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Research Article

Lower Bound for the Class Number of $\mathbb{Q}\left(\sqrt{n^2+4}\right)$

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In this paper, we give an explicit lower bound for the class number of real quadratic field $\mathbb{Q}(\sqrt{d})$, where $d = n^2 + 4$ is a square-free integer, using $\omega(n)$ which is the number of odd prime divisors of n.

1. Introduction

Let d be a positive square-free integer and let $h(d)$ and C_k denote the class number and the class group of a real quadratic field $k = \mathbb{Q}(\sqrt{d})$, respectively.

The class number problem of quadratic fields is one of the most intriguing unsolved problems in Algebraic Number Theory and has for a long time inspired the study of lower bounds of $h(d)$.

Many fruitful research studies have been conducted in this direction. Hasse [1] and Yokoi [2, 3] studied lower bounds for class numbers of certain real quadratic fields. Mollin [4, 5] generalized their results for certain real quadratic and biquadratic fields.

In this work, we give a lower bound for $h(n^2+4)$, and also we find a necessary and sufficient condition for $k = \mathbb{Q}(\sqrt{n^2+4})$ to have class number $\omega(n)+1$.

2. Notation and Preliminaries

Let k be a real quadratic field and $\zeta_k(s)$ be its Dedekind zeta function. Siegel [6] developed a method of computing $\zeta_k(1-2n)$, where n is a positive integer. By specializing Siegel's formula for a real quadratic field, we obtain the following result.

Theorem 1 (Zagier [7]). *Let k be a real quadratic field with discriminant D. Then*

$$\zeta_k(-1) = \frac{1}{60} \sum_{\substack{|t|<\sqrt{D} \\ t^2 \equiv D \pmod 4}} \sigma_1\left(\frac{D-t^2}{4}\right), \tag{1}$$

where $\sigma_1(r)$ denotes the sum of divisors of r.

However, there is another method, according to Lang, of computing special values of $\zeta_k(s)$ if k is a real quadratic field.

Let $k = \mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant D and H an ideal class of k. Let I be any integral ideal belonging to H^{-1} with an integral basis $\{r_1, r_2\}$. We put

$$\delta(I) = r_1 r_2' - r_1' r_2, \tag{2}$$

where r_1' and r_2' are the conjugates of r_1 and r_2 respectively.

Let ε be the fundamental unit of k. Then, $\{\varepsilon r_1, \varepsilon r_2\}$ is also integral basis of I, and thus we can find a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries satisfying

$$\varepsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M . \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{3}$$

Now, we can state Lang's formula.

Theorem 2 (Lang [8]). *By keeping the abovementioned notation, we have*

$$\begin{aligned}\zeta_k(-1,H) = \frac{\operatorname{sgn}\delta(I)\, r_2 r_2'}{360\, N(I) c^3}\Big\{&(a+d)^3 - 6(a+d)N(\varepsilon)\\ &- 240c^3(\operatorname{sgn} c)S^3(a,c) + 180ac^3(\operatorname{sgn} c)S^2(a,c)\\ &-240c^3(\operatorname{sgn} c)S^3(d,c) + 180\,dc^3(\operatorname{sgn} c)S^2(d,c)\Big\},\end{aligned} \tag{4}$$

where $N(I)$ denotes the norm of an ideal I, $N(\varepsilon)$ is the norm of ε, and $S^i(-,-)$ denotes the generalized Dedekind sum as defined in [9].

To use Lang's formula, we need to determine the values of a, b, c, d, and generalized Dedekind sum.

Lemma 1 (see ref [10]). *The entries of M are given by*

$$\begin{aligned} a &= \operatorname{tr}\left(\frac{r_1 r_2' \varepsilon}{\delta(I)}\right),\\ b &= \operatorname{tr}\left(\frac{r_1 r_1' \varepsilon'}{\delta(I)}\right),\\ c &= \operatorname{tr}\left(\frac{r_2 r_2' \varepsilon}{\delta(I)}\right),\\ d &= \operatorname{tr}\left(\frac{r_1 r_2' \varepsilon'}{\delta(I)}\right).\end{aligned} \tag{5}$$

Moreover, $\det M = N(\varepsilon)$ and $bc \neq 0$.

Kim [11] obtained the following expressions for generalized Dedekind sum. These expressions are also needed to compute the values of zeta functions for ideal classes of the respective real quadratic fields.

Lemma 2 (Kim [11]). *Let m be a positive integer. Then, we have*

(i) $S^3(\pm 1, m) = \pm\left((-m^4 + 5m^2 - 4)/(120m^3)\right)$

(ii) $S^2(\pm 1, m) = \left((m^4 + 10m^2 - 6)/(180m^3)\right)$

3. Main Results

Let n be a positive integer and let $d = n^2 + 4$ is a square-free integer. Clearly, $d \equiv 5 \pmod 8$ and n is odd. In this case, the fundamental unit of k is $\varepsilon = (n + \sqrt{d})/2$ and $N(\varepsilon) = -1$. If $p \mid n$, then p splits in $k = \mathbb{Q}(\sqrt{d})$ as

$$\langle p\rangle = \left\langle p, \frac{p+2+\sqrt{d}}{2}\right\rangle \left\langle p, \frac{p+2-\sqrt{d}}{2}\right\rangle. \tag{6}$$

By [12], Theorem 2.4, we know that

$$\zeta_k(-1,A) = \frac{n^3 + 11n}{360}, \tag{7}$$

where A will always denote the principal ideal class in k.

In this section, we will prove our main results. As a start, we record the following proposition.

Proposition 1. *Let $k = \mathbb{Q}(\sqrt{d})$, where $d = n^2 + 4$ is a square-free integer. Let p be an odd prime divisor of n, and let C be the ideal class containing $\langle p, ((p+2+\sqrt{d})/2)\rangle$ or $\langle p, ((p+2-\sqrt{d})/2)\rangle$. Then*

$$\zeta_k(-1,C) = \frac{n^3 + n(p^4 + 10p^2)}{360p^2}. \tag{8}$$

Proof. Let us assume $I := \langle p, ((p+2+\sqrt{d})/2)\rangle \in C^{-1}$. Then, $\{r_1 = ((p+2+\sqrt{d})/2), r_2 = p\}$ is an integral basis for I, and thus $\delta(I) = p\sqrt{d}$. By Lemma 1, we get

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} \dfrac{n+p}{2} + 1 & \dfrac{d-(p+2)^2}{4p}\\ p & \dfrac{n-p}{2} - 1\end{pmatrix}. \tag{9}$$

Since $p \mid n$, $((n \pm p)/2) \pm 1 \equiv \pm 1 \pmod p$. Hence, by Lemma 2, we obtain

$$\begin{aligned} 240c^3(\operatorname{sgn} c)S^3(a,c) &= 240 \times p^3 S^3(1,p)\\ &= 2\left(-p^4 + 5p^2 - 4\right),\\ 240c^3(\operatorname{sgn} c)S^3(d,c) &= 240 \times p^3 S^3(-1,p)\\ &= -2\left(-p^4 + 5p^2 - 4\right),\\ 180ac^3(\operatorname{sgn} c)S^2(a,c) &= 180\left(\frac{n+p}{2} + 1\right)p^3 S^2(1,p)\\ &= \frac{(p^4 + 10p^2 - 6)(n+p+2)}{2},\\ 180\,dc^3(\operatorname{sgn} c)S^2(d,c) &= 180\left(\frac{n-p}{2} - 1\right)p^3 S^2(-1,p)\\ &= \frac{(p^4 + 10p^2 - 6)(n-p-2)}{2}.\end{aligned} \tag{10}$$

By Theorem 2, we get

$$\zeta_k(-1,C) = \frac{n^3 + n(p^4 + 10p^2)}{360p^2}. \tag{11}$$

□

We use Proposition 1 in order to prove the following theorem, which gives a lower bound for $h(n^2 + 4)$.

Theorem 3. *Let n be a positive integer, $\omega(n) \geq 3$ and let $d = n^2 + 4$ is a square-free integer. Then*

$$h(d) \geq \omega(n) + 1. \tag{12}$$

Proof. By [12], Proposition 2.3, it follows that (2) remains prime in $k = \mathbb{Q}(\sqrt{d})$.

We assume that A denotes the principal ideal class in k, and then $\zeta_k(-1, A)$ is given by (7). If $p_i \mid n$ $(1 \leq i \leq \omega(n))$, then p_i splits in k as in (6), so let C_i $(1 \leq i \leq \omega(n))$ be the ideal class in k such that $\langle p_i, ((p_i + 2 + \sqrt{d})/2)\rangle \in C_i$; then, by Proposition 1, we obtain

$$\zeta_k(-1, C_i) = \frac{n^3 + n(p_i^4 + 10p_i^2)}{360p_i^2}. \tag{13}$$

By computing $\zeta_k(-1, C_i) = \zeta_k(-1, A)$, we arrive at $n = p_i$. This contradicts the fact that $\omega(n) \geq 3$. Similarly, $\zeta_k(-1, C_i) = \zeta_k(-1, C_j)$ for $i \neq j$, and then $n = p_i p_j$. This again contradicts the fact that $\omega(n) \geq 3$.

We notice $\{C_1, \ldots, C_{\omega(n)}\}$ are a distinct nonprincipal ideal class in k. This completes the proof. □

Now, we give a necessary and sufficient condition for $k = \mathbb{Q}(\sqrt{n^2 + 4})$ to have class number $\omega(n) + 1$.

Theorem 4. *By keeping the abovementioned notation, we have $h(d) = \omega(n) + 1$ if and only if*

$$\sum_{m=0}^{((n-1)/2)} \sigma_1\left(1 + mn - m^2\right) = \frac{n^3 + 11n}{12} + \sum_{i=1}^{\omega(n)} \frac{n^3 + n(p_i^4 + 10p_i^2)}{12p_i^2}. \tag{14}$$

Proof. Now, by Theorem 1,

$$\begin{aligned}
\zeta_k(-1) &= \frac{1}{60} \sum_{\substack{|t| < \sqrt{n^2+4} \\ t^2 \equiv n^2+4 \pmod 4}} \sigma_1\left(\frac{n^2 + 4 - t^2}{4}\right), \\
&= \frac{1}{60} \sum_{\substack{|t| \leq n \\ t \text{ is odd}}} \sigma_1\left(1 + \frac{n^2 - t^2}{4}\right), \\
&= \frac{1}{30} \sum_{\substack{1 \leq t \leq n \\ t \text{ is odd}}} \sigma_1\left(1 + \frac{(n-t)(n+t)}{4}\right).
\end{aligned} \tag{15}$$

If we replace t by $n - r$, where r is even, then

$$\zeta_k(-1) = \frac{1}{30} \sum_{\substack{0 \leq r \leq n-1 \\ r \text{ is even}}} \sigma_1\left(1 + \frac{r(2n - r)}{4}\right). \tag{16}$$

Now, we replace r by $2m$, and then we have

$$\zeta_k(-1) = \frac{1}{30} \sum_{m=0}^{(n-1)/2} \sigma_1\left(1 + mn - m^2\right). \tag{17}$$

Necessary: let $h(d) = \omega(n) + 1$. Then, the class group of k is $C_k = \{A, C_1, \ldots, C_{\omega(n)}\}$. Now, by definition, we have

$$\begin{aligned}
\zeta_k(-1) &= \sum_{J \in C_k} \zeta_k(-1, J), \\
\zeta_k(-1) &= \zeta_k(-1, A) + \sum_{i=1}^{\omega(n)} \zeta_k(-1, C_i).
\end{aligned} \tag{18}$$

This implies

$$\zeta_k(-1) = \frac{n^3 + 11n}{360} + \sum_{i=1}^{\omega(n)} \frac{n^3 + n(p_i^4 + 10p_i^2)}{360p_i^2}. \tag{19}$$

Finally, by (17), we obtain

$$\sum_{m=0}^{(n-1)/2} \sigma_1\left(1 + mn - m^2\right) = \frac{n^3 + 11n}{12} + \sum_{i=1}^{\omega(n)} \frac{n^3 + n(p_i^4 + 10p_i^2)}{12p_i^2}. \tag{20}$$

Sufficiency: let

$$\sum_{m=0}^{(n-1)/2} \sigma_1\left(1 + mn - m^2\right) = \frac{n^3 + 11n}{12} + \sum_{i=1}^{\omega(n)} \frac{n^3 + n(p_i^4 + 10p_i^2)}{12p_i^2}. \tag{21}$$

Hence, by (17), we find

$$\zeta_k(-1) = \frac{n^3 + 11n}{360} + \sum_{i=1}^{\omega(n)} \frac{n^3 + n(p_i^4 + 10p_i^2)}{360p_i^2}. \tag{22}$$

By Theorem 3, we get $h(d) \geq \omega(n) + 1$.

Suppose $h(d) > \omega(n) + 1$. Then, there exist at least $\omega(n) + 2$ ideal classes in k.

Since for any ideal class E, $\zeta_k(-1, E) > 0$; thus,

$$\zeta_k(-1) > \zeta_k(-1, A) + \sum_{i=1}^{\omega(n)} \zeta_k(-1, C_i). \tag{23}$$

It is a contradiction. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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