned} \tag{69}$$

where

$$M(t_{2m}, t_{2m-1}) \in \left\{ d(b_{2m}, b_{2m-1}), d(b_{2m}, b_{2m+1}), d(b_{2m-1}, b_{2m}), \frac{d(b_{2m}, b_{2m}) + d(b_{2m-1}, b_{2m+1})}{2s} \right\} \left\{ d(b_{2m}, b_{2m-1}), d(b_{2m}, b_{2m+1}), \frac{d(b_{2m+1}, b_{2m-1})}{2s} \right\}. \tag{70}$$

To show that $d(b_{2m,}, b_{2m+1}) \le d(b_{2m-1}, b_{2m})$, suppose that

$$d(b_{2m-1}, b_{2m}) < d(b_{2m}, b_{2m+1}). \tag{71}$$

Here, we discuss three possible cases of $M(t_{2m}, t_{2m-1})$.

Case (1): if $M(t_{2m}, t_{2m-1}) = d(b_{2m}, b_{2m-1})$, then

$$\begin{aligned} \psi(s^{\epsilon} d(b_{2m}, b_{2m+1})) &\le \psi(d(b_{2m}, b_{2m-1})) \\ &\quad - \phi(d(b_{2m}, b_{2m-1})) \\ &\le \psi(d(b_{2m}, b_{2m-1})). \end{aligned} \tag{72}$$

As ψ is nondecreasing, therefore

$$s^{\epsilon} d(b_{2m}, b_{2m+1}) \le d(b_{2m}, b_{2m-1}), \tag{73}$$

which contradicts assumption (69). Thus,

$$d(b_{2m+1}, b_{2m}) \le d(b_{2m}, b_{2m-1}). \tag{74}$$

Case (2): if $M(t_{2m}, t_{2m-1}) = d(b_{2m}, b_{2m+1})$, then

$$\begin{aligned} \psi(s^{\epsilon} d(b_{2m}, b_{2m+1})) &\le \psi(d(b_{2m}, b_{2m+1})) - \phi(d(b_{2m}, b_{2m+1})) \\ &\le \psi(s^{\epsilon} d(b_{2m}, b_{2m+1})) - \phi(d(b_{2m}, b_{2m+1})). \end{aligned} \tag{75}$$

This is only possible if $\phi(d(b_{2m}, b_{2m+1})) = 0$. However, $\phi(n) = 0$ if $n = 0$. Hence, $d(b_{2m}, b_{2m+1}) = 0$, which implies that $b_{2m} = b_{2m+1}$ which contradicts (68). Hence,

$$d(b_{2m+1}, b_{2m}) \le d(b_{2m}, b_{2m-1}). \tag{76}$$

Case (3): if $M(t_{2m}, t_{2m-1}) = d(b_{2m-1}, b_{2m+1})/2s$. Using triangle inequality and (69), we have Then,

$$\begin{aligned} M(t_{2m}, t_{2m-1}) &\le \frac{s(d(b_{2m-1}, b_{2m})) + d(b_{2m}, b_{2m+1})}{2s} \\ &\le d(b_{2m}, b_{2m+1}). \end{aligned} \tag{77}$$

$$\psi(s^{\epsilon}d(b_{2m},b_{2m+1}))\le\psi(d(b_{2m},b_{2m+1}))-\phi\left(\frac{d(b_{2m-1},b_{2m+1})}{2s}\right)$$
$$\le\psi(s^{\epsilon}d(b_{2m},b_{2m+1}))-\phi\left(\frac{d(b_{2m-1},b_{2m+1})}{2s}\right), \tag{78}$$

which implies that $\phi(d(b_{2m-1},b_{2m+1})/2s)=0$, Therefore, $d(b_{2m-1},b_{2m+1})=0$. So, $b_{2m-1}=b_{2m+1}$ contradicts (68).

Therefore, in all three cases, we concluded that

$$d(b_{2m+1},b_{2m})\le d(b_{2m},b_{2m-1}). \tag{79}$$

Step II: in this step, we will show that the sequence b_m is b-Cauchy sequence by using Lemma 1 for all three cases.

Case (1): if $M(t_{2m},t_{2m-1})=d(b_{2m},b_{2m-1})$.

From (69), one can write

$$\psi(s^{\epsilon}d(b_{2m},b_{2m+1}))\le\psi(d(b_{2m},b_{2m-1}))-\phi(d(b_{2m},b_{2m-1})). \tag{80}$$

Since $\phi(n)>0$ for $n>0$, therefore

$$\psi(s^{\epsilon}d(b_{2m},b_{2m+1}))\le\psi(d(b_{2m},b_{2m-1})). \tag{81}$$

Since ψ is nondecreasing, therefore

$$s^{\epsilon}d(b_{2m},b_{2m+1})\le d(b_{2m},b_{2m-1}),$$
$$d(b_{2m},b_{2m+1})\le\frac{1}{s^{\epsilon}}d(b_{2m},b_{2m-1}), \tag{82}$$
$$d(b_{2m},b_{2m+1})\le Kd(b_{2m},b_{2m-1}),$$

where $K=1/s^{\epsilon}$. Thus,

$$d(b_{2m},b_{2m+1})\le Kd(b_{2m},b_{2m-1}). \tag{83}$$

Case(2): when $M(t_{2m},t_{2m-1})=d(b_{2m},b_{2m+1})$, then from (79), one can write

$$M(t_{2m},t_{2m-1})=d(b_{2m},b_{2m+1})\le d(b_{2m},b_{2m-1}). \tag{84}$$

From (69), one has

$$\psi(s^{\epsilon}d(b_{2m},b_{2m+1}))\le\psi(d(b_{2m},b_{2m-1}))-\phi(d(b_{2m},b_{2m+1})). \tag{85}$$

Since $\phi(n)>0$ for $n>0$, therefore

$$\psi(s^{\epsilon}d(b_{2m},b_{2m+1}))\le\psi(d(b_{2m},b_{2m-1})). \tag{86}$$

From nondecreasing property of ψ, we have

$$s^{\epsilon}d(b_{2m},b_{2m+1})\le d(b_{2m},b_{2m-1}),$$
$$d(b_{2m},b_{2m+1})\le K\,d(b_{2m},b_{2m-1}), \tag{87}$$

where $K=1/s^{\epsilon}$ with $K\in[0,1/s)$.

Case (3): when $M(t_{2m},t_{2m-1})=d(b_{2m-1},b_{2m+1})/2s$.

Then, by using triangle inequality and (79) it will be converted to case (2). Therefore, (2) holds for all cases.

Similarly, if $Ft_{2m+1}=b_{2m+1}$ and $Gt_{2m+2}=b_{2m+2}$, again we have

$$d(b_{2m+1},b_{2m+2})\le Kd(b_{2m},b_{2m+1}). \tag{88}$$

Hence,

$$d(b_{m+1},b_{m+2})\le Kd(b_m,b_{m+1}),\quad \text{for } b_m\ne b_{m+1}\ \forall m\in N. \tag{89}$$

Let $d_k=d(b_k,b_{k+1})$. Assume that $d_{k_0}=0$ for k_0, then $b_{k_0}=b_{k_{0+1}}$.

When $k_0=2m$, it implies that $b_{2m}=b_{2m+1}$; therefore,

$$M(t_{2m},t_{2m+1})\in\left\{0,d(b_{2m+1},b_{2m+2}),0,\frac{0+d(b_{2m},b_{2m+2})}{2s}\right\}$$
$$=\left\{0,d(b_{2m+2},b_{2m+1}),\frac{d(b_{2m+2},b_{2m})}{2s}\right\}. \tag{90}$$

If

$$M(t_{2m},t_{2m+1})=d(b_{2m+2},b_{2m+1}). \tag{91}$$

Then, from (64),

$$\psi(s^{\epsilon}d(b_{2m+1},b_{2m+2}))\le\psi(d(b_{2m+2},b_{2m+1}))-\phi(d(b_{2m+2},b_{2m+1})). \tag{92}$$

From the above, we can obtain

$$\psi(s^{\epsilon}d(b_{2m+1},b_{2m+2}))\le\psi(d(b_{2m+1},b_{2m+2})). \tag{93}$$

Since ψ is nondecreasing, therefore

$$s^{\epsilon}d(b_{2m+1},b_{2m+2})\le d(b_{2m+1},b_{2m+2}). \tag{94}$$

It implies that $d(b_{2m+1},b_{2m+2})=0$. Therefore, $b_{2m+1}=b_{2m+2}$.

Hence, $b_{2m}=b_{2m+1}=b_{2m+2}$.

Therefore, the sequence $\{b_k\}$ is constant sequence for $k\ge k_0$. Therefore, (2) also holds for constant sequence $\{b_k\}$. Thus, by Lemma 1, $\{b_m\}$ is a b-Cauchy sequence. From the completeness of X one can say that b-Cauchy sequence $\{b_m\}$ converges to some b in X.

Consequently, $At_{2m}\longrightarrow b, Bt_{2m+1}\longrightarrow b$, $Ft_{2m+1}\longrightarrow b$, and $Gt_{2m}\longrightarrow b$.

Step III: now, we show that b is a coincidence point of A and F:

$$\lim_{m\longrightarrow\infty}d(At_{2m},b)=\lim_{m\longrightarrow\infty}d(Gt_{2m+1},b)$$
$$=\lim_{m\longrightarrow\infty}d(b_{2m+1},b)=0,$$
$$\lim_{m\longrightarrow\infty}d(Bt_{2m+1},b)=\lim_{m\longrightarrow\infty}d(Ft_{2m+2},b) \tag{95}$$
$$=\lim_{m\longrightarrow\infty}d(b_{2m+2},b)=0.$$

Since the pair (A, F) is compatible, so we have

$$\lim_{m\longrightarrow\infty} d(FAt_{2m}, AFt_{2m}) = 0. \tag{96}$$

Moreover, $\lim_{m\longrightarrow\infty} d(At_{2m}, b) = 0$ and $\lim_{m\longrightarrow\infty} d(Ft_{2m}, b) = 0$. Also, A and F are continuous mapping; therefore,

$$\lim_{m\longrightarrow\infty} d(FAt_{2m}, Fb) = \lim_{m\longrightarrow\infty} d(AFt_{2m}, Ab) = 0. \tag{97}$$

Using triangle inequality, one can write

$$d(Fb, Ab) \le s(d(Fb, FAt_{2m}) + d(FAt_{2m}, Ab)). \tag{98}$$

Again, applying triangle inequality on the second term, we have

$$\begin{aligned} d(Fb, Ab) \le{}& sd(Fb, FAt_{2m}) + s^2(d(FAt_{2m}, AFt_{2m})) \\ &+ d(AFt_{2m}, Ab). \end{aligned} \tag{99}$$

By taking limit $m \longrightarrow \infty$ and using (96) and (97), we can write

$$\lim_{m\longrightarrow\infty} d(Fb, Ab) \le 0. \tag{100}$$

Therefore, $d(Fb, Ab) = 0$. Thus, $Ab = Fb$.

Similarly, we can show that $Bb = Gb$.

Since Gb and Fb are comparable, therefore from (64), we can obtain

$$\psi(s^{\epsilon} d(Ab, Bb)) \le \psi(M(b, b)) - \phi(M(b, b)), \tag{101}$$

where

$$\begin{aligned} M(b, b) \in{}& \left\{ d(Gb, Fb), d(Gb, Ab), d(Fb, Bb), \right. \\ &\left. \frac{d(Gb, Bb) + d(Fb, Ab)}{2s} \right\} \\ ={}& \{d(Ab, Bb), d(Bb, Ab), d(Ab, Bb), 0,\} = d(Ab, Bb). \end{aligned} \tag{102}$$

Thus,

$$\psi(s^{\epsilon} d(Ab, Bb)) \le \psi(d(Ab, Bb)) - \phi(d(Ab, Bb)), \tag{103}$$

which implies that $Ab = Bb$. Hence, $Ab = Bb = Fb = Gb$.

In the forthcoming, result condition of continuity and compatibility for mapping is relaxed. □

Theorem 9. *Let A, B, G, F: $X \longrightarrow X$, with $A(X) \subseteq F(X)$, $B(X) \subseteq G(X)$ and $F(X)$, $G(X)$ are b-closed subsets of regular partially ordered complete b-metric space $(X, \prec, d)$. Assume that the pairs (A, B) and (B, A) are partially weakly increasing with respect to F and G, respectively, whereas (A, G) and (B, F) are weakly compatible and satisfy the following condition:*

$$\psi(s^{\epsilon} d(At, Bb)) \le \psi(M(t, b)) - \phi(M(t, b)), \quad \forall t, b \in X. \tag{104}$$

Then, these four mappings have a coincidence point. If Gb and Fb are comparable then $Ab = Bb = Fb = Gb$.

Proof. As Theorem 8, one can easily construct a sequence $\{b_m\}$ which converges to some $b \in X$. Thus,

$$\lim_{m\longrightarrow\infty} d(b_m, b) = 0. \tag{105}$$

If $\{b_{2m+1}\} \subseteq F(X)$, $\{b_{2m+2}\} \subseteq G(X)$. Since $F(X)$ and $G(X)$ are b-closed subsets X, therefore, there exist some $x, k \in X$, such that $b = Fx$ and $b = Gk$. Hence,

$$\begin{aligned} \lim_{m\longrightarrow\infty} Ft_{2m+1} &= \lim_{m\longrightarrow\infty} b_{2m+1} = b = Gk, \\ \lim_{m\longrightarrow\infty} Bt_{2m+1} &= \lim_{m\longrightarrow\infty} b_{2m+2} = b = Gk. \end{aligned} \tag{106}$$

Now, we prove that $Ab = Gb$.

Since $(X, \prec, d)$ is regular, therefore

$$\begin{aligned} Ft_{2m+1} &\prec Gk, \\ Bt_{2m+1} &\prec Gk. \end{aligned} \tag{107}$$

Hence, (104) implies that

$$\psi(s^{\epsilon} d(Ak, Bt_{2m+1})) \le \psi(M(k, t_{2m+1})) - \phi(M(k, t_{2m+1})), \tag{108}$$

where

$$\begin{aligned} M(k, t_{2m+1}) \in{}& \left\{ d(Gk, Ft_{2m+1}), d(Gk, Ak), d(Ft_{2m+1}, Bt_{2m+1}), \frac{d(Gk, Bt_{2m+1}) + d(Ft_{2m+1}, Ak)}{2s} \right\} \\ \le{}& \left\{ d(Gk, Gk), d(Gk, Ak), d(Gk, Gk), \frac{d(Gk, Gk) + d(Ft_{2m+1}, Gk) + d(Gk, Ak)}{2s} \right\} \\ ={}& \left\{ 0, d(Gk, Ak), 0, \frac{d(Gk, Ak)}{2s} \right\}. \end{aligned} \tag{109}$$

From (108) and (109), we have

$$\begin{aligned}\psi\left(s^{\epsilon}d(Ak,Gk)\right) &\le \psi\left(\frac{d(Gk,Ak)}{2s}\right)-\phi\left(\frac{d(Gk,Ak)}{2s}\right)\\ &\le \psi\left(s^{\epsilon}\frac{d(Gk,Ak)}{2s}\right)-\phi\left(\frac{d(Gk,Ak)}{2s}\right)\\ &\le \psi\left(s^{\epsilon}d(Gk,Ak)\right)-\phi\left(\frac{d(Gk,Ak)}{2s}\right),\end{aligned} \tag{110}$$

which implies that $d(Gk,Ak)=0$. Hence, $Gk=Ak$. Thus, $b=Gk=Ak$. Now, from weak compatibility of Ab and G, one can write $Ab=AGk=GAk=Gb$. Thus, $Ab=Gb$. Similarly, we can show $Fb=Bb$. By following the last lines of the proof of Theorem 8 we have $Ab=Bb=Fb=Gb$.

By taking $G=F$, in Theorem 8, the following corollary is obtained with

$$M(t,b)\in\left\{d(Ft,Fb),d(Ft,At),d(Fb,Bb),\frac{d(Ft,Bb)+d(Fb,At)}{2s}\right\}. \tag{111}$$

□

Corollary 2. *Let A, B, F: $X\longrightarrow X$ be three continuous mappings on $(X,\prec,d)$ where $A(X)\cup B(X)\subseteq F(X)$. Let pairs (A,F) and (B,F) be compatible, whereas (A,B) be partially weakly increasing with respect to F and satisfying the following condition:*

$$\psi\left(s^{\epsilon}d(At,Bb)\right)\le\psi(M(t,b))-\phi(M(t,b)),\quad \forall t,b\in X, \tag{112}$$

then A,B, and F have a coincidence point.

If we put $A=B$ in Theorem 8, the following corollary is obtained with

$$M(t,b)\in\left\{d(Gt,Fb),d(Gt,At),d(Fb,Ab),\frac{d(Gt,Ab)+d(Fb,At)}{2s}\right\}. \tag{113}$$

Corollary 3. *Let A, F, G: $X\longrightarrow X$ be three continuous mappings on $(X,\prec,d)$ such that $A(X)\subseteq F(X)\cup G(X)$. Assume that pairs (A,F) and (A,G) are compatible, whereas A is partially weakly increasing with respect to G, F and satisfies the following condition:*

$$\psi\left(s^{\epsilon}d(At,Ab)\right)\le\psi(M(t,b))-\phi(M(t,b)),\quad \forall t,b\in X, \tag{114}$$

then A,F, and G have a coincidence point.

When we put $A=B$ and $G=F$ in Theorem 8, the following corollary is obtained with

$$M(t,b)\in\left\{d(Ft,Fb),d(Ft,At),d(Fb,Ab),\frac{d(Ft,Ab)+d(Fb,At)}{2s}\right\}. \tag{115}$$

Corollary 4. *Let A, F: $X\longrightarrow X$ be mappings on $(X,\prec,d)$ such that $A(X)\subseteq F(X)$. Assume that pair (A,F) is compatible, whereas A is partially weakly increasing with respect to F and satisfies the following condition:*

$$\psi\left(s^{\epsilon}d(At,Ab)\right)\le\psi(M(t,b))-\phi(M(t,b)),\quad \forall t,b\in X, \tag{116}$$

either A and F are continuous or space X is regular. Then, A,F have a coincidence point.

If we put $F=G=I$ (identity mapping) in Theorem 8, then the following corollary is obtained with

$$M(t,b)\in\left\{d(t,b),d(t,At),d(b,Bb),\frac{d(t,Bb)+d(b,At)}{2s}\right\}. \tag{117}$$

Corollary 5. *Let A, B: $X\longrightarrow X$ be mappings on $(X,\prec,d)$ such that the pair (A,B) is partially weakly increasing and satisfies the following condition:*

$$\psi\left(s^{\epsilon}d(At,Bb)\right)\le\psi(M(t,b))-\phi(M(t,b)),\quad \forall t,b\in X, \tag{118}$$

either A and B are continuous or space X is regular. Then, A,B have a common fixed point.

Example 2. Let $t\prec b$ if and only if $b\le t$, for all t,b in X, is partially ordering on $X=[0,\infty)$. Define b-metric on X by $d(t,b)=|t-b|^2$, with $s=2$. Suppose A,B,F, and G be continuous mappings defined by

$$\begin{aligned}A(t)&=\sinh^{-1}t,\\ B(t)&=\sinh^{-1}\frac{t}{2},\\ G(t)&=\sinh 4t,\\ F(t)&=\sinh 8t.\end{aligned} \tag{119}$$

First, we check the compatibility of (A,F). Let $\{t_a\}$ be sequence which converges to some $a\in X$. Then,

$$\begin{aligned}\lim_{a\longrightarrow\infty}d(a,At_a)&=\lim_{a\longrightarrow\infty}d(a,Ft_a)=0,\\ \lim_{a\longrightarrow\infty}\left|\sinh^{-1}t_a-a\right|^2&=\lim_{a\longrightarrow\infty}\left|\sinh 8t_a-a\right|^2=0.\end{aligned} \tag{120}$$

The continuity of A, F implies that

$$\lim_{a\longrightarrow\infty}\left|t_a-\sinh a\right|^2=\lim_{a\longrightarrow\infty}\left|t_a-\frac{\sinh^{-1}a}{8}\right|^2=0. \tag{121}$$

Since limit of the sequence is unique, therefore $\sinh a = (\sinh^{-1} a/8)$, which implies that $a = 0$. Also, A and F are continuous, so

$$\lim_{a\longrightarrow\infty} d(AFt_a, FAt_a) = \lim_{a\longrightarrow\infty} |AFt_a - FAt_a|^2 = 0. \quad (122)$$

Hence, the pair (A, F) is compatible. Similarly, one can check the compatibility of the pair (B, G).

Now, we have to prove that (A, B) is partially weakly increases with respect to G. Assume that $b \in G^{-1}(At)$, for $t, b \in S$,

$$A(t) = G(b) \Longrightarrow \sinh^{-1} t = \sinh 4b. \quad (123)$$

Therefore, $b = (\sinh^{-1}(\sinh^{-1} t))/4$. We know that

$$\sinh 8t \geq \sinh^{-1} t \Longrightarrow 8t \geq \sinh^{-1}\left(\sinh^{-1} t\right). \quad (124)$$

$t \geq (\sinh^{-1}(\sinh^{-1} t))/8$. Hence,

$$\begin{aligned} A(t) &= \sinh^{-1} t \geq \sinh^{-1}\left(\frac{\sinh^{-1}\left(\sinh^{-1} t\right)}{8}\right) \\ &= \sinh^{-1}\frac{b}{2} = Bb. \end{aligned} \quad (125)$$

Therefore,

$$A(t) \prec B(b). \quad (126)$$

With the same process one can show that (B, A) is partially weakly increases with respect to F.

Next, we check the validity of the following inequity:

$$\psi\left(s^{\epsilon} d(At, Bb)\right) \leq \psi(M(t, b)) - \phi(M(t, b)), \quad \text{for all } t, b \in X. \quad (127)$$

Let $\psi(m) = bm$ and $\phi(m) = (b-1)m$, where $1 < b < 17$ and $1 < \epsilon < 2$.

Apply mean value theorem to $\sinh^{-1}$ in intervals $[t, b/2]$ and sinh in intervals $[8t, 4b]$. Take

$$\begin{aligned} \psi\left(2^{\epsilon} d(At, Bb)\right) &= 2^{\epsilon} b|At - Bb|^2 \\ &\leq 2^{\epsilon} b\left|\sinh^{-1} t - \sinh^{-1}\frac{b}{2}\right|^2 \\ &\leq 2^{\epsilon} b\left|t - \frac{b}{2}\right|^2 \\ &\leq 2^{\epsilon} b\frac{|8t - 4b|^2}{8^2} \\ &\leq 2^{\epsilon-6} b|\sinh 8t - \sinh 4b|^2 \\ &= 2^{\epsilon-6} b|Ft - Gb|^2 \\ &\leq d(Ft, Gb) \\ &= \psi(M(t, b)) - \phi(M(t, b)). \end{aligned} \quad (128)$$

All conditions of Theorem 8 are satisfied, and consequently 0 is a coincidence point.

Remark 4. Note that the conditions of our Theorem 8 holds for example of [19] and the corresponding conclusion holds. However, when $1 < \epsilon < 2$, then inequality of Theorem 3 does not hold for our Example 2. Hence, Theorem 8 is a genuine generalization of Theorem 3.

Remark 5. By substitution, $\epsilon = 2$ in Theorem 8 and 9 and corollaries 3.7, 3.8, and 3.9, and we get the results of [19].

Remark 6. Theorems 2.4 and 2.6 of [45] are special case of Corollary 4. By putting $G = I$ in Corollary 2, one can get Corollary 2.7 of [45].

Remark 7. Theorems 3.1 and 4.3 and Corollary 3.3 of [46] are special cases of Corollary 5.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All the authors have contributed equally in all parts.

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