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Research Article

Relative Gottlieb Groups of Embeddings between Complex Grassmannians

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Let $\mathrm{Gr}(k,n)$ be the complex Grassmann manifold of k-linear subspaces in $\mathbb{C}^n$. We compute rational relative Gottlieb groups of the embedding $i\colon \mathrm{Gr}(k,n) \longrightarrow \mathrm{Gr}(k,n+r)$ and show that the G-sequence is exact if $r \geq k(n-k)$.

1. Introduction

We work in the category of spaces having the homotopy type of simply connected CW complexes of finite type. We denote by $h\colon X \longrightarrow X_{\mathbb{Q}}$ the rationalization of X [1, 2]. Let $f\colon (X,x_0) \longrightarrow (Y,y_0)$ be a pointed continuous mapping and $\mathrm{map}(X,Y;f)$ be the component of f in the space of all continuous maps $g\colon X \longrightarrow Y$. Consider the evaluation map $\mathrm{ev}\colon \mathrm{map}(X,Y;f) \longrightarrow Y$ at the base point x_0, that is, $\mathrm{ev}(g) = g(x_0)$. The nth evaluation subgroup of f, $G_n(Y,X;f)$, is the image of $\pi_n(\mathrm{ev})$ in $\pi_n(Y)$ [3]. In the special case where $X = Y$ and $f = 1_X$, one obtains the Gottlieb group $G_n(X)$ of X [4]. Gottlieb groups play an important role in topology. For instance, if $G_n(X) = 0$, then any fibration $X \longrightarrow E \longrightarrow S^{n+1}$ admits a section (Corollary 2–7 in [4]).

In [2], Lee and Woo introduce relative evaluation groups $G_n^{\mathrm{rel}}(Y,X;f)$ and obtain a long sequence,

$$\begin{aligned} \cdots &\longrightarrow G_{n+1}^{\mathrm{rel}}(Y,X;f) \longrightarrow G_n(X) \longrightarrow G_n(Y,X;f) \\ &\longrightarrow G_n^{\mathrm{rel}}(Y,X;f) \longrightarrow \cdots, \end{aligned} \tag{1}$$

called G-sequence [5]. This sequence is exact in some cases, for instance, if f is a homotopy monomorphism [6].

2. Rational Relative Gottlieb Groups

The rationalization $h\colon Y \longrightarrow Y_{\mathbb{Q}}$ induces a rationalization $h_*\colon \mathrm{map}(X,Y;f) \longrightarrow \mathrm{map}(X,Y;h\circ f)$ [7]. Therefore,

$$\mathrm{ev}_*(\pi_*(\mathrm{map}(X,Y;f))\otimes\mathbb{Q}) \cong \mathrm{ev}_*(\pi_*(\mathrm{map}(X,Y_{\mathbb{Q}};h\circ f))). \tag{2}$$

In this paper, we study the G-sequence of the natural inclusion $\mathrm{Gr}(k,n) \longrightarrow \mathrm{Gr}(k,n+r)$ using models of function spaces in rational homotopy [8, 9]. In particular, we show that the G-sequence is exact if $r \geq k(n-k)$. We work with algebraic models in rational homotopy theory introduced by Sullivan and Quillen [10, 11]. In this section, we give relevant definitions and fix notation. Details can be found in [1]. All vector spaces and algebras are over the field of rational numbers $\mathbb{Q}$.

Let (A,d) be a cochain algebra. The degree of an homogeneous element $a \in A^p$ is written $|a|$. We assume that

(A, d) is 1-connected, that is, $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. The algebra A is called commutative if $ab = (-1)^{|a||b|}ba$ for homogeneous elements $a, b \in A$.

Definition 1. A commutative differential graded algebra (cdga, for short) (A, d) is called a Sullivan algebra if $A = S(V^{\text{even}}) \otimes E(V^{\text{odd}})$, where $V = \oplus_{k\geq 2} V^k$. It will be denoted by $(\wedge V, d)$.

Moreover, a Sullivan algebra $(\wedge V, d)$ is called minimal if $dV \subset \wedge^{\geq 2} V$. A Sullivan model of (A, d) is given by a Sullivan algebra $(\wedge V, d)$ together with a quasi-isomorphism $f\colon (\wedge V, d) \longrightarrow (A, d)$. It is unique up to isomorphism.

Definition 2. If X is a simply connected space of finite type, then the (minimal) Sullivan model of X is the (minimal) Sullivan model of cdga $A_{\text{PL}}(X)$ of polynomial differential forms on X [1, 5]. A simply connected topological space X is called formal if there exists a quasi-isomorphism $(\wedge V, d) \longrightarrow H^*(X, \mathbb{Q})$, where $(\wedge V, d)$ is a Sullivan model of X. Formal spaces include homogeneous spaces G/H, where G and H have the same rank.

The complex Grassmann manifold $\text{Gr}(k, n)$ is the space of k-dimensional subspaces of $\mathbb{C}^n$. Moreover, $G(k, n) \simeq U(n)/(U(k) \times U(n-k))$, where $U(n)$ is the unitary group. Hence, $G(k, n)$ is formal (see also [11, 12]). As $G(k, n) \cong G(n-k, n)$, we will assume that $k \leq n/2$. As $G(k, n)$ is a formal, its Sullivan model can be computed from its cohomology algebra. Precisely,

$$H^*(\text{Gr}(k, n)) = \frac{\wedge(x_2, x_4, \ldots, x_{2k})}{(h_{n-k+1}, \ldots, h_{n-1}, h_n)}, \tag{3}$$

where h_j is the polynomial of degree $2j$ in the Taylor expansion of the expression $1/(1 + x_2 + \cdots + x_{2k})$ [13]. A Sullivan model is given by

$$(\wedge(x_2, \ldots, x_{2k}, x_{2n-2k+1}, \ldots, x_{2n-1}), d), \tag{4}$$

where $dx_{2i} = 0$ and $dx_{2n-2k+2i-1} = h_{n-k+i}$, $i = 1, \ldots, k$. Moreover, this model is minimal.

Let

$$\begin{aligned} (\wedge V, d) &= (\wedge(x_2, \ldots, x_{2k}, x_{2n+2r-2k+1}, \ldots, x_{2n+2r-1}), d), \\ (\wedge W, d) &= (\wedge(y_2, \ldots, y_{2k}, y_{2n-2k+1}, \ldots, y_{2n-1}), d), \end{aligned} \tag{5}$$

be respective minimal Sullivan models of $\text{Gr}(k, n+r)$ and $\text{Gr}(k, n)$. A Sullivan model of the inclusion $i\colon \text{Gr}(k, n) \longrightarrow \text{Gr}(k, n+r)$ is then

$$\phi\colon (\wedge V, d) \longrightarrow (\wedge W, d), \tag{6}$$

which is defined by

$$\begin{aligned} \phi(x_2) &= y_2, \ldots, \phi(x_{2k}) = y_{2k}, \\ \phi(x_{2n+2r-2k+2i+1}) &= \sum_{j=0}^{k-1} p_{ij} y_{2n-2k+2j+1}, \end{aligned} \tag{7}$$

where p_{ij} is a polynomial of degree $2(r+i-j)$ in $y_2, \ldots, y_{2k}$, for $i, j = 0, 1, 2, \ldots, k-1$, provided that $r + i - j \geq 0$.

The polynomials p_{ij} encode the relationships between h_i's. They can be explicitly expressed from the equality:

$$(1 + x_2 + \cdots + x_{2k})(1 + h_1 + h_2 + \cdots) = 1. \tag{8}$$

For instance, for $k = 2$,

$$\begin{aligned} h_1 &= -x_2, \\ h_2 &= x_2^2 - x_4, \\ h_3 &= -x_2^3 + 2x_2x_4, \\ h_4 &= x_2^4 - 3x_2^2x_4 + x_4^2, \\ h_5 &= -x_2h_4 - x_4h_3, \\ h_6 &= (x_2^2 - x_4)h_4 + x_2x_4h_3, \\ h_7 &= h_4h_3 + (-x_2^2x_4 + x_4^2)h_3. \end{aligned} \tag{9}$$

Example 1. The inclusion $\text{Gr}(2, 4) \longrightarrow \text{Gr}(2, 7)$ has a Sullivan model:

$$\phi\colon (\wedge(x_2, x_4, x_{11}, x_{13}), d) \longrightarrow (\wedge(y_2, y_4, y_5, y_7), d), \tag{10}$$

where

$$\begin{aligned} dx_2 &= dx_4 = 0, \\ dx_{11} &= (x_2^2 - x_4)h_4 + x_2x_4h_3, \\ dx_{13} &= h_4h_3 + (-x_2^2x_4 + x_4^2)h_3, \\ dy_2 &= dy_4 = 0, \\ dy_5 &= -y_2^3 + 2y_2y_4, \\ dy_7 &= y_2^4 - 3y_2^2y_4 + y_4^2, \\ \phi(x_2) &= y_2, \\ \phi(x_4) &= y_4, \\ \phi(x_{11}) &= y_2y_4y_5 + (y_2^2 - y_4)y_7, \\ \phi(x_{13}) &= (-y_2^2y_4 + y_4^2)y_5 + (-y_2^3 + 2y_2y_4)y_7. \end{aligned} \tag{11}$$

We note that $-y_2^2y_4 + y_4^2 = d(y_2y_5 + y_7)$; therefore,

$$\phi(x_{13}) = d(y_2y_5 + y_7)y_5 + d(y_5)y_7. \tag{12}$$

Recall that if $\phi\colon (A, d_A) \longrightarrow (B, d_B)$ is a map of chain complexes; the mapping cone of ϕ, denoted by $\text{Rel}(\phi)$, is defined by

$$\text{Rel}(\phi)_* = (sA_{*-1} \oplus B_*, D), \tag{13}$$

where the differential is defined by $D(sa, b) = (-sd_A(a), \phi(a) + d_Bb)$ [9] or p. 46 in [14]. Define chain maps $J\colon B_n \longrightarrow \text{Rel}_n(\phi)$ and $P\colon \text{Rel}_n(\phi) \longrightarrow A_{n-1}$ by $J(b) = (0, b)$ and $P(sa, b) = a$. There is an exact sequence of chain complexes:

$$0 \longrightarrow B_* \xrightarrow{J} \mathrm{Rel}_*(\phi) \xrightarrow{P} A_{*-1} \longrightarrow 0, \quad (14)$$

which induces a long exact sequence:

$$\longrightarrow H_n(B) \xrightarrow{H_n(J)} H_n(\mathrm{Rel}(\phi)) \xrightarrow{H_n(P)} H_{n-1}(A) \xrightarrow{H_{n-1}(\phi)} H_{n-1}(B) \longrightarrow, \quad (15)$$

(see Proposition 4.3 in [14]).

Definition 3. Let $\phi: (A,d) \longrightarrow (B,d)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping $\theta: A^n \longrightarrow B^{n-k}$ such that $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b)$. We denote by $\mathrm{Der}_n(A,B;\phi)$ the vector space of ϕ-derivations of degree n and by $\mathrm{Der}(A,B;\phi) = \oplus_n \mathrm{Der}_n(A,B;\phi)$ the $\mathbb{Z}$-graded vector space of all ϕ-derivations. The differential on $\mathrm{Der}(A,B;\phi)$ is defined by $\delta\theta = d\theta - (-1)^k \theta\, d$. We will restrict to derivations of positive degree; however, in degree one, we only consider those derivations which are cycles.

If $\phi: A \longrightarrow A$ is the identity mapping, we simply write $\mathrm{Der}A$ for $\mathrm{Der}(A,A;1_A)$. Moreover, if $A = \wedge V$, where $\{v_1, v_2, \ldots\}$ is a basis of V and $\phi: (\wedge V, d) \longrightarrow (B,d)$ is a morphism of cdga's, we denote by (v_i, b) the unique ϕ-derivation θ such that $\theta(v_i) = b$ and zero on other elements of the basis.

Define the Gottlieb group of $(\wedge V, d)$:

$$G_n(\wedge V) = \{[\theta] \in H_n(\mathrm{Der}\wedge V, \delta): \theta(v) = 1, v \in V^n\}. \quad (16)$$

Hence, $G_*(\wedge V) \cong \mathrm{im}\, H_*(\epsilon_*)$, where $\epsilon_*: \mathrm{Der}\wedge V \longrightarrow \mathrm{Der}(\wedge V, \mathbb{Q}; \epsilon)$ is the postcomposition with the augmentation map $\epsilon: \wedge V \longrightarrow \mathbb{Q}$. If X is simply connected and $(\wedge V, d)$ is the minimal Sullivan model of X, then $G_n(\wedge V) \cong G_n(X_{\mathbb{Q}})$, where $h: X \longrightarrow X_{\mathbb{Q}}$ is the rationalization (Propostion 29.8 in[1]).

Similarly, if $\phi: (\wedge V, d) \longrightarrow (\wedge W, d)$ is a map between Sullivan algebras, then the Gottlieb group $G_*(\wedge V, \wedge W; \phi)$ is defined as $H_*(\epsilon_*)$, where $\epsilon_*: \mathrm{Der}(\wedge V, \wedge W; \phi) \longrightarrow \mathrm{Der}(\wedge V, \mathbb{Q}; \epsilon)$ is the postcomposition with $\epsilon: (\wedge W, d) \longrightarrow \mathbb{Q}$. Moreover, if ϕ is a Sullivan model of a map $f: X \longrightarrow Y$, where Y is finite, then $G_*(\wedge V, \wedge W; \phi) \cong G_n(Y_{\mathbb{Q}}, X; h \circ f)$, where $h: Y \longrightarrow Y_{\mathbb{Q}}$ is the rationalization map.

Let $\phi: (\wedge V, d) \longrightarrow (\wedge W, d)$ be a Sullivan model of a map $f: X \longrightarrow Y$ between simply connected spaces. It induces a chain map $\phi^*: \mathrm{Der}(\wedge W) \longrightarrow \mathrm{Der}(\wedge V, \wedge W; \phi)$ by precomposition by ϕ. We get the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Der}\wedge W & \xrightarrow{\phi^*} & \mathrm{Der}(\wedge V, \wedge W; \phi) & \xrightarrow{J} & \mathrm{Rel}(\phi^*) \\ \downarrow \epsilon_* & & \downarrow \epsilon_* & & \downarrow (\epsilon_*; \epsilon_*) \\ \mathrm{Der}(\wedge W, \mathbb{Q}; \epsilon) & \xrightarrow{\widehat{\phi^*}} & \mathrm{Der}(\wedge V, \mathbb{Q}; \epsilon) & \xrightarrow{\widehat{J}} & \mathrm{Rel}\left(\widehat{\phi^*}\right) \end{array}. \quad (17)$$

Then, rational evaluation subgroups are corresponding images in the lower ladder induced in homology by vertical maps. Therefore, there is a long sequence:

$$\cdots \longrightarrow G_n(\wedge W) \xrightarrow{H(\widehat{\phi^*})} G_n(\wedge V, \wedge W; \phi) \xrightarrow{H(\widehat{J})} G_n^{\mathrm{rel}}(\wedge V, \wedge W; \phi) \xrightarrow{H(\widehat{P})} \cdots. \quad (18)$$

We will use the following result for our computations (Theorem 2.1 in [9] or Corollary 1 in [15]).

Theorem 1 (see [9]). *Let $f: X \longrightarrow Y$ be a map between simply connected CW complexes, where X is of finite type and $\phi: (\wedge V, d) \longrightarrow (\wedge W, d)$ its Sullivan model. The long exact sequence induced by the map $f_*: map(X, X; 1_X) \longrightarrow map(X, Y; f)$ on rational homotopy groups is equivalent to the long exact sequence of*

$$\phi^*: \mathrm{Der}(\wedge W, d) \longrightarrow \mathrm{Der}(\wedge V, \wedge W; \phi). \quad (19)$$

We consider the particular case, where f is the inclusion $i: \mathrm{Gr}(k,n) \longrightarrow \mathrm{Gr}(k, n+r)$, where $r \geq 1$ and its Sullivan model $\phi: (\wedge V, d) \longrightarrow (\wedge W, d)$ as given in equation (6).

Theorem 2. *Let $\phi: (\wedge V, d) \longrightarrow (\wedge W, d)$ be a Sullivan model of the inclusion $i: Gr(k,n) \longrightarrow Gr(k, n+r)$, where $r \geq k(n-k)$:*

(1) $G_*(\wedge V, \wedge W; \phi) \cong V^{\#}$, the dual of V

(2) $G_*^{rel}(\wedge V, \wedge W; \phi) \cong sG_*(\wedge W) \oplus G_*(\wedge V, \wedge W; \phi) \cong sG_*(\wedge W) \oplus V^{\#}$

Proof

(1) Recall that $\wedge V = \wedge(x_2, \ldots, x_{2k}, x_{2n+2r-2k+1}, \ldots, x_{2n+2r-1})$, $\wedge W = \wedge(y_2, \ldots, y_{2k}, y_{2n-2k+1}, \ldots, y_{2n-1})$, and ϕ: $(\wedge V, d) \longrightarrow (\wedge W, d)$ are defined by $\phi(x_2) = y_2, \ldots, \phi(x_{2k}) = y_{2k}$, $\phi(x_{2n+2r-2k+2i-1}) = \sum_{j=1}^{k} p_{ij} y_{2n-2k+2j-1}$, and p_{ij} is a polynomial of degree $2(r + i - j)$ in $y_2, \ldots, y_{2k}$ and $i \in \{1, \ldots, k\}$.

We consider the composition φ: $(\wedge V, d) \longrightarrow^{\phi} (\wedge W, d) \longrightarrow^{p} H^*(\wedge W, d)$. As p is a quasi-isomorphism, then the G-sequence of the inclusion is computed from the long exact sequence induced by the cone of the map:

$$\phi^*: \operatorname{Der}(\wedge W, H^*(\wedge W); p) \longrightarrow \operatorname{Der}(\wedge V, H^*(\wedge W); \varphi). \quad (20)$$

Each of the derivations $x^*_{2n+2r-2k+2i+1} = (x_{2n+2r-2k+2i-1}, 1) \in \operatorname{Der}(\wedge V, H^*(\wedge W); \varphi)$ is a cycle of degree at least $2k + 2r + 2i - 1 > 2k + 2r$ and cannot be boundary as all even degree derivations in $\operatorname{Der}(\wedge V, H^*(\wedge W), \varphi)$ are of degree at most $2k$. Hence, $[x^*_{2n+2r-2k+2i+1}]$ is nonzero in $G_*(\wedge V, H^*(\wedge W); \varphi)$

Consider the derivations $x^*_{2i} = (x_{2i}, 1) \in \operatorname{Der}(\wedge V, H^*(\wedge W, d), \varphi)$, for $i = 1, \ldots, k$. Then,

$$(\delta x^*_{2i})(x_{2n+2r-2k+2j-1}) \in H^{2(n+r-k+j-i)}(\wedge W, d). \quad (21)$$

Moreover, as $1 \le 1, j \le k$, then $j - i \ge -k + 1$. Therefore,

$$\begin{aligned} 2(n + r - k + j - i) &\ge 2(n + r - k - k + 1) \\ &\ge 2(r + 1), \quad \text{as } n \ge 2k. \end{aligned} \quad (22)$$

Therefore, $(\delta x^*_{2i})(x_{2n+2r-2k+2j-1}) \in H^{\ge 2k(n-k)+2} = 0$. Hence, x^*_{2i} is a cycle for $i = 1, \ldots, k$. Moreover, x^*_{2i} cannot be a boundary as all odd degree derivations are of degree at least $2n + 2r - 2k + 1 - 2k(n - k) > 2(n - k) + 1 \ge 2k + 1$. Therefore, $x^*_{2n+2r-2k+2i-1}$ are cycles which cannot be boundaries for degree reasons. Hence, $G_*(\wedge V, H^*(\wedge W, d), \varphi) \cong V^{\#}$.

(2) First, we note that $H_{\text{even}}(\operatorname{Der}(\wedge W, \wedge H^*(\wedge W, d); p)) = 0$, and consequently, $G_{\text{even}}(\wedge W, \wedge H^*(\wedge W, d); p) = 0$ [1, 16]. Moreover, a straightforward calculation shows that

$$G_{\text{odd}}(\wedge W, \wedge H^*(\wedge W, d), p) \cong \langle y^*_{2n-2k+1}, \ldots, y^*_{2n-1} \rangle. \quad (23)$$

We consider the vector space:

$$\operatorname{Rel}(\phi^*) = s\operatorname{Der}(\wedge W, H^*(\wedge W); p) \oplus \operatorname{Der}(\wedge V, \wedge W; \varphi), \quad (24)$$

where the differential is defined by $D(s\alpha, \beta) = (-s\delta\alpha, \phi^*(\alpha) + \delta\beta)$. Consider $W_1^{\#} = \langle y^*_{2n-2k+1}, \ldots, y^*_{2n-1} \rangle$ in $\operatorname{Der}(\wedge W, H^*(\wedge W); p)$. For degree reasons, $\phi^*(W_1^{\#}) = 0$. Therefore, $D(sy^*, 0) = 0$, for $y^* \in W_1^{\#}$. Hence, $sy^*_{2n-2k+1}, \ldots, sy^*_{2n-1}$ represent nonzero homology classes in $G^{\text{rel}}_*(\wedge V, H^*(\wedge W); \varphi)$. We conclude that
$G^{\text{rel}}_*(\wedge V, H^*(\wedge W); \varphi)$-
$= sG_*(\wedge W, H^*(\wedge W); p) \oplus G_*(\wedge V, H^*(\wedge W); \varphi)$. □

Corollary 1. *If $r \ge k(n - k)$, then the rational G-sequence of the inclusion i: $Gr(k, n) \longrightarrow Gr(k, n + r)$ is exact.*

Proof. It comes from the previous lemma that the G-sequence is

$$\begin{aligned} 0 \longrightarrow G_*(\wedge V, H^*(\wedge W); \varphi) &\longrightarrow G_*(\wedge V, H^*(\wedge W); \varphi) \oplus sG_*(\wedge W, H^*(\wedge W); p) \\ &\longrightarrow G_*(\wedge W, H^*(\wedge W); p) \longrightarrow 0, \end{aligned} \quad (25)$$

which is exact. □

3. Inclusion Gr$(k, n) \longrightarrow$ Gr$(k, n + 1)$

In the range $1 \le r < k(n - k)$, the G-sequence of the inclusion $\mathrm{Gr}(k, n) \longrightarrow \mathrm{Gr}(k, n + r)$ is more challenging to characterize, as shown in the following example.

Example 2. Consider the inclusion $\mathrm{Gr}(2, 4) \longrightarrow \mathrm{Gr}(2, 7)$ of which a Sullivan model is given by

$$\phi: A = (\wedge(x_2, x_4, x_{11}, x_{13}), d) \longrightarrow (\wedge(y_2, y_4, y_5, y_7), d) = B, \quad (26)$$

where ϕ is defined in Example 1. We compose with the quasi-isomorphism p: $(B, d) \longrightarrow (H^*(B), 0)$ and consider

$\phi^*\colon \mathrm{Der}(B, H^*(B); p) \longrightarrow \mathrm{Der}(A, H^*(B); \varphi)$, where $\varphi = p \circ \phi$. Moreover, $G_*(B, H^*(B); p) = \langle [y_5^*], [y_7^*] \rangle$, where $y_5^* = (y_5, 1)$ and similarly $y_7^* = (y_7, 1)$. Furthermore, $\delta x_2^* = 0$; hence, $[x_2^*]$ represents a nonzero homology class in $\mathrm{Der}(A, H^*(B); \varphi)$. A simple calculation shows that $\delta x_4^* = (x_{11}, \omega/2)$, where $\omega = [x_2^4]$. Hence,

$$G_*(A, H^*(B); \varphi) \cong \langle [x_2^*], [x_{11}^*], [x_{13}^*] \rangle. \quad (27)$$

Consider

$$\mathrm{Rel}_*(\phi^*) = \left(s\mathrm{Der}(B, H^*(B); p) \oplus \mathrm{Der}(A, H^*(B); \varphi), D\right). \quad (28)$$

Then,

$$D(sy_5^*, 0) = (0, \alpha_5),$$
$$D(sy_7^*, 0) = (0, \alpha_7), \quad (29)$$

where $\alpha_5 = (x_{11}, [y_2 y_4])$ and $\alpha_7 = (x_{11}, [y_2^2 - y_4])$. Therefore, the image of

$$H_*(P)\colon G_*^{\mathrm{rel}}(A, H^*(B), \varphi) \longrightarrow G_{*-1}(B, H^*(B); p), \quad (30)$$

is zero. Hence, the sequence

$$G_6^{\mathrm{rel}}(A, H^*(B); \varphi) \xrightarrow{H_*(P)} G_5(B, H^*(B); p) \xrightarrow{H_5(\phi^*)} G_5(A, H^*(B); \varphi), \quad (31)$$

reduces to

$$0 \longrightarrow \langle [y_5^*] \rangle \longrightarrow 0, \quad (32)$$

which is not exact.

In the same way,

$$G_8^{\mathrm{rel}}(A, H^*(B); \varphi) \xrightarrow{H_*(P)} G_7(B, H^*(B); p) \xrightarrow{H_7(\phi^*)} G_5(A, H^*(B); \varphi) \quad (33)$$

is not exact. Moreover, $H_*(J)\colon G_*(A, H^*(B); \varphi) \longrightarrow G_*^{\mathrm{rel}}(A, H^*(B); \varphi)$ is an isomorphism.

Although the G-sequence of the inclusion $\mathrm{Gr}(k, n) \longrightarrow \mathrm{Gr}(k, n+r)$ might not be exact for some values of $1 \le r < k(n-k)$, we have the following result for $r = 1$.

Theorem 3. *Let $\phi\colon (\wedge V, d) \longrightarrow (\wedge W, d)$ be a Sullivan model of the inclusion $Gr(k, n) \longrightarrow Gr(k, n+1)$:*

(1) $G_*^{rel}(\wedge V, \wedge W; \phi)$ has dimension 1

(2) The G-sequence of the inclusion $Gr(k, n) \longrightarrow Gr(k, n+1)$ is not exact

Proof. Recall from Section 2 that the minimal Sullivan model of $\mathrm{Gr}(k, n)$ is $(\wedge W, d)$, where

$$W = \langle y_2, y_4, \ldots, y_{2k}, y_{2(n-k)+1}, \ldots, y_{2n-1} \rangle,$$
$$\mathrm{d}y_2 = \cdots = \mathrm{d}y_{2k} = 0, \quad (34)$$
$$\mathrm{d}y_{2(n-k+i)-1} = h_{n-k+i}, \quad \text{for } i = 1, \ldots, k.$$

Similarly, a model of $G(k, n+1)$ is $(\wedge V, d)$, where

$$V = \langle x_2, \ldots, x_{2k}, x_{2(n-k)+3}, \ldots, x_{2n+1} \rangle,$$
$$\mathrm{d}x_2 = \cdots = \mathrm{d}x_{2k} = 0, \quad (35)$$
$$\mathrm{d}x_{2(n-k+i)+1} = h_{n-k+i+1}, \quad \text{for } i = 1, \ldots, k.$$

Moreover, a model of the inclusion $i\colon \mathrm{Gr}(k, n) \longrightarrow \mathrm{Gr}(k, n+1)$ is given by $\phi\colon (\wedge V, d) \longrightarrow (\wedge W, d)$ and defined by

$$\phi(x_{2i}) = y_{2i},$$
$$\phi\left(x_{2(n-k)+3}\right) = y_{2(n-k)+3}, \ldots, \phi(x_{2n-1}) = y_{2n-1},$$
$$\phi(x_{2n+1}) = -y_2 y_{2n-1} - y_4 y_{2n-3} - \cdots - y_{2k} y_{2(n-k)+1}. \quad (36)$$

We consider the quasi-isomorphism

$$p\colon (\wedge W, d) \longrightarrow H^*(\wedge W, d) = \frac{\wedge(y_2, \ldots, y_{2k})}{\left(\mathrm{d}y_{2(n-k)+1}, \ldots, \mathrm{d}y_{2n-1}\right)}, \quad (37)$$

and set $\varphi = p \circ \phi$. Consider

$$\phi^*\colon \mathrm{Der}(\wedge W, H^*(\wedge W, d); p) \longrightarrow \mathrm{Der}(\wedge V, H^*(\wedge W); \varphi). \quad (38)$$

We have the following relations:

$$\phi^*(y_{2i}^*) = x_{2i}^*, \quad \text{for } i = 1, \ldots, n$$
$$\phi^*(y_{2n-1}^*) = x_{2n-1}^* - (x_{2n+1}, y_2)$$
$$\phi^*(y_{2n-3}^*) = x_{2n-3}^* - (x_{2n+1}, y_4),$$
$$\cdots \quad (39)$$
$$\phi^*\left(y_{2(n-k)+3}^*\right) = x_{2(n-k)+3}^* - (x_{2n+1}, y_{2k-2})$$
$$\phi^*\left(y_{2(n-k)+1}^*\right) = -(x_{2n+1}, y_{2k}).$$

As a result, in

$$\mathrm{Rel}(\phi^*) = s\mathrm{Der}(\wedge W, H^*(\wedge W, d); p) \oplus \mathrm{Der}(\wedge V, H^*(\wedge W); \varphi), \quad (40)$$

we have the following relations:

$$\begin{aligned} D\left(0, x^*_{2(n-k)+2i+1}\right) &= 0, \quad \text{for } i = 1, \ldots, k \\ D\left(sy^*_{2n-1}, 0\right) &= \left(0, x^*_{2n-1} - \left(x_{2n+1}, y_2\right)\right) \\ D\left(sy^*_{2n-3}, 0\right) &= \left(0, x^*_{2n-3} - \left(x_{2n+1}, y_4\right)\right) \\ &\cdots \\ D\left(sy^*_{2n-2k+3}, 0\right) &= \left(0, x^*_{2n-2k+3} - \left(x_{2n+1}, y_{2k-2}\right)\right) \\ D\left(sy^*_{2n-2k+1}, 0\right) &= \left(0, -\left(x_{2n+1}, y_{2k}\right)\right). \end{aligned} \quad (41)$$

We consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Der}(\wedge W, H^*(\wedge W); p) & \xrightarrow{\phi^*} & \mathrm{Der}(\wedge V, H^*(\wedge W); \varphi) \\ \downarrow \epsilon_* & & \downarrow \epsilon_* \\ \mathrm{Der}(\wedge W, \mathbb{Q}; \epsilon) & \xrightarrow{\widehat{\phi}^*} & \mathrm{Der}(\wedge V, \mathbb{Q}; \epsilon) \end{array} . \quad (42)$$

Let $\widehat{y_i^*} = \epsilon_*(y_i^*)$ and $\widehat{x_j^*} = \epsilon_*(x_j^*)$. Consider

$$\mathrm{Rel}\widehat{\phi}^* = (s\mathrm{Der}(\wedge W, \mathbb{Q}; \epsilon) \oplus \mathrm{Der}(\wedge V, \mathbb{Q}; \epsilon), \widehat{D}). \quad (43)$$

Then,

$$\begin{aligned} \widehat{D}\left(s\widehat{y}^*_{2n-2k+1}, 0\right) &= (0, 0) \\ \widehat{D}\left(s\widehat{y}^*_{2n-2k+3}, 0\right) &= \left(0, \widehat{x}^*_{2n-2k+3}\right) \\ &\cdots \\ \widehat{D}\left(s\widehat{y}^*_{2n-1}, 0\right) &= \left(0, \widehat{x}^*_{2n-1}\right). \end{aligned} \quad (44)$$

Hence,

$$H_*\left(\mathrm{Rel}\widehat{\phi}^*\right) = \langle [(s\widehat{y}^*_{2n-2k+1}, 0)], [(0, \widehat{x}^*_{2n+1})] \rangle. \quad (45)$$

Moreover, the image of $H_*(\epsilon_*, \epsilon_*)$: $H_*(\mathrm{Rel}\phi^*) \longrightarrow H_*(\mathrm{Rel}\widehat{\phi}^*)$ is $\langle [(0, \widehat{x}^*_{2n+1})] \rangle$. Therefore,

$$G^{\mathrm{rel}}_*(\wedge V, H^*(\wedge W); \varphi) = \langle [(0, \widehat{x}^*_{2n+1})] \rangle. \quad (46)$$

This shows the first part of the theorem and corrects Theorem 3 in [17] and Theorem 3 [18].

Moreover,

$$H(\widehat{P})\colon G^{\mathrm{rel}}_*(\wedge V, H^*(\wedge W, d); \varphi) \longrightarrow G_{*-1}(\wedge W, H^*(\wedge W, d); p) \quad (47)$$

is the zero map. The G-sequence then reduces to exact portions

$$G_{2n-2k+2i+1}(\wedge W, H^*(\wedge W); p) \underset{\simeq}{\xrightarrow{H(\widehat{\phi}^*)}} G_{2n-2k+2i+1}(\wedge V, H^*(\wedge W); \varphi), \quad (48)$$

for $i = 1, \ldots, k-1$, and

$$G_{2n+1}(\wedge V, H^*(\wedge W); p) \underset{\simeq}{\xrightarrow{H(\widehat{J})}} G^{\mathrm{rel}}_{2n+1}(\wedge V, H^*(\wedge W); \varphi), \quad (49)$$

and a nonexact part,

$$0 \longrightarrow G_{2n-2k+1}(\wedge W, H^*(\wedge W); p) \longrightarrow 0. \quad (50)$$

□

Example 3. We consider a model of the inclusion $\mathrm{Gr}(2,4) \longrightarrow \mathrm{Gr}(2,5)$ which is of the form

$$\begin{aligned} \phi\colon (\wedge V, d) &= (\wedge(x_2, x_4, x_7, x_9), d) \longrightarrow (\wedge(y_2, y_4, y_5, y_7), d) \\ &= (\wedge W, d), \end{aligned} \quad (51)$$

where $dx_2 = dx_4 = 0$, $dx_7 = x_2^4 - 3x_2^2 x_4 + x_4^2$, $dx_9 = -x_2(x_2^4 - 3x_2^2 x_4 + y_4^2) - x_4(-x_2^3 + 2x_2 x_4)$, $dy_2 = dy_4 = 0$, $dy_5 = -y_2^3 + 2y_2 y_4$, $dy_7 = y_2^4 - 3y_2^2 y_4 + y_4^2$, $\phi(x_2) = y_2$, $\phi(x_4) = y_4$, $\phi(x_7) = y_7$, and $\phi(x_9) = -y_2 y_7 - y_4 y_5$.

We compose which the quasi-isomorphism $p\colon (\wedge W, d) \longrightarrow H^*(\wedge W, d)$ to get $\varphi\colon (\wedge V, d) \longrightarrow H^*(\wedge W, d)$. In

$$\mathrm{Rel}(\phi)_* = s\mathrm{Der}(\wedge W, H^*(\wedge W, d); p) \oplus \mathrm{Der}(\wedge V, H^*(\wedge W, d), \varphi), \quad (52)$$

we have the following relations:

$$D((sy_5^*, 0)) = (0, (x_9, -y_4)),$$
$$D((sy_7^*, 0)) = (0, x_7^* + (x_9, -y_2)). \quad (53)$$

Consider

$$\mathrm{Rel}\widehat{\phi}^* = (s\mathrm{Der}(\wedge W, \mathbb{Q}; \epsilon) \oplus \mathrm{Der}(\wedge V, \mathbb{Q}, \epsilon), D) \cong \left(sW^{\#} \oplus V^{\#}, D\right), \quad (54)$$

where

$$D(s\widehat{y}_5^*, 0) = (0, 0),$$
$$D(s\widehat{y}_7^*, 0) = (0, \widehat{x}_7^*), \quad (55)$$
$$D(0, \widehat{x}_7^*) = D(0, \widehat{x}_9) = (0, 0).$$

Hence,

$$H_*\left(\mathrm{Rel}\widehat{\phi}^*\right) \cong \langle [(s\widehat{y}_5^*, 0)], [(0, \widehat{x}_9^*)] \rangle. \quad (56)$$

However, $\mathrm{im} H(\epsilon_*, \epsilon_*) = \langle [(0, \widehat{x}_9^*)] \rangle$. Therefore, $G_*^{\mathrm{rel}}(\wedge V, H^*(\wedge W, d); \varphi) \cong \langle [(0, \widehat{x}_9^*)] \rangle$. As $G_*(\wedge V, H^*(\wedge W, d); \varphi) \cong \langle [\widehat{x}_7^*], [\widehat{x}_9^*] \rangle$ and $G_*(\wedge W, H^*(\wedge W, d), p) = \langle [\widehat{y}_5^*], [\widehat{y}_7^*] \rangle$, then the G-sequence reduces to exact nonzero fragments:

$$0 \longrightarrow G_9(\wedge V, H^*(\wedge W, d); \varphi) \xrightarrow{\cong} G_9^{\mathrm{rel}}(\wedge V, H^*(\wedge W, d); \varphi) \longrightarrow 0, 0 \longrightarrow G_7(\wedge W, H^*(\wedge W, d); p) \xrightarrow{\cong} G_7(\wedge V, H^*(\wedge W, d); \varphi) \longrightarrow 0, \quad (57)$$

and a nonexact sequence,

$$0 \longrightarrow G_5(\wedge W, H^*(\wedge W, d); p) \longrightarrow 0. \quad (58)$$

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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