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*Research Article*

# On a Characterization of Convergence in Banach Spaces with a Schauder Basis

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We extend the well-known characterizations of convergence in the spaces $l_p$ $(1 \le p < \infty)$ of $p$-summable sequences and $c_0$ of vanishing sequences to a general characterization of convergence in a Banach space with a Schauder basis and obtain as instant corollaries characterizations of convergence in an infinite-dimensional separable Hilbert space and the space $c$ of convergent sequences.

*"The method in the present paper is abstract and is phrased in terms of Banach spaces, linear operators, and so on. This has the advantage of greater simplicity in proof and greater generality in applications."*
Jacob T. Schwartz

## 1. Introduction

In normed vector spaces of sequences, termwise convergence, being a necessary condition for convergence of a sequence (of sequences), falls short of being characteristic (see, e.g., [1]). Thus, the natural question is as follows: what conditions are required to be, along with termwise convergence, necessary and sufficient for convergence of a sequence in such spaces?

It turns out that, in the Banach spaces $l_p$ $(1 \le p < \infty)$ of $p$-summable sequences with $p$-norm,

$$x := (x_k)_{k\in\mathbb{N}} \mapsto \|x\|_p := \left[\sum_{k=1}^{\infty} |x_k|^p\right]^{1/p}, \tag{1}$$

($\mathbb{N} := \{1, 2, \ldots\}$ is the set of natural numbers) and $c_0$ of vanishing sequences with $\infty$-norm,

$$x := (x_k)_{k\in\mathbb{N}} \mapsto \|x\|_\infty := \sup_{k\in\mathbb{N}} |x_k|, \tag{2}$$

only one additional condition is needed. The following characterizations of convergence in the foregoing spaces are well known.

**Proposition 1** (characterization of convergence in $l_p$ $(1 \le p < \infty)$). *In the (real or complex) space $l_p$ $(1 \le p < \infty)$,*

$$\left(x_k^{(n)}\right)_{k\in\mathbb{N}} =: x^{(n)} \longrightarrow x := (x_k)_{k\in\mathbb{N}}, \quad n \longrightarrow \infty, \tag{3}$$

*iff*

*(1) $\forall k \in \mathbb{N}$: $x_k^{(n)} \longrightarrow x_k$, $n \longrightarrow \infty$,*
*(2) $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n \in \mathbb{N}$: $\sum_{k=K+1}^{\infty} |x_k^{(n)}|^p < \varepsilon$.*

See, e.g., Proposition 2.16 in [2] and Proposition 2.17 in [1].

*Remarks 1*

(i) Condition (1) is termwise convergence.
(ii) Condition (2) signifies the uniform convergence of the series,

$$\sum_{k=1}^{\infty} \left|x_k^{(n)}\right|^p, \tag{4}$$

to their respective sums over $n \in \mathbb{N}$.

**Proposition 2** (characterization of convergence in $c_0$). *In the (real or complex) space $c_0$,*

$$\left(x_k^{(n)}\right)_{k\in\mathbb{N}} =: x^{(n)} \longrightarrow x := \left(x_k\right)_{k\in\mathbb{N}}, \quad n \longrightarrow \infty, \tag{5}$$

*iff*

*(1) $\forall k \in \mathbb{N}$: $x_k^{(n)} \longrightarrow x_k$, $n \longrightarrow \infty$,*

*(2) $\forall \varepsilon > 0\, \exists K \in \mathbb{N}\, \forall n \in \mathbb{N}$: $\sup_{k\geq K+1}|x_k^{(n)}| < \varepsilon$.*

See, e.g., Proposition 2.15 in [2] and Proposition 2.16 in [1].

*Remarks 2*

(i) Condition (1) is termwise convergence.

(ii) Condition (2) signifies the uniform convergence of the sequences $(x_k^{(n)})_{k\in\mathbb{N}}$ to 0 over $n \in \mathbb{N}$.

One cannot but notice that both characterizations share the same condition (1) and that condition (2) in each can be reformulated in the following equivalent form:

(2C) $\forall \varepsilon > 0\, \exists K_0 \in \mathbb{N}\, \forall K \geq K_0\, \forall n \in \mathbb{N}$: $\|R_K x^{(n)}\| < \varepsilon$,

where $\|\cdot\|$ stands for $p$-norm $\|\cdot\|_p$ $(1 \leq p < \infty)$ or $\infty$-norm, respectively, and the mapping $R_K: X \longrightarrow X$, $K \in \mathbb{N}$, ($X := l_p$ $(1 \leq p < \infty)$ or $X := c_0$) is defined as follows:

$$x := \left(x_k\right)_{k\in\mathbb{N}} \mapsto R_K x := \Bigg(\underbrace{0,\ldots,0}_{K \text{ terms}}, x_{K+1}, x_{K+2}, \ldots\Bigg), \quad K \in \mathbb{N}. \tag{6}$$

Thus, we have the following combined characterization encompassing both $l_p$ $(1 \leq p < \infty)$ and $c_0$.

**Proposition 3** (combined characterization of convergence). *In the (real or complex) space $X := l_p$ $(1 \leq p < \infty)$ or $X := c_0$,*

$$\left(x_k^{(n)}\right)_{k\in\mathbb{N}} =: x^{(n)} \longrightarrow x := \left(x_k\right)_{k\in\mathbb{N}}, \quad n \longrightarrow \infty, \tag{7}$$

*iff*

*(1) $\forall k \in \mathbb{N}$: $x_k^{(n)} \longrightarrow x_k$, $n \longrightarrow \infty$,*

*(2C) $\forall \varepsilon > 0\, \exists K_0 \in \mathbb{N}\, \forall K \geq K_0\, \forall n \in \mathbb{N}$: $\|R_K x^{(n)}\| < \varepsilon$,*

*where $\|\cdot\|$ stands for $p$-norm $\|\cdot\|_p$ $(1 \leq p < \infty)$ or $\infty$-norm, respectively, and the mapping $R_K: X \longrightarrow X$, $K \in \mathbb{N}$, is defined by (6).*

In view of the fact that both $l_p$ $(1 \leq p < \infty)$ and $c_0$ are Banach spaces with a Schauder basis, our goal to show that a two-condition characterization of convergence, similar to the foregoing combined characterization, holds for all such spaces appears to be amply motivated. We establish a general characterization of convergence in a Banach space with a Schauder basis and obtain as instant corollaries characterizations of convergence in an infinite-dimensional separable Hilbert space and the Banach space $c$ of convergent sequences.

## 2. Preliminaries

Here, we briefly outline certain preliminaries essential for our discourse.

*Definition 1* (Schauder basis). A Schauder basis (also a countable basis) of a (real or complex) Banach space $(X, \|\cdot\|)$ is a countably infinite set $\{e_n\}_{n\in\mathbb{N}}$ in $X$ such that

$$\forall x \in X\, \exists!\, \left(c_k(x)\right)_{k\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}}: x = \sum_{k=1}^{\infty} c_k(x) e_k, \tag{8}$$

($\mathbb{F} := \mathbb{R}$ or $\mathbb{F} := \mathbb{C}$) the series called the Schauder expansion of $x$ and the numbers $c_k(x) \in \mathbb{F}$, $k \in \mathbb{N}$, the coordinates of $x$ relative to $\{e_n\}_{n\in\mathbb{N}}$.

See, e.g., [1–4].

For an infinite-dimensional separable Hilbert space $(X, (\cdot,\cdot), \|\cdot\|)$ ($(\cdot,\cdot)$ stands for inner product and $\|\cdot\|$ for inner product norm), an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ is a Schauder basis, and for an arbitrary $x \in X$,

$$x = \sum_{k=1}^{\infty} c_k(x) e_k, \quad \text{with } c_k(x) = \left(x, e_k\right), k \in \mathbb{N}, \tag{9}$$

(see, e.g., [1, 2]).

As we mention above, the sequence spaces $l_p$ $(1 \leq p < \infty)$, $c_0$, and $c$ are examples of Banach spaces with a Schauder basis. For $l_p$ $(1 \leq p < \infty)$ and $c_0$, the standard Schauder basis is the set

$$\left\{e_n := \left(\delta_{nk}\right)_{k\,\in\,\mathbb{N}}\right\}_{n\in\mathbb{N}} \tag{10}$$

($\delta_{nk}$ is the Kronecker delta) and for an arbitrary $x := (x_k)_{k\in\mathbb{N}}$ in the foregoing spaces,

$$x = \sum_{k=1}^{\infty} c_k(x) e_k, \quad \text{with } c_k(x) = x_k, k \in \mathbb{N}, \tag{11}$$

(see, e.g., [1–4]).

For the Banach space $c$ of convergent sequences equipped with $\infty$-norm (see (2)), the standard Schauder basis is $\{e_n\}_{n\in\mathbb{Z}_+}$ ($\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the set of nonnegative integers) with

$$e_0 := (1, 1, 1, \ldots), \tag{12}$$

and for an arbitrary $x := (x_k)_{k\in\mathbb{N}} \in c$,

$$\begin{aligned} x = \sum_{k=0}^{\infty} c_k(x) e_k, \quad &\text{with } c_0(x) = \lim_{m\longrightarrow\infty} x_m, c_k(x) \\ &= x_k - c_0(x), k \in \mathbb{N}, \end{aligned} \tag{13}$$

see, e.g., [1–4].

Banach spaces with more sophisticated Schauder bases encompass $L_p(a, b)$ $(1 \leq p < \infty)$ and $C[a, b]$ $(-\infty < a < b < \infty)$ with $\infty$-norm,

$$C[a, b] \ni x \mapsto \|x\|_\infty := \max_{a\leq t\leq b} |x(t)|, \tag{14}$$

(see, e.g., [3, 4]).

A Banach space with a Schauder basis is infinite-dimensional and separable (see, e.g., [1–3]). However, an infinite-dimensional separable Banach space need not have a Schauder basis (see [5]).

The set of $\mathbb{F}$-termed sequences,

$$Y := \left\{ y := (c_k)_{k\in\mathbb{N}} \in \mathbb{F}^{\mathbb{N}} \,\middle|\, \sum_{k=1}^{\infty} c_k e_k \text{ converges in } X \right\}, \quad (15)$$

with termwise linear operations and the norm,

$$Y \ni y := (c_k)_{k\in\mathbb{N}} \mapsto \|y\|_Y := \sup_{n\in\mathbb{N}} \left\| \sum_{k=1}^{n} c_k e_k \right\|, \quad (16)$$

is a Banach space and the linear operator,

$$Y \ni y := (c_k)_{k\in\mathbb{N}} \mapsto Ay := \sum_{k=1}^{\infty} c_k e_k \in X, \quad (17)$$

is subject to the inverse mapping theorem (see, e.g., [1–3, 6]). The boundedness of the inverse operator $A^{-1}: X \longrightarrow Y$ implies boundedness, and hence, continuity, for the linear Schauder coordinate functionals,

$$X \ni x = \sum_{k=1}^{\infty} c_k(x) e_k \mapsto c_n(x) \in \mathbb{F}, \quad n \in \mathbb{N}, \quad (18)$$

with

$$\|c_n\| \le \frac{2\|A^{-1}\|}{\|e_n\|}, \quad n \in \mathbb{N}, \quad (19)$$

(see, e.g., [1–3]) as well as for the linear operators:

$$\begin{aligned} X \ni x = \sum_{k=1}^{\infty} c_k(x) e_k &\mapsto S_n x := \sum_{k=1}^{n} c_k(x) e_k, \\ R_n x &:= \sum_{k=n+1}^{\infty} c_k(x) e_k, \quad n \in \mathbb{N}, \end{aligned} \quad (20)$$

with

$$I = S_n + R_n, \quad n \in \mathbb{N}, \quad (21)$$

($I$ is the identity operator on $X$) and

$$\|S_n\| \le \|A^{-1}\|, \|R_n\| \le 2\|A^{-1}\|, \quad n \in \mathbb{N}, \quad (22)$$

(see, e.g., [3]).

*Remark 3.* Here and henceforth, we use the notation $\|\cdot\|$ for the operator norm.

## 3. General Characterization

The following statement appears to be a perfect illustration of the profound observation by Schwartz found in [7] and chosen as the epigraph.

**Theorem 1** (general characterization of convergence). *Let $(X, \|\cdot\|)$ be a (real or complex) Banach space with a Schauder basis $\{e_n\}_{n\in\mathbb{N}}$ and corresponding coordinate functionals $c_n(\cdot)$, $n \in \mathbb{N}$.*

*For a sequence $(x_n)_{n\in\mathbb{N}}$ and a vector $x$ in $X$,*

$$x_n \longrightarrow x, \quad n \longrightarrow \infty, \quad (23)$$

*iff*

*(1) $\forall k \in \mathbb{N}$: $c_k(x_n) \longrightarrow c_k(x)$, $n \longrightarrow \infty$,*

*(2) $\forall \varepsilon > 0\ \exists K_0 \in \mathbb{N}\ \forall K \ge K_0\ \forall n \in \mathbb{N}$: $\|R_K x_n\| < \varepsilon$.*

*Proof.* "*Only if*" part.

Suppose that, for a sequence $(x_n)_{n\in\mathbb{N}}$ and a vector $x$ in $X$,

$$x_n \longrightarrow x, \quad n \longrightarrow \infty. \quad (24)$$

Then, by the continuity of the Schauder coordinate functionals $c_n(\cdot)$, $n \in \mathbb{N}$, we infer that condition (1) holds.

Let $\varepsilon > 0$ be arbitrary. Then,

$$\exists N \in \mathbb{N}\ \forall n \ge N: \|x_n - x\| < \frac{\varepsilon}{4\|A^{-1}\|}. \quad (25)$$

Since $x \in X$,

$$R_K x := \sum_{k=K+1}^{\infty} c_k(x) e_k \longrightarrow 0, \quad K \longrightarrow \infty, \quad (26)$$

and hence,

$$\exists K_0 \in \mathbb{N}\ \forall K \ge K_0: \|R_K x\| < \frac{\varepsilon}{2}. \quad (27)$$

In view of (22), (25), and (27), we have

$$\begin{aligned} \forall K \ge K_0, \forall n \ge N: \|R_K x_n\| &= \|R_K x_n - R_K x + R_K x\| \\ &\le \|R_K(x_n - x)\| + \|R_K x\| \\ &\le \|R_K\|\|x_n - x\| + \|R_K x\| \\ &< 2\|A^{-1}\| \frac{\varepsilon}{4\|A^{-1}\|} + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (28)$$

Furthermore, since $x_n \in X$, $n = 1, \ldots, N-1$, we can regard $K_0 \in \mathbb{N}$ in (27) to be large enough so that

$$\forall K \ge K_0, \forall n = 1, \ldots, N-1: \|R_K x_n\| = \left\| \sum_{k=K+1}^{\infty} c_k(x_n) e_k \right\| < \varepsilon. \quad (29)$$

Thus, condition (2) holds as well.

This completes the proof of the "only if" part.

"*If*" part. Suppose that, for a sequence $(x_n)_{n\in\mathbb{N}}$ and a vector $x$ in $X$, conditions (1) and (2) are met.

For an arbitrary $\varepsilon > 0$ and $K_0 \in \mathbb{N}$, from condition (2), by condition (1),

$$\exists N \in \mathbb{N}\ \forall n \ge N: \|S_{K_0}(x_n - x)\| \le \sum_{k=1}^{K_0} |c_k(x_n) - c_k(x)| \|e_k\| < \frac{\varepsilon}{3}. \quad (30)$$

Since $x \in X$, we can also regard that $K_0 \in \mathbb{N}$ in condition (2) to be large enough so that

$$\|R_{K_0} x\| < \frac{\varepsilon}{3}. \quad (31)$$

Then, in view of (21), (30), and (31) and by condition (2),

$$\begin{aligned}\forall n \ge N:\ \|x_n - x\| &= \|S_{K_0}(x_n - x) + R_{K_0}(x_n - x)\| \\ &\le \|S_{K_0}(x_n - x)\| + \|R_{K_0}x_n\| + \|R_{K_0}x\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned} \tag{32}$$

This concludes the proof of the "if" part and the entire statement. □

*Remarks 4*

(i) Condition (1) is the convergence of the coordinates of $x_n$ to the corresponding coordinates of $x$ relative to $\{e_k\}_{k\in\mathbb{N}}$.

(ii) Condition (2) signifies the uniform convergence of the Schauder expansions,

$$\sum_{k=1}^{\infty} c_k(x_n)e_k, \tag{33}$$

of $x_n$ relative to $\{e_k\}_{k\in\mathbb{N}}$ over $n \in \mathbb{N}$.

Now, the combined characterization of convergence (Proposition 3) is an instant corollary of the foregoing general characterization.

## 4. Characterization of Convergence in an Infinite-Dimensional Separable Hilbert Space

For an infinite-dimensional separable Hilbert space $(X, (\cdot,\cdot), \|\cdot\|)$ relative to an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$, in view of (9), the general characterization of convergence (Theorem 1) acquires the following form.

**Corollary 1** (characterization of convergence in a separable Hilbert space). *Let* $(X, (\cdot,\cdot), \|\cdot\|)$ *be a (real or complex) infinite-dimensional separable Hilbert space with an orthonormal basis* $\{e_n\}_{n\in\mathbb{N}}$*.*

*For a sequence* $(x_n)_{n\in\mathbb{N}}$ *and a vector* $x$ *in* $X$,

$$x_n \longrightarrow x, \quad n \longrightarrow \infty, \tag{34}$$

*iff*

*(1)* $\forall k \in \mathbb{N}$: $(x_n, e_k) \longrightarrow (x, e_k)$, $n \longrightarrow \infty$,

*(2)* $\forall \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall n \in \mathbb{N}$: $\sum_{k=K+1}^{\infty} |(x_n, e_k)|^2 < \varepsilon$.

*Remarks 5*

(i) Condition (1) is the convergence of the Fourier coefficients of $x_n$ to the corresponding Fourier coefficients of $x$ relative to $\{e_k\}_{k\in\mathbb{N}}$.

(ii) Condition (2) signifies the uniform convergence of the Fourier series expansions,

$$\sum_{k=1}^{\infty} (x_n, e_k)e_k, \tag{35}$$

of $x_n$ relative to $\{e_k\}_{k\in\mathbb{N}}$ over $n \in \mathbb{N}$.

(iii) The characterization of convergence in $l_p$ (Proposition 1) for $p = 2$ is now a particular case of the prior characterization.

## 5. Characterization of Convergence in $c$

Another immediate corollary of the general characterization of convergence (Theorem 1) is the realization of the latter in the space $c$ of convergent sequences equipped with $\infty$-norm (see (2)) relative to the standard Schauder basis $\{e_n\}_{n\in\mathbb{Z}_+}$ (see Section 2).

Indeed, in $c$ relative to $\{e_n\}_{n\in\mathbb{Z}_+}$, for an arbitrary $x := (x_k)_{k\in\mathbb{N}}$,

$$\begin{aligned}x = \sum_{k=0}^{\infty} c_k(x)e_k, \quad &\text{with } c_0(x) = \lim_{m\longrightarrow\infty} x_m, c_k(x) \\ &= x_k - c_0(x), k \in \mathbb{N},\end{aligned} \tag{36}$$

(see (13)) and

$$\begin{aligned}S_K x &:= \sum_{k=0}^{K} c_k(x)e_k = \left(\lim_{m\longrightarrow\infty} x_m, x_1 - \lim_{m\longrightarrow\infty} x_m, \ldots, x_K - \lim_{m\longrightarrow\infty} x_m, 0, \ldots\right), \\ R_K x &:= \sum_{k=K+1}^{\infty} c_k(x)e_k = \left(\underbrace{0, \ldots, 0}_{K+1 \text{ terms}}, x_{K+1} - \lim_{m\longrightarrow\infty} x_m, \ldots\right), \quad K \in \mathbb{Z}_+,\end{aligned} \tag{37}$$

(cf. (20)).

Thus, the general characterization of convergence (Theorem 1), in view of the obvious circumstance that, for any $x := (x_k)_{k\in\mathbb{N}} \in c$, the sequence,

$$\|R_K x\| = \sup_{k\ge K+1} \left| x_k^{(n)} - \lim_{m\longrightarrow\infty} x_m^{(n)} \right|, \quad K \in \mathbb{Z}_+, \tag{38}$$

is decreasing, acquires the following form.

**Corollary 2** (characterization of convergence in $c$). *In the (real or complex) space* $c$,

$$\left(x_k^{(n)}\right)_{k\in\mathbb{N}} =: x^{(n)} \longrightarrow x := (x_k)_{k\in\mathbb{N}}, \quad n \longrightarrow \infty, \tag{39}$$

*iff*

(1) $\lim_{m\longrightarrow\infty} x_m^{(n)} \longrightarrow \lim_{m\longrightarrow\infty} x_m, \quad n \longrightarrow \infty,$ *and* $\forall k \in \mathbb{N}$: $x_k^{(n)} \longrightarrow x_k$, $n \longrightarrow \infty$,

(2) $\forall \varepsilon > 0 \exists K \in \mathbb{Z}_+ \forall n \in \mathbb{N}: \sup_{k\geq K+1} |x_k^{(n)} - \lim_{m\longrightarrow\infty} x_m^{(n)}| < \varepsilon$.

*Remarks 6*

(i) Condition (1), beyond termwise convergence, includes convergence of the limits.

(ii) Condition (2) signifies the uniform convergence of the sequences $(x_k^{(n)})_{k\in\mathbb{N}}$ to their respective limits over $n \in \mathbb{N}$.

(iii) The characterization of convergence in $c_0$ (Proposition 2) is a mere restriction of the prior characterization to the subspace $c_0$ of $c$.

## 6. Concluding Remark

As is easily seen, the general characterization of convergence (Theorem 1) is consistent with the following characterization of compactness, which underlies the results of [8].

**Theorem 2** (characterization of compactness, Theorem III.7.4 in [3]). *In a (real or complex) Banach space* $(X, \|\cdot\|)$ *with a Schauder basis, a set C is precompact (a closed set C is compact) iff*

(1) *C is bounded,*

(2) $\forall \varepsilon > 0\, \exists K_0 \in \mathbb{N}\, \forall K \geq K_0\, \forall x \in C$: $\|R_K x\| < \varepsilon$.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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