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Research Article

Controllability of a Family of Nonlinear Population Dynamics Models

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Considering a nonlinear dynamical system, we study the nonlinear infinite-dimensional system obtained by grafting an operator $\mathbf{A}$ and an age structure. This system is such that the nonlinearity is at the level of births. We show that there is a time T dependent on the constraints on the age and the observability minimal time T_0 of the pair $(\mathbf{A}, \mathbf{B})$ ($\mathbf{B}$ is the control operator), from which the system is null controllable. We first establish an observability inequality useful for the proof of the null controllability of an auxiliary system. We also apply Schauder's fixed point in the proof of the null controllability of the nonlinear system..

1. Introduction and Main Results

In this paper, we study the null controllability of an infinite-dimensional nonlinear system describing the dynamics of age-structured population.

Let V, H, and U be a four separable Hilbert space. We identify H and U with their duals. We suppose V is dense in H and that the following inclusions are checked:

$$V \subseteq H \subseteq V', \tag{1}$$

where the inclusion of V in H is continuous.

Now, we consider the operator $\mathbf{A}$ is defined by

$$\mathbf{A}\colon V \longrightarrow V', \tag{2}$$

where V is the Hilbert space and $V \subseteq H \subseteq V'$, for example, $H^1(\Omega) \subseteq L^2(\Omega) \subseteq H^{-1}(\Omega)$.

And, we suppose that

$$\begin{cases} \mathbf{A}\colon V \longrightarrow V', \\ \forall \phi, \psi \in V \longrightarrow \langle \mathbf{A}\phi, \psi \rangle_{V',V}, \end{cases} \tag{3}$$

is measurable. Let y be a solution of the following system:

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \mathbf{A}y + \mu_0 y = \chi_{(a_1,a_2)}\mathbf{B}u, & \text{in } (0, A) \times (0, T), \\ y(0,t) = \displaystyle\int_0^A \beta(a) y da \Phi\left(\int_0^A \beta(a) y da\right), & \text{in } (0, T), \\ y(a,0) = y_0(a), & \text{in } (0, A), \end{cases} \tag{4}$$

where $\Phi \in L^\infty(\mathbb{R})$, and y_0 is given in

$$K = L^2(0, A; H) \text{ and } B \in \mathscr{L}\left(L^2(0, T; \mathrm{U}), L^2(0, T; H)\right) \text{ a bounded operator.} \tag{5}$$

Moreover, the positive function μ_0 denotes the natural mortality rate of individuals of age a, supposed to be independent of the times t.

The control function is u, depending on t and a, where m is the characteristic function.

We denote by β the positive function describing the fertility rate that depends only on the age. The birth law ensures the survival of the species. In several works, one denotes $\int_0^A \beta y da$, the number of newborn individuals at time t. If we consider an oviparous species, that is to say whose birth process goes through egg laying, we see that $\int_0^A \beta y da$ is the number of eggs laid at time t. Since all the eggs do not reach maturity, we introduce a function giving the proportion of eggs who will arrive there. Let Φ this function. Then, we have

$$y(0,t) = \int_0^A \beta y da \Phi\left(\int_0^A \beta y da\right), \tag{6}$$

the distribution of newborn individuals at time t.

We assume that the fertility rate β and the mortality rate μ_0 satisfy the demographic property:

$$\begin{aligned}
(H_1) &= \begin{cases} (\mathrm{i}) - \mu_0(a) \geq 0, & \text{for every } a \in (0, A), \\ (\mathrm{ii}) - \mu_0 \in L^1(0, A^*), & \text{for every } A^* \in [0, A), \\ (\mathrm{iii}) - \displaystyle\int_0^A \mu(a) da = +\infty, \end{cases} \\
(H_2) &= \begin{cases} (\mathrm{i}) - \beta \in L^\infty([0, A]), \\ (\mathrm{ii}) - \beta(a) = 0, \quad \forall a \in (0, \widehat{a}) \text{ where } \widehat{a} \in (0, A). \end{cases}
\end{aligned} \tag{7}$$

We define the probability of survival of an individual of age given by

$$\pi(a) = e^{-\left(\int_0^a \mu_0(s) ds\right)}. \tag{8}$$

Thus, with hypothesis (H_1) − (iii), the probability of survival at maximal age A is zero. Moreover, hypothesis (H_2) − (ii) means that individuals of age below $\widehat{a}$ are not fertile. This hypothesis, in addition to being a demographic assumption, is also very important in establishing the observability inequality.

For more details about the modelling of such system and the biological significance of the hypotheses, we refer to Webb [1].

Moreover, there exist constants $\alpha > 0$, $\gamma > 0$, and λ such that, for every ϕ, $\psi \in V$, we have

$$\begin{aligned}
(H_3) &= \begin{cases} \left|\langle \mathbf{A}\phi, \psi\rangle_{V',V}\right| \leq \gamma \|\phi\|_V \|\psi\|_V, \\ \langle -\mathbf{A}\phi, \phi\rangle_{V',V} + \lambda \|\phi\|_H^2 \geq \alpha \|\phi\|_V^2, \end{cases} \\
&(H_4)\colon \beta\mu_0 \in L^1(0, A).
\end{aligned} \tag{9}$$

In the following, we recall that the operator $\mathbf{A}$ is independent of t and a and generator of a C_0 semigroup on H.

Assumptions (H_3) and (H_4) are necessary for the existence of the fixed point. Indeed, they make it possible to establish an adequate regularity on the system for the existence of a fixed point.

Theorem 1. *Let $y_0 \in L^2(0, A; H)$ and $\mathscr{B}u \in L^2((a_1, a_2) \times (0, T); U)$; under assumptions (H_1), (H_2), and (H_3), system (4) admits a unique solution $y \in L^2((0, T) \times (0, A); H)$.*

Proof. For the proof of the existence, we consider the following auxiliary system:

$$\begin{cases} \dfrac{\partial y_1}{\partial t} + \dfrac{\partial y_1}{\partial a} - \mathbf{A} y_1 + \mu_0 y_1 = e^{-\lambda_0 t} \mathscr{B}u, & \text{in } (0, A) \times (0, T), \\ y_1(0, t) = \displaystyle\int_0^A \beta(a) z da \Phi\left(\int_0^A \beta(a) e^{\lambda_0 t} z da\right), & \text{in } (0, T), \\ y_1(a, 0) = y_0(a), & \text{in } (0, A), \end{cases} \tag{10}$$

where $z \in L^2(0, T; H)$ and $y_1 = e^{-\lambda_0 t} y$.

Using Theorem 2.8 of [2], the auxiliary system (10) admits a unique solution $y_1 \in C([0, A]; L^2(0, T; H))$.

Now, we consider the operator Λ defined in $L^2((0, A) \times (0, T); H)$ on $L^2((0, A) \times (0, T); H)$ by $\Lambda(z) = y_1(z)$. Let $y_1(z_1)$ and $y_1(z_2)$ be the solutions of (10), where z is

replaced, respectively, by z_1 and z_2. We denote by $Y = y_1(z_1) - y_1(z_2)$. The function Y verifies

$$\begin{cases} \dfrac{\partial Y}{\partial t} + \dfrac{\partial Y}{\partial a} - \mathbf{A}Y + (\mu_0 + \lambda_0)Y = 0, & \text{in } (0, A) \times (0, T), \\ Y(0,t) = \displaystyle\int_0^A \beta(a) z_1 \mathrm{d}a \Phi\left(\int_0^A \beta(a) e^{\lambda_0 t} z_1 \mathrm{d}a\right) - \int_0^A \beta(a) z_2 \mathrm{d}a \Phi\left(\int_0^A \beta(a) e^{\lambda_0 t} z_2 \mathrm{d}a\right), & \text{in } (0, T), \\ Y(a,0) = y_0(a), & \text{in } (0, A). \end{cases} \quad (11)$$

By calculating the scalar product of the first equation of (11) by Y and integrating with respect to the age a and the time t, we obtain

$$\begin{aligned} &\int_0^T \|Y(T,)\|_H^2 \mathrm{d}a + \int_0^A \|Y(.,A)\|_H^2 \mathrm{d}t - \int_0^T \int_0^A \langle AY, Y\rangle_{V',V} \mathrm{d}a\mathrm{d}t + \int_0^T \int_0^A (\mu(a) + \lambda_0)\|Y\|_H^2 \mathrm{d}a\mathrm{d}t \\ &= \int_0^T \left\| \int_0^A \beta(a) z_1 \mathrm{d}a \Phi\left(\int_0^A \beta(a) e^{\lambda_0 t} z_1 \mathrm{d}a\right) - \int_0^A \beta(a) z_2 \mathrm{d}a \Phi\left(\int_0^A \beta(a) e^{\lambda_0 t} z_2 \mathrm{d}a\right)\right\|_H^2 \mathrm{d}t. \end{aligned} \quad (12)$$

Then,

$$\begin{aligned} &\int_0^T \|Y(T,)\|_H^2 \mathrm{d}a + \int_0^A \|Y(.,A)\|_H^2 \mathrm{d}t + (\lambda_0 - \lambda) \int_0^T \int_0^A \|Y\|_H^2 \mathrm{d}a\mathrm{d}t \\ &\leq M^2 A \|\beta\|_\infty^2 \|z_1 - z_2\|_{L^2((0,A)\times(0,T);H)}^2. \end{aligned} \quad (13)$$

Choosing λ_0 such that $\lambda_0 - \lambda = 2M^2 A\|\beta\|_\infty^2$ (λ is given in assumption (H_3)), we obtain

$$\|Y\|_{L^2((0,A)\times(0,T);H)}^2 \leq \frac{1}{2}\|z_1 - z_2\|_{L^2((0,A)\times(0,T);H)}^2. \quad (14)$$

Therefore, the operator

$$\|\Lambda(z_1) - \Lambda(z_2)\|_{L^2((0,A)\times(0,T);H)}^2 \leq \frac{1}{2}\|z_1 - z_2\|_{L^2((0,A)\times(0,T);H)}^2. \quad (15)$$

We conclude that, by the Banach fixed point that Λ admits a fixed point, we have

$$\Lambda(y_1) = y_1. \quad (16)$$

Finally, replacing z by $y_1 = e^{-\lambda_0 t} y$ in system (10), we obtain that y is solution of system (4).

Before we state our main result, let us introduce the notion of null controllability of the pair $(\mathbf{A}, \mathbf{B})$. We have the following result. □

Definition 1. We say that a pair $(\mathbf{A}, \mathbf{B})$ is null controllable in time T_0 if, for every $z_0 \in H$, there exists a control $u \in L^2(0, T; U)$ such that the solution of the system

$$z'(t) = \mathbf{A}z(t) + \mathbf{B}u(t), \quad t \in [0, T],\ z(0) = z_0, \quad (17)$$

satisfies $z(T) = 0$, for every $T > T_0$.

The main result of this paper is as follows.

Theorem 2. *Assume that β and μ satisfy conditions $(H_1) - (H_2) - (H_3)$ above.*

Assume that $(\mathbf{A}, \mathbf{B})$ is null controllable in any time $T > T_0$, with

$$T_0 < \widehat{T}, \widehat{T} = \min\{\widehat{a} - a_1, a_2 - a_1\} \quad (18)$$

and $a_1 < \widehat{a}$. Then, for every $T > a_1 + A - a_2 + 2T_0$ and for every $y_0 \in L^2(0, A; H)$, there exists a control $u_1 \in L^2((a_1, a_2) \times (0, T); U)$ such that the solution y of (4) satisfies

$$y(a, T) = 0, \quad \text{for all } a \in (0, A). \quad (19)$$

This result can be seen as a generalization of those obtained by Traoré [3] and Simporé [4] in the case when $\mathbf{A}$ is an elliptic operator with Neumann or Dirichlet homogeneous boundary conditions and Echarroudi and Maniar [5] when $\mathbf{A}$ is a degenerate elliptic operator.

Let us now mention some related works in controllability and observability from the literature. In the Network, Information, and Computer Security, Alcaraz and Wolthusen studied strategies for the efficient restoration of controllability following attacks and attacker-defender interactions in power-law networks in [6]. In [7], Alcaraz presents a network infrastructure based on three layers, where the redundant support is primarily concentrated on a fog-based structure to protect a specific subset of cyber-physical control devices.

In the anticancer therapy, Swierniak and Kalmka study a local controllability of a model of combined anticancer therapy with delay in control (see [8]).

For linear and nonlinear population dynamics models (Lotka–Mckendrick model), many results of local or global controllability have been established.

Indeed, Ainseba and Iannelli, in [9], have worked on the controllability of two nonlinear dynamic model problems, the nonlinearity being on mortality and fertility. In the first model, the control corresponds to a supply of individuals on a small age interval. The second one is age and space structured, and the control corresponds to a supply of individuals on a small subdomain ω of the whole space domain Ω.

In [3], Traoré proved the nonlinear Lotka–McKendrick is null controllable except for a small interval of ages near zero where the controls is localized with respect to the space variable but active for all ages.

In [2], Maity et al. study the null controllability of the linear infinite-dimensional system obtained by grafting an age structure.

To be in line with the studied models, many authors have also worked on constrained controllability problems. Indeed, the authors Maity et al., in [10], solved a problem of null controllability with positivity of the Lotka-Mckendrick system with spatial diffusion where the control is localized in the space variable as well as with respect to the age. The method combines final-state observability estimates with the use of characteristics and the associated semigroup.

In [11], Kalmka worked in constrained approximate controllability IEEE Transactions on Automatic Control.

In this paper, we extend the result of [2] to the nonlinear case. The nonlinearity is at the level of births and can be justified as follows.

If we consider an oviparous species, that is to say whose birth process goes through egg-laying, we see that $\int_0^A \beta(a) y da$ is the number of eggs laid at time t. Since all the eggs do not reach maturity, we introduce a function giving the proportion of eggs which will arrive there. Let Φ be this function. Then, we have,

$$\Phi\left(\int_0^A \beta(a) y \mathrm{d}a\right)\int_0^A \beta(a) y \mathrm{d}a, \tag{20}$$

the distribution of newborn individuals at time t.

For the establishment of the observability inequality, we use the same technique as in [2, 10]. One of the advantages of this method is that the conditions $\int_0^A \mu_0(a)\mathrm{d}a = +\infty$ on the mortality function do not interfere in establishing of the observability inequality which allows us to have a global controllability, which is not the case in [3, 9]. Moreover, we improve the observation domain, and better still we have null controllability for all ages. Indeed, in [3, 5, 9], the control is localized with respect to the space variable but active for all ages, and we do not have necessarily the extinction on the small interval of ages near zero. In this paper, the control is also localized with respect to the age variable, and we obtain the state of the system null for all ages.

The remaining part of this work is organized as follows.

In Section 2, we establish the observability inequality of the adjoint system of an auxiliary linear system. This allows us to prove its approximate null controllability. Section 3 is devoted to the proof of the main result of this paper using a fixed point of Schauder, see for example [12]. To finish, we give some perspectives in the conclusion in Section 4.

2. Approximate Null Controllability of Auxiliary System

2.1. Adjoint System. We suppose that the pair $(\mathbf{A}, \mathbf{B})$ is final-state observable at time T_0, and we consider the following auxiliary system (well chosen so as to obtain the main result of the paper by rotating a fixed point) given by

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \mathbf{A} y + \mu_0(a) y = \mathscr{B} u, & \text{in } (0, A) \times (0, T), \\ y(0, t) = G \displaystyle\int_0^A \beta y \mathrm{d}a, & \text{in } \Omega \times (0, T), \\ y(a, 0) = y_0, & \text{in } \Omega \times (0, A). \end{cases} \tag{21}$$

Applying the scalar product of the first equation of (21) by q and integrating with respect to the age a and time t, we obtain the adjoint system of (21):

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \dfrac{\partial q}{\partial a} - \mathbf{A}^* q + \mu_0(a) q - G\beta q(0,t) = 0, & \text{in } (0,A)\times(0,T), \\ q(A,t) = 0, & \text{in } (0,T), \\ q(a,T) = q_T(a), & \text{in } (0,A). \end{cases} \tag{22}$$

For every $q_0 \in K$, under assumption (H_1) and (H_2), system (22) admits a unique solution q. Moreover, integrating along the characteristic lines, the solution q of (22) is given by

$$q(t) = \begin{cases} \dfrac{\pi(a+T-t)}{\pi(a)} e^{(T-t)\mathbf{A}^*} q_T(a+T-t) + \displaystyle\int_t^T \left(\dfrac{\pi(a+s-t)}{\pi(a)} e^{(s-t)\mathbf{A}^*} G(s)\beta(a+s-t) q(0,s) \right) \mathrm{d}s, & \text{if } T-t \leq A-a, \\ \displaystyle\int_t^{t+A-a} \left(\dfrac{\pi(a+s-t)}{\pi(a)} e^{(s-t)\mathbf{A}^*} G(s)\beta(a+s-t) q(0,s) \right) \mathrm{d}s, & \text{if } A-a < T-t, \end{cases} \tag{23}$$

where $\pi(a) = e^{-\int_0^a \mu_0(\alpha) d\alpha}$.

We have the following result.

Theorem 3. *Let us assume assumptions $(H_1)-(H_2)$, and suppose the pair $(\mathbf{A},\mathbf{B})$ in null controllable for every time $T > T_0$. For $T > A - a_2 + a_1 + 2T_0$, there exists a control u_ε such that the solution y of system (21) verifies*

$$\|y(a,T)\|_{L^2(0,A;H)} \leq \varepsilon, \quad \text{for every } \varepsilon > 0. \tag{24}$$

For the proof of Theorem 1, we need the results of observability.

2.2. Observability Inequality

Theorem 4. *Under the assumption of Theorem 3, for every $q_T \in K$, the solution q of system (22) is final-state observable for every $T > a_1 + A - a_2 + 2T_0$. In other words, for every $T > a_1 + A - a_2 + 2T_0$, there exists $K_T > 0$ such that the solution q of (22) satisfies*

$$\|q(a,0)\|^2_{L^2(0,A;H)} \leq K_T\left(\|G\|_\infty, T, a_1, a_2, A\right) \int_0^T \int_{a_1}^{a_2} \left\|\mathbf{B}^* q(a,t)\right\|_U^2 \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}t. \tag{25}$$

For the proof, we need the following results.

Proposition 1. *Let us assume assumption $(H_1)-(H_2)$, and let $a_1 < \hat{a}$, $T > T_0 + a_1$, and $a_1 + T_0 < \eta < T$; then, there exists a constant $C > 0$ such that, for every $q_T \in K$, the solution q of system (22) verifies the following inequality:*

$$\int_0^{T-\eta} \|q(0,t)\|_H^2 \mathrm{d}t \leq C \int_0^T \int_{a_1}^{a_2} \left\|\mathbf{B}^* q(a,t)\right\|_U^2 \mathrm{d}a\, \mathrm{d}t. \tag{26}$$

Proposition 2. *Let us assume assumptions $(H_1)-(H_2)$, and let $a_1 < \hat{a}$, $T > T_0 + a_1$, and $a_1 < a_0 < a_2 - T_0$; there exists $C_T > 0$ such that the solution q of system (26) verifies the following inequality:*

$$\|q(a,0)\|^2_{L^2(0,a_0;H)} \leq C_T \int_0^T \int_{a_1}^{a_2} \left\|\mathbf{B}^* q(a,t)\right\|_U^2 \mathrm{d}a\, \mathrm{d}t. \tag{27}$$

For the proofs of Propositions 1 and 2, we have the following proposition.

Proposition 3. *Let us recall that the pair $(\mathbf{A}^*,\mathbf{B}^*)$ is final-state observable in any time $T > T_0$ with $T_0 \geq 0$. Let $C(T)$ be a observability cost with $C(T) \longrightarrow +\infty$ as $T \longrightarrow T_0$. Let T_1, T_2, and T_3 be three real numbers such that $0 \leq T_1 \leq T_2 \leq T_3$ with $T_2 - T_1 > T_0$. Then, for every $w_0 \in \mathrm{D}(\mathbf{A}^*)$, the solution w of the problem*

$$\begin{cases} \dfrac{\partial w(\lambda)}{\partial \lambda} - \mathbf{A}^* w(\lambda) = 0, & \text{in } [T_1, T_3], \\ w(T_1) = w_0, \end{cases} \tag{28}$$

satisfies the estimate

$$\left\|w(T_3)\right\|_H^2 \le M(T_1,T_2,T_3)\int_{T_1}^{T_2}\left\|\mathscr{B}^* w(\lambda)\right\|_U^2 d\lambda, \quad (29)$$

where the constants c_1 and c_2 depend on T and Ω.

Proof (see [2, 13]).

For $a \in (0,\hat{a})$, we have $\beta(a)=0$; therefore, system (22) can be written by

$$\begin{cases} -\dfrac{\partial q}{\partial t}-\dfrac{\partial q}{\partial a}-\mathbf{A}^* q+\mu_0 q=0, & \text{in } (0,\hat{a})\times(0,T), \\ q(a,T)=q_T(a), & \text{in } (0,\hat{a}). \end{cases} \quad (30)$$

We denote by $\tilde{q}(a,t)=q(a,t)e^{-\int_0^a \mu_0(\alpha)d\alpha}$. Then, $\tilde{q}$ satisfies

$$\frac{\partial \tilde{q}}{\partial t}+\frac{\partial \tilde{q}}{\partial a}-\mathbf{A}^*\tilde{q}=0 \quad \text{in } (0,\hat{a})\times(0,T). \quad (31)$$

Proving inequality (26) also leads to show that there exits a constant $C>0$ such that the solution $\tilde{q}$ of (31) satisfies

$$\int_0^{T-\eta}\|\tilde{q}(0,t)\|_H^2 dt \le C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*\tilde{q}(a,t)\right\|_U^2 da dt. \quad (32)$$

Indeed, we have

$$\int_0^{T-\eta}\|q(0,t)\|_H^2 dt=\int_0^{T-\eta}\|\tilde{q}(0,t)\|_H^2 dt \le C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*\tilde{q}(a,t)\right\|_U^2 da dt \le C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^* q(a,t)\right\|_U^2 da dt. \quad (33)$$

□

Proof of Proposition 1. We consider the following characteristics trajectory $\gamma(\lambda)=(T-\lambda,T+t-\lambda)$. If $T-\lambda=0$, the backward characteristics starting from $(0,t)$. If $T<a_1$, the trajectory $\gamma(\lambda)$ never reaches the observation region (a_1,a_2) (see Figure 1). So, we choose $T>a_1$.

Moreover, we estimate $q(0,t)$ on $(0,T-\eta)$, where $\eta>a_1$. Indeed, all the characteristic starting on $(0,t)$ with $t\in(T-a_1,T)$ never intersects the observation domain (see Figure 1).

Without loss of the generality, let us assume here $\eta<a_2\le\hat{a}<T$. □

2.2.1. *Case 1: For $t\in(0,T-\hat{a})$.* We denote by

$$w(\lambda)=\tilde{q}(T-\lambda,T+t-\lambda); \quad (\lambda\in(0,t). \quad (34)$$

Then, w satisfies

$$\begin{cases} \dfrac{\partial w(\lambda)}{\partial\lambda}-\mathbf{A}^* w(\lambda)=0, & \text{in } (T-\hat{a},T), \\ w(0)=\tilde{q}(T,T+t). \end{cases} \quad (35)$$

Using Proposition 3 with $0<t_0<t_1<t$, we obtain

$$\|w(T)\|_H^2 \le \left\|w(t_1)\right\|_H^2 \le c_1 e^{c_2/t_1-t_0}\int_{t_0}^{t_1}\left\|\mathbf{B}^* w(\lambda)\right\|_U^2 d\lambda, \quad (36)$$

that is equivalent to

$$\|\tilde{q}(0,t)\|_H^2 \le c_1 e^{c_2/t_1-t_0}\int_{t_0}^{t_1}\left\|\mathbf{B}^*\tilde{q}(T-\lambda,t+T-`\lambda)\right\|_U^2 d\lambda$$
$$C\int_{T-t_1}^{T-t_0}\left\|\mathbf{B}^*\tilde{q}(a,t+a)\right\|_U^2 da. \quad (37)$$

Then, for $t_0=T-\hat{a}$ and $t_1=T-a_1$, we obtain

$$\left\|\tilde{q}^2(0,t)\right\|_H^2 \le C\int_{a_1}^{\hat{a}}\left\|\mathbf{B}^*\tilde{q}^2(a,t+a)\right\|_U^2 da. \quad (38)$$

Integrating with respect to t over $(0,T-\hat{a})$, we obtain

$$\int_0^{T-\hat{a}}\left\|\tilde{q}^2(0,t)\right\|_H^2 dt \le C\int_{a_1}^{\hat{a}}\int_a^{T-\hat{a}+a}\left\|\mathbf{B}^*\tilde{q}^2(a,t)\right\|_U^2 dt da. \quad (39)$$

Finally,

$$\int_0^{T-\hat{a}}\left\|\tilde{q}^2(0,t)\right\|_H^2 dt \le C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*\tilde{q}(a,t)\right\|_U^2 da dt. \quad (40)$$

2.2.2. *Case 2: For $t\in(T-\hat{a},T-\eta)$.* We denote by

$$w(\lambda)=\tilde{q}(T-\lambda,T+t-\lambda); \quad (\lambda\in(T-\hat{a},T). \quad (41)$$

Then, w satisfies

$$\begin{cases} \dfrac{\partial w(\lambda)}{\partial\lambda}-\mathbf{A}^* w(\lambda,x)=0, & \text{in } (T-\hat{a},T), \\ w(T-\hat{a})=\tilde{q}(\hat{a},t+\hat{a}), & \text{in } \Omega. \end{cases} \quad (42)$$

Using Proposition 3 with $T-\hat{a}<t_0<t_1<T$, we obtain

$$\|w(T)\|_H^2 \le \left\|w(t_1)\right\|_H^2 \le c_1 e^{c_2/t_1-t_0}\int_{t_0}^{t_1}\left\|\mathbf{B}^* w(\lambda)\right\|_U^2 d\lambda, \quad (43)$$

that is equivalent to

$$\|\tilde{q}(0,t)\|_H^2 \le c_1 e^{c_2/t_1-t_0}\int_{t_0}^{t_1}\left\|\mathbf{B}^*\tilde{q}(T-\lambda,T+t-\lambda)\right\|_U d\lambda. \quad (44)$$

Then, for $t_0=T-\eta$ and $t_1=T-a_1$, we obtain

$$\|\tilde{q}(0,t)\|_H^2 \le C\int_{a_1}^{\eta}\left\|\mathbf{B}^*\tilde{q}(a,t+a)\right\|_U^2 da. \quad (45)$$

Integrating with respect to t over $(T-\widehat{a}, T-\eta)$, we obtain

$$\int_{T-\widehat{a}}^{T-\eta} \|\widetilde{q}(0,t)\|_H^2 \mathrm{d}t \leq C \int_{a_1}^{\eta} \int_{a+T-\widehat{a}}^{a+T-\eta} \left\|\mathbf{B}^*\widetilde{q}(\alpha,t)\right\|_U^2 \mathrm{d}t\mathrm{d}a. \quad (46)$$

Finally,

$$\int_{T-\widehat{a}}^{T-\eta} \|\widetilde{q}(0,)\|_H^2 \mathrm{d}t \leq C \int_0^T \int_{a_1}^{a_2} \left\|\mathbf{B}^*\widetilde{q}\right\|_U^2 \mathrm{d}a\mathrm{d}t. \quad (47)$$

Combining (40) and (47), we obtain the result.

Proving inequality (27) also leads to show that there exits a constant $C>0$ such that the solution $\widetilde{q}$ of (31) satisfies

$$\int_0^{a_0} \|\widetilde{q}(a,0)\|_H^2 \mathrm{d}t \leq C \int_0^T \int_{a_1}^{a_2} \left\|\mathbf{B}^*\widetilde{q}(a,t)\right\|_U^2 \mathrm{d}a\mathrm{d}t. \quad (48)$$

Indeed, we have

$$\begin{aligned} \int_0^{a_0} \|q(a,0)\|_H^2 \mathrm{d}t &\leq e^{2\|\mu\|_{L^1(0,a_0)}} \int_0^{a_0} \|\widetilde{q}(a,0)\|_H^2 \mathrm{d}t \leq Ce^{2\|\mu\|_{L^1(0,a_0)}} \int_0^T \int_{a_1}^{a_2} \left\|\mathbf{B}^*\widetilde{q}(a,t)\right\|_U^2 \mathrm{d}a\mathrm{d}t \\ &\leq Ce^{2\|\mu\|_{L^1(0,a_0)}} \int_0^T \int_{a_1}^{a_2} \left\|\mathbf{B}^*q(a,t)\right\|_U^2 \mathrm{d}a\mathrm{d}t. \end{aligned} \quad (49)$$

Proof of Proposition 1. The proof will be done in two parts.

2.2.3. The First Part: Interval $(0,a_1)$. We consider in this proof the characteristics $\gamma(\lambda)=(a+\lambda,\lambda)$. For $\lambda=0$, the characteristics start from $(a,0)$.

For $T>a_1$ and $a\in(0,a_1)$, all the characteristics starting on $(a,0)$ enter the observation domain (see Figure 2). We have three cases.

Case 1: $T<a_2$ and $a_2\leq\widehat{a}$.

Two situations can arise:

(i) $b_0=a_2-T<a_1<a_0$: in this situation, we split the interval $(0,a_1)$ as

$$(0,a_1)=(0,b_0)\cup(b_0,a_1). \quad (50)$$

(ii) $a_1<b_0<a_0$: in this situation, we keep the interval $(0,a_1)$.

Case 2: $T\geq a_2$ and $a_2\leq\widehat{a}$.

In this case, we keep the interval $(0,a_1)$.

Case 3: $a_2>\widehat{a}$.

In this case, we prove similarly the observability in $(a_1,\widehat{a})$ and expand to (a_1,a_2).

Here, we give the proof in the only situation, where

$$\begin{aligned} T&<a_2,\\ b_0&=a_2-T<a_1. \end{aligned} \quad (51)$$

In the remaining part of the proof, we give upper bounds for $\int_I\|\mathbf{B}^*\widetilde{q}(a,0)\|_H^2\mathrm{d}a$, where I is successively each one of the intervals appearing in decomposition (50) (see Figure 2).

Upper bound on $(0,b_0)$: for $a\in(0,b_0)$, we first set $w(\lambda)=\widetilde{q}(T+a-\lambda,T-\lambda)\,(\lambda\in(0,T)\text{ and }x\in\Omega)$, where $\widetilde{q}=e^{-\int_0^a\mu_0(\alpha)\mathrm{d}\alpha}q$.

Then, w verifies

$$\begin{cases} \dfrac{\partial w(\lambda)}{\partial\lambda}-\mathbf{A}^*w(\lambda)=0 & \text{in } (0,T), \\ w(T)=\widetilde{q}(a+T,T). \end{cases} \quad (52)$$

By applying Proposition 3 with $t_0=0$ and $t_1=a+T-a_1$, we obtain

$$\|w(T)\|_H^2 \leq c_1e^{c_2/a+T-a_1}\int_0^{a+T-a_1}\left\|\mathbf{B}^*w(\lambda)\right\|_U^2\mathrm{d}\lambda. \quad (53)$$

Then, we have

$$\begin{aligned} \|\widetilde{q}(a,0)\|_H^2 &\leq c_1e^{c_2/a+T-a_1}\int_{a_1}^{a+T}\left\|\mathbf{B}^*\widetilde{q}(\alpha,\alpha-a)\right\|_U^2\mathrm{d}\lambda \\ &= C\int_{a_1}^{a+T}\left\|\mathscr{B}^*\widetilde{q}(\alpha,\alpha-a)\right\|_U^2\mathrm{d}\alpha \leq C\int_{a_1}^{a_2}\left\|\mathbf{B}^*\widetilde{q}(\alpha,\alpha-a)\right\|_U^2\mathrm{d}\alpha. \end{aligned} \quad (54)$$

Integrating with respect a over $(0,b_0)$, we obtain

$$\int_0^{b_0}\|\widetilde{q}(a,0)\|_H^2\mathrm{d}a \leq C\int_0^{b_0}\int_{a_1}^{a_2}\left\|\mathbf{B}^*\widetilde{q}(\alpha,\alpha-a)\right\|_U^2\mathrm{d}\alpha\mathrm{d}a. \quad (55)$$

As

$$\int_0^{b_0}\int_{a_1}^{a_2}\left\|\mathscr{B}^*\widetilde{q}(\alpha,\alpha-a)\right\|_U^2\mathrm{d}\alpha\mathrm{d}a=\int_{a_1}^{a_2}\int_0^{b_0}\left\|\mathbf{B}^*\widetilde{q}(\alpha,\alpha-a)\right\|_U^2\mathrm{d}a\mathrm{d}\alpha, \quad (56)$$

then

$$\int_0^{b_0}\|\widetilde{q}(a,0)\|_H^2\mathrm{d}a \leq C\int_{a_1}^{a_2}\int_{\alpha-b_0}^{\alpha}\left\|\mathbf{B}^*\widetilde{q}(\alpha,t)\right\|_U^2\mathrm{d}t\mathrm{d}\alpha. \quad (57)$$

Finally,

$$\int_0^{b_0}\|\widetilde{q}(a,0)\|_H^2\mathrm{d}a \leq C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*\widetilde{q}(a,t)\right\|_U^2\mathrm{d}a\mathrm{d}t. \quad (58)$$

Upper bound (b_0,a_1): for $a\in(b_0,a_1)$, we consider always system (52), but $\lambda\in(T+a-a_2,T)$.

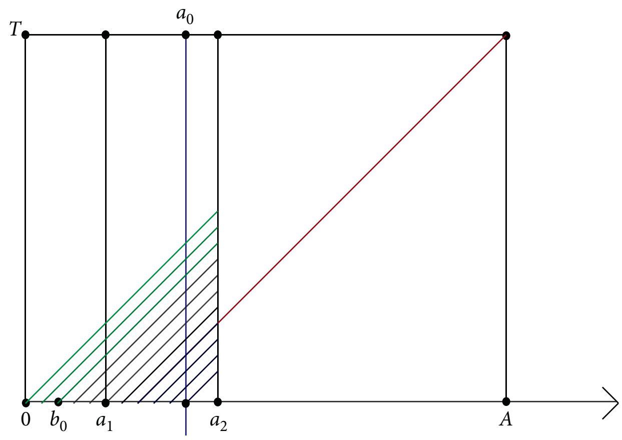

Figure 1: An illustration of the estimation of $q(a,0)$ between 0 and a_0 for case 1: green region corresponds to the interval $(0,b_0)$, black corresponds to the interval (b_0,a_1), and blue corresponds to the interval (a_1,a_0).

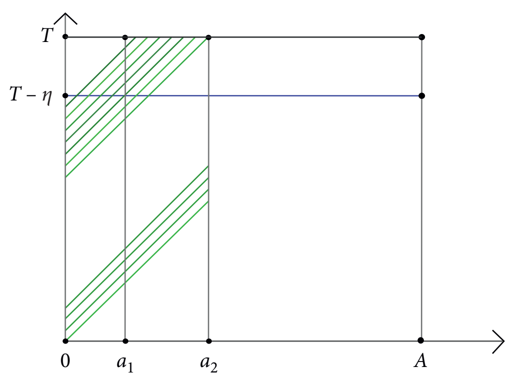

Figure 2: An illustration of the estimate of $q(0,t)$. Here, we have chosen $a_2=\hat{a}$. Since $t\in(0,T-\eta)$ with $a_1+T_0<\eta<T$, all the backward characteristics starting from $(0,t)$ enters the observation domain.

Applying Proposition 3 with $t_0=a+T-a_2$ and $t_1=a+T-a_1$, we obtain

$$\|\tilde{q}(a,0)\|_H^2\le C\int_{a_1}^{a_2}\left\|\mathbf{B}^*\tilde{q}(\alpha,\alpha-a)\right\|_U^2 d\alpha. \quad (59)$$

And as before, we obtain

$$\int_{b_0}^{a_1}\|\tilde{q}(.0)\|_H^2 da\le C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*\tilde{q}^2\right\|_U^2 da dt. \quad (60)$$

2.2.4. The Second Part: Interval (a_1,a_0). Here, we suppose $T>T_0$ and $a_1<a_0<a_2-T_0$.

And as in [2], we prove the existence of $C>0$ such that

$$\int_{a_1}^{a_0}\|\tilde{q}(.0)\|_H^2 da\le C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*\tilde{q}\right\|_U^2 da dt. \quad (61)$$

Consequently, combining (58), (60), and (61), we obtain

$$\int_0^{a_0}\|q(.0)\|_H^2 da\le C\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*q\right\|_U^2 da dt. \quad (62)$$

Lemma 1 (see Figure 3). *Let us suppose that $T>A-a_2+a_1+2T_0$ with $T_0<\hat{T}$. Then, there exists $a_0\in(a_1,\min\{\hat{a},ta_2\}-T_0)$ and $\kappa>0$ such that*

$$T-(a_1+T_0+\kappa)>A-a_0. \quad (63)$$

Proof. Without loosing the generality, we suppose that $a_2=\hat{a}$.

Suppose that $T>a_1+A-a_2+2T_0\Leftrightarrow T-a_1+T_0>A-a_2+T_0$. Then, there exists $\kappa>0$ (well chosen) such that $T-a_1-2\kappa-T_0>A-a_2+T_0\Leftrightarrow T-(a_1+\kappa+T_0)>A-(a_2-\kappa-T_0)$.

We denote by $a_0=a_2-\kappa-T_0$; then, $T-(a_1+\kappa+T_0)>A-a_0$.

We are in position to prove the observability inequality. □

Proof (proof of theorem 2). Let $a_0\in(a_1,\min\{\hat{a},ta_2\}-T_0)$, as in the previous Lemma. We have

$$\int_0^A\|q(a,0)\|_H^2 da=\int_0^{a_0}\|q(.0)\|_H^2 da+\int_{a_0}^A\|q(.0)\|_H^2 da. \quad (64)$$

From the result of (23) and applying the previous lemma, there exists $a_0\in(0,\hat{a})$ such that

$$q(a,0)=\int_0^{A-a}\left(\frac{\pi_1(a+s)}{\pi_1(a)}G(s)\beta(a+s)q(0,s)\right)ds, \quad (65)$$

and then

$$\int_{a_0}^A\|q(a,0)\|_H^2 da\le K_2\int_0^{T-(a_1+\kappa+T_0)}\|q(0,)\|_H^2 dt. \quad (66)$$

Finally, with the result of Proposition 1, we obtain

$$\int_{a_0}^A\|q(.0)\|_H^2 da\le K_3\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*q\right\|_U^2 da dt. \quad (67)$$

With the result of Proposition 2 and inequality (67), we obtain the result of Theorem 4 (see Figure 3).

The figures below illustrate, respectively, the estimate of the nonlocal term, the estimate of $q(a,0)$, and the observation time. □

Proof of the Theorem 3. For $\varepsilon>0$, we consider the functional J_ε define by

$$J_\varepsilon(u)=\frac{1}{2}\int_0^T\int_{a_1}^{a_2}\|u\|_U^2 da dt+\frac{1}{2\varepsilon}\int_0^A\|y(.T)\|_H^2 da, \quad (68)$$

where y is the solution of the following system:

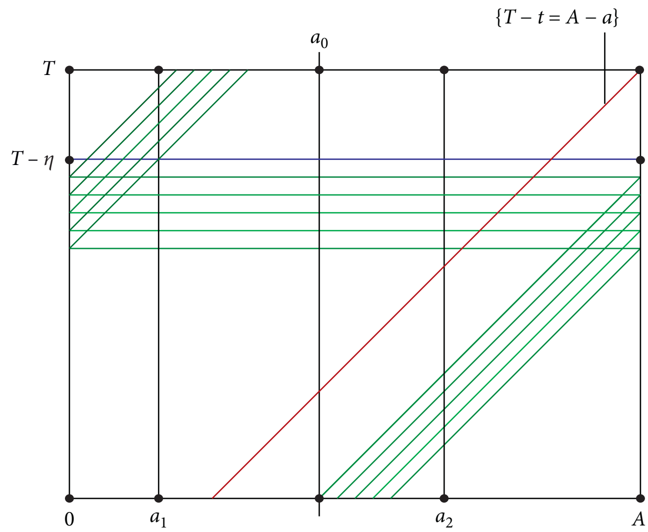

FIGURE 3: The backward characteristics starting from $(a,0)$ with $a\in(a_2,A)$ (green lines) hits the line $(a=A)$, gets renewed by the renewal condition $\beta(a)q(0,t)$, and then enters the observation domain. More precisely, in addition to the observation time T_0 of the pair $(\mathbf{A},\mathbf{B})$, the backward characteristics need at most $A-a_2$ time to hit the line $a=A$, get renewed by the renewal condition $\beta(a)q(0,t)$, and take maximum T_0+a_1 time to enter the observation domain. Thus, we need in all, at least, $T=A-a_2+a_1+2T_0$ time to obtain the observability inequality. So, with the conditions $T>A-a_2+a_1+2T_0$, $T_0>\widehat{T}$, and $a_1+T_0<\eta<T$, all the characteristics starting at $(a,0)$ with $a\in(a_2,A)$ get renewed by the renewal condition $\beta(a)q(0,t)$ in $t\in(0,T-\eta)$ and enter the observation domain.

$$\begin{cases}\dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}-\mathbf{A}y+\mu_0 y=\mathscr{B}u, & \text{in } (0,A)\times(0,T),\\[2ex] y(0,t)=G(l)\displaystyle\int_0^A\beta(a)y\,\mathrm{d}a, & \text{in } (0,T),\\[2ex] y(a,0)=y_0, & \text{in } (0,A),\end{cases} \tag{69}$$

with $l\in L^2(0,T;H)$. □

Lemma 2. *The functional J_ε is continuous, strictly convex, and coercive. Consequently, J_ε reaches its minimum at a point $u_\varepsilon\in L^2((a_1,a_2)\times(0,T);U)$.*

Moreover, setting y_ε, the associated solution of (69), and q_ε, the solution (21) with $q_\varepsilon(a,T)=(1/\varepsilon)y_\varepsilon(a,T)$, has $u_\varepsilon=\mathscr{B}^*q_\varepsilon$, and there exist $C_1>0$ and $C_2>0$ independent of ε and l, such that

$$\begin{gathered}\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*q_\epsilon\right\|_U^2\mathrm{d}a\mathrm{d}t\le C_1\left\|y_0\right\|^2_{L^2(0,A;H)}\\ \left\|y_\varepsilon(.T)\right\|^2_{L^2(0,A;H)}\le\varepsilon C_2\left\|y_0\right\|^2_{L^2(0,A;H)}.\end{gathered}\tag{70}$$

Proof (proof of lemma). It is easy to check that J_ε is coercive, continuous, and strictly convex. Then, it admits a unique minimizer u_ε. The maximum principle gives that $u_\varepsilon=\mathscr{B}^*q_\varepsilon$.

By calculating the dot product of the first equation of (69) with $u=\mathscr{B}^*q_\varepsilon$ by Y and integrating with respect to the age a and the time t, we obtain

$$\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*q_\epsilon\right\|_U^2\mathrm{d}a\mathrm{d}t+\frac{1}{\varepsilon}\int_0^A\left\|y_\varepsilon^2(.T)\right\|_H^2\mathrm{d}a=\int_0^A\langle y_0,q_\varepsilon(.0)\rangle_H\mathrm{d}a. \tag{71}$$

Using the inequality of Young, we obtain, for any $\delta>0$,

$$\begin{aligned}\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*q_\varepsilon\right\|_U^2\mathrm{d}a\mathrm{d}t+\frac{1}{\varepsilon}\int_0^A\left\|y_\epsilon^2(.T)\right\|_H^2\mathrm{d}a&\le\frac{1}{2}\int_0^A\left\|y_0\right\|_H^2\mathrm{d}a\\ &+\frac{1}{2}\int_0^A\left\|q_\varepsilon(.0)\right\|_H^2\mathrm{d}a.\end{aligned}\tag{72}$$

Using the observability inequality (25) and choosing $K_T=\delta$, where K_T is given in (25), we obtain

$$\frac{1}{2}\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^*q_\varepsilon\right\|_U^2\mathrm{d}a\mathrm{d}t+\frac{1}{2\varepsilon}\int_0^A\left\|y_\varepsilon^2(.T)\right\|_H^2\mathrm{d}a\le\frac{1}{2}\int_0^A\left\|y_0\right\|_H^2\mathrm{d}a. \tag{73}$$

Then, the result is obtained. □

3. Null Controllability of Nonlinear System

Now, we consider the following system:

$$\begin{cases} \dfrac{\partial y(l)}{\partial t} + \dfrac{\partial y(l)}{\partial a} - \mathbf{A}y(l) + \mu_0 y(l) = \chi_{(a_1,a_2)}\mathbf{B}\mathbf{B}^* q, & \text{in } (0,A)\times(0,T), \\ y(l)(0,t) = G(l)\displaystyle\int_0^A \beta(a) y(l)\mathrm{d}a, & \text{in } (0,T), \\ y(l)(a,0) = y_0, & \text{in } (0,A), \end{cases} \tag{74}$$

where $G(l) = \Phi(l), l \in L^2(0,T;H)$.

We have the following.

Lemma 3. *Under assumption* (H_3)*, there exists* $C > 0$ *and* $K > 0$ *such that the solution* $y_\varepsilon(l)$ *of (69) verifies*

$$\begin{aligned} &\|y_\varepsilon(l)\|^2_{L^2((0,A)\times(0,T);H)} + \int_0^T\int_0^A \mu_0 \|y_\varepsilon(l)\|^2_H \mathrm{d}a\mathrm{d}t \leq K\|y_0\|^2_{L^2(0,A;H)} \\ &\|y_\varepsilon(l)(.A)\|^2_{L^2(0,T;H)} \leq C\|y_0\|^2_{L^2(0,A;H)}. \end{aligned} \tag{75}$$

Proof. (proof of the lemma). For the proof, we denote by $p_\varepsilon(l) = y_\varepsilon(l)e^{\lambda_0 t}$. Then, $p_\varepsilon(l)$ satisfies

$$\begin{cases} \dfrac{\partial p_\varepsilon(l)}{\partial t} + \dfrac{\partial p_\varepsilon(l)}{\partial a} - \mathbf{A}p_\varepsilon(l) + \mu_0 p_\varepsilon(l) = 1_{(a_1,a_2)} e^{-\lambda_0 t}\mathbf{B}\mathbf{B}^* q_\varepsilon(l), & \text{in } (0,A)\times(0,T), \\ p_\varepsilon(l)(0,t) = G(l)\displaystyle\int_0^A \beta p_\varepsilon(l)\mathrm{d}a, & \text{in } (0,T), \\ p_\varepsilon(l)(a,0) = y_0(a), & \text{in } (0,A). \end{cases} \tag{76}$$

The function $p_\varepsilon(l)$ verifies

$$\begin{aligned} &\frac{\partial p_\varepsilon(l)}{\partial t} + \frac{\partial p_\varepsilon(l)}{\partial a} + \mathbf{A}p_\varepsilon(l) + (\lambda_0 + \mu_0)p_\varepsilon(l) \\ &= 1_{(a_1,a_2)} e^{-\lambda_0 t}\mathbf{B}\mathbf{B}^* q_\varepsilon(l). \end{aligned} \tag{77}$$

By calculating the dot product of the equation of (77) by $p_\varepsilon(l)$ and integrating with respect the age a and the times t, we obtain

$$\begin{aligned} &\int_0^A \|p_\varepsilon(l)(T,)\|^2_H \mathrm{d}a + \int_0^T \|p_\varepsilon(l)(.A)\|^2_H \mathrm{d}t - \int_0^T\int_0^A \langle \mathbf{A}p_\varepsilon(l), p_\varepsilon(l)\rangle_{V',V}\mathrm{d}a\mathrm{d}t + \int_0^T\int_0^A (\lambda_0+\mu_0)\|p_\varepsilon(l)\|^2_H \mathrm{d}a\mathrm{d}t \\ &= \int_0^T\int_0^A \langle p_\varepsilon(l), 1_{(a_1,a_2)} e^{-\lambda_0 t}\mathbf{B}\mathbf{B}^* q_\varepsilon(l)\rangle_{H,U}\mathrm{d}a\mathrm{d}t + \int_0^A \|p_\varepsilon(l)(0,)\|^2_H \mathrm{d}a + \int_0^T \|p_\varepsilon(l)(.0)\|^2_H \mathrm{d}t. \end{aligned} \tag{78}$$

We have

$$\int_0^T\int_0^A \langle \mathbf{B}^* p_\varepsilon(l), 1_{(a_1,a_2)} e^{-\lambda_0 t}\mathbf{B}^* q_\varepsilon(l)\rangle_{U,U}\mathrm{d}a\mathrm{d}t \leq \frac{1}{2\delta}\int_0^T\int_0^A \|\mathbf{B}^* p_\varepsilon(l)\|^2_U \mathrm{d}a\mathrm{d}t + \frac{\delta}{2}\int_0^T\int_{a_1}^{a_2} \|\mathbf{B}^* q_\epsilon(l)\|^2_U. \tag{79}$$

As

$$\int_0^T\int_0^A \langle \mathbf{B}^* p_\varepsilon(l), 1_{(a_1,a_2)} e^{-\lambda_0 t}\mathbf{B}^* q_\varepsilon(l)\rangle_{U,U} \mathrm{d}a\mathrm{d}t \le \frac{C_1}{2\delta}\int_0^T\int_0^A \|p_\varepsilon(l)\|_H^2 \mathrm{d}a\mathrm{d}t + \frac{\delta}{2}\int_0^T\int_{a_1}^{a_2}\|\mathbf{B}^* q_\varepsilon(l)\|_U^2, \tag{80}$$

moreover

$$\int_0^T \|p_\varepsilon(l)(.0)\|_H^2 \mathrm{d}t \le C_2(\|\beta\|_\infty, A)\|p_\varepsilon(l)\|^2_{L^2((0,T)\times(0,A);H)}. \tag{81}$$

Using assumption (H_3) and choosing λ_0 and δ such that

$$\lambda_0 = \left(\frac{C_1}{2\delta} + C_2(\|\beta\|_\infty, A) + \gamma + 1\right), \tag{82}$$

we obtain

$$\|p_\varepsilon(.A)\|^2_{L^2(0,T)} + \int_0^T\int_0^A \mu_0\|p_\varepsilon(l)\|_H^2 \mathrm{d}a\mathrm{d}t + \alpha\|p_\varepsilon(l)\|^2_{L^2(0,T;V)} + \|p_\varepsilon(l)\|^2_{L^2((0,T)\times(0,A);V)} \le C\|y_0\|^2_{L^2(0,A;H)}, \tag{83}$$

(with α and λ the parameters given in the assumption (H_3)).

Therefore, we have the result.

Let Λ be an operator defined on $L^2(0,T;H)$ by

$$\Lambda(l) = \int_0^A \beta y_\varepsilon(l)\mathrm{d}a, \tag{84}$$

where $y_\varepsilon(l)$ is the solution of the following system. □

Proposition 4. *We suppose that operator* $\mathbf{A}$ *verifies the hypotheses* (H_3). *Then, the operator* Λ *is continuous, bounded, and compact of that* $L^2(0,T;H)$. *Therefore,* Λ *admits a fixed point.*

Proof (proof of the proposition 3). Let

$$M = \int_0^A \beta y_\varepsilon(l)\mathrm{d}a. \tag{85}$$

Then, $Y = e^{-\lambda_0 t}M$ is the solution of the following system:

$$\begin{cases} \dfrac{\partial Y(l)}{\partial t} - \mathbf{A}Y(l) + \lambda_0 Y(l) = e^{-\lambda_0 t}\left(\displaystyle\int_{a_1}^{a_2}\beta\mathbf{B}\mathbf{B}^* q_\varepsilon da + R\right), & \text{in } (0,A)\times(0,T), \\ Y(l)(.0) = \displaystyle\int_0^A \beta y_0 da, & \text{in } (0,T), \end{cases} \tag{86}$$

where $R = \int_0^A \mu_0\beta e^{\lambda_0 t} y_\varepsilon(l)\mathrm{d}a + \int_0^A \beta' y_\varepsilon(l)\mathrm{d}a - \beta(A)y_\varepsilon(l)(.A)$.

From the results of the previous lemma, we have

$$\|y_\varepsilon(l)\|_{L^2((0,A)\times(0,T);H)} \le K\|y_0\|_{L^2(0,T;H)}. \tag{87}$$

By calculating the dot product of the first equation of (87) by $Y(l)$ and integrating with respect the age a and the times t, we obtain

$$\begin{aligned} &\|Y(l)(.T)\|_H^2 - \int_0^T \langle \mathbf{A}Y(l), Y(l)\rangle_{V',V}\mathrm{d}t + \lambda_0\int_0^T \|Y(l)\|_H^2\mathrm{d}t \\ &= \int_0^T \left\langle Y(l), \int_{a_1}^{a_2} e^{-\lambda_0 t}\mathbf{B}\mathbf{B}^* q_\varepsilon(l)\right\rangle_{H,U}\mathrm{d}t + \int_0^T \langle R, Y(l)\rangle_{H,H}\mathrm{d}t + \|Y(l)(.0)\|_H^2. \end{aligned} \tag{88}$$

We have, using Young inequality,

$$\begin{aligned} &\int_0^T \left\langle \mathbf{B}^* Y(l), \int_{a_1}^{a_2} e^{-\lambda_0 t}\mathbf{B}^* q_\varepsilon(l)\right\rangle_{U,U}\mathrm{d}t + \int_0^T \langle R, Y(l)\rangle_{H,H}\mathrm{d}t \\ &\le \frac{\delta}{2}\int_0^T \|\mathbf{B}^* Y(l)\|_U^2\mathrm{d}t + \frac{1}{2\delta}\int_0^T\int_{a_1}^{a_2}\|\mathbf{B}^* q_\varepsilon(l)\|_U^2\mathrm{d}a\mathrm{d}t + \frac{1}{2\delta}\int_0^T \|R\|_H^2\mathrm{d}t + \frac{\delta}{2}\int_0^T\|Y(l)\|_H^2\mathrm{d}t. \end{aligned} \tag{89}$$

Then,

$$\int_0^T \langle \mathbf{B}^* Y(l), \int_{a_1}^{a_2} e^{-\lambda_0 t}\mathbf{B}^* q_\varepsilon(l)\rangle_{U,U}\mathrm{d}t + \int_0^T \langle R, Y(l)\rangle_{H,H}\mathrm{d}t \\ \le \frac{\delta(C+1)}{2}\int_0^T \|Y(l)\|_H^2\mathrm{d}t + \frac{1}{2\delta}\int_0^T\int_{a_1}^{a_2}\left\|\mathbf{B}^* q_\varepsilon(l)\right\|_U^2\mathrm{d}a\mathrm{d}t + \frac{1}{2\delta}\int_0^T \|R\|_H^2\mathrm{d}t. \tag{90}$$

Boundness of R: we have

$$\|R\|_{L^2(0,T:\,H)}^2 \le 2e^{\lambda_0 T}\int_0^T \left\|\int_0^A \mu_0\beta y_\varepsilon(l)\mathrm{d}a\right\|_H^2\mathrm{d}t \\ +2\left\|\beta'\right\|_\infty \left\|y_\varepsilon(l)\right\|_{L^2((0,A)\times(0,T);H)}^2 + 2\beta A\left\|y_\varepsilon(l)(.A,)\right\|_{L^2(0,T;H)}^2. \tag{91}$$

As $\mu_0\beta \in L^1(0,1)$, using Cauchy Schwarz inequality, we obtain

$$\|R\|_{L^2(0,T:\,H)}^2 \le 2e^{\lambda_0 T}\int_0^A \mu_0\beta\mathrm{d}a\int_0^T \left\|\int_0^A \sqrt{\mu_0\beta}\left|\sqrt{\mu_0\beta}\, y_\varepsilon(l)\right|\mathrm{d}a\right\|_H^2\mathrm{d}a\mathrm{d}t \\ + 2\left\|\beta'\right\|_\infty \left\|y_\varepsilon(l)\right\|_{L^2((0,A)\times(0,T);H)}^2 + 2\beta(A)\left\|y_\varepsilon(l)(.A,)\right\|_{L^2(0,T;H)}^2. \tag{92}$$

Then,

$$\|R\|_{L^2(0,T:\,H)}^2 \le 2e^{\lambda_0 T}\int_0^A \mu_0\beta\mathrm{d}a\int_0^T\int_0^A \mu_0\beta\left\|y_\varepsilon(l)\right\|_H^2\mathrm{d}a\mathrm{d}t \\ + 2\left\|\beta'\right\|_\infty \left\|y_\varepsilon(l)\right\|_{L^2((0,A)\times(0,T);H)}^2 + 2\beta(A)\left\|y_\varepsilon(l)(.A,)\right\|_{L^2(0,T;H)}^2. \tag{93}$$

Finally, in Lemma 3, there exists $K_T > 0$ such that

$$\|R\|_{L^2(0,T:\,H)}^2 \le K_T\left\|y_0\right\|_{L^2(0,T;H)}^2. \tag{94}$$

Now, using assumption (H_3) and choosing $\lambda_0 = (\delta(C+1)/2) + 1 + \lambda$, we obtain

$$\|Y(l)(T)\|_H^2 + \alpha\|Y(l)\|_{L^2(0,T;V)}^2 + \|Y(l)\|_{L^2(0,T,H)}^2 \\ \le K\left\|y_0\right\|_{L^2((0,A)\times(0,T);H)}^2, \tag{95}$$

(with α and λ the parameters given in the assumption (H_3)).

Then, M is bounded in $L^2(0,T;V)$ independently of l and ε.

Proceeding in the same way by calculating the dot product of the first equation of (86) by a test function ψ, we obtain

$$\int_0^T \left|\langle\frac{\partial Y}{\partial t}, \psi\rangle\right|\mathrm{d}t \le C_T\|\psi\|_{L^2(0,T;V)}\left\|y_0\right\|_{L^2(0,A;H)}. \tag{96}$$

Then,

$$\left\|\frac{\partial Y}{\partial t}\right\|_{L^2(0,T;V')} \le C_T\left\|y_0\right\|_{L^2(0,A;H)}. \tag{97}$$

Then, M is bounded in $L^2(0,T;V)$ and $\partial M/\partial t$ is bounded in $L^2(0,T;V')$ independently of l and ε.

Hence, using the Lions–Aubin lemma, we conclude that Λ is bounded and compact in $L^2(0,T;H)$. □

3.1. Step 2: Continuity of the Operator Λ. Let $(m_{1,\varepsilon}(p))$ and $(m_{2,\varepsilon}(p))$ be the solution of the following cascade system:

$$\begin{cases} \dfrac{\partial m_{1,\varepsilon}(l)}{\partial t} + \dfrac{\partial m_{1,\varepsilon}(l)}{\partial a} - \mathbf{A} m_{1,\varepsilon}(l) + \mu_0 m_{1,\varepsilon}(l) = \chi_{(a_1,a_2)} \mathbf{B}^* v_\varepsilon, & \text{in } (0,A)\times(0,T), \\ m_{1,\varepsilon}(l)(a,0) = 0, & \text{in } (0,A), \\ m_{1,\varepsilon}(l)(0,t) = \Phi(l) \displaystyle\int_0^A \beta m_{1,\varepsilon}(l) \mathrm{d}a, & \text{in } (0,T), \end{cases} \tag{98}$$

$$\begin{cases} \dfrac{\partial m_{0,\varepsilon}(l)}{\partial t} + \dfrac{\partial m_{0,\varepsilon}(l)}{\partial a} - \mathbf{A} m_{0,\varepsilon}(l) + \mu_0 m_{0,\varepsilon}(l) = 0, & \text{in } (0,A)\times(0,T), \\ m_{0,\varepsilon}(l)(a,0) = m_0, & \text{in } (0,A), \\ m_{0,\varepsilon}(l)(0,t) = \Phi(l) \displaystyle\int_0^A \beta(a) m_{0,\varepsilon}(l) \mathrm{d}a, & \text{in } (0,T). \end{cases} \tag{99}$$

Due to Lemma 2, there exists $v_\varepsilon \in L^2((a_1,a_2)\times(0,T);\mathbf{U})$ such that

$$\left\| m_{1,\varepsilon}(l)(.T) + m_{0,\varepsilon}(l)(.T) \right\| \le \kappa, \tag{100}$$

for any $\varepsilon > 0$.

Any suitable control can be chosen, in particular, the control of minimum norm in $L^2((a_1,a_2)\times(0,T);\mathbf{U})$. Let us then consider

$$y_\varepsilon(l) = m_{1,\varepsilon}(l) + m_{0,\varepsilon}(l), \tag{101}$$

where $v_{1,\varepsilon}$ is the optimal control characterised by Lemma 2 such that (100) holds, and let us consider the operator:

$$\Lambda' : \begin{cases} L^2(0,T;H) \longrightarrow L^2((0,A)\times(0,T);H), \\ l \longrightarrow y_\varepsilon(l). \end{cases} \tag{102}$$

The function $\Phi(l)$ is bounded, when l spans $L^2(0,T;H)$. Since the solution $m_{1,\varepsilon}$ on system (33) depends continuously on the data $v_{1,\varepsilon}$, it follows that the operator Λ' is continuous. Moreover, the operator $\Lambda''(l') = \int_0^A \beta(a) l' \mathrm{d}a$ is continuous on $L^2(0,T;H)$, and therefore, we conclude that the operator Λ is continuous.

Schauder's fixed-point theorem (see [4, 12]) implies that Λ admits a fixed point. Then, there exists $l \in H$ such that

$$\Lambda(l) = l = \int_0^A \beta y_\varepsilon \mathrm{d}a, \tag{103}$$

and y_ε solves

$$\begin{cases} \dfrac{\partial y_\varepsilon}{\partial t} + \dfrac{\partial y_\varepsilon}{\partial a} - \mathbf{A} y_\varepsilon + \mu_0(a) y_\varepsilon = 1_{(a,a_2)} \mathbf{B}\mathbf{B}^* q_\varepsilon, & \text{in } (0,A)\times(0,T), \\ y_\varepsilon(0,t) = \displaystyle\int_0^A \beta y_\varepsilon \mathrm{d}a \Phi\left(\int_0^A \beta y_\varepsilon \mathrm{d}a \right), & \text{in } (0,T), \\ y_\varepsilon(a,0) = y_0(a), & \text{in } (0,A). \end{cases} \tag{104}$$

Moreover, we have

$$\begin{aligned} &\int_0^T \int_{a_1}^{a_2} \left\| \mathbf{B}^* q_\varepsilon \right\|_U^2 \mathrm{d}a\mathrm{d}t \le C_1 \left\| y_0 \right\|^2_{L^2(0,A;H)}, \\ &\left\| y_\varepsilon(.T) \right\|^2_{L^2(0,A;H)} \le \varepsilon C_2 \left\| y_0 \right\|^2_{L^2(0,A;H)}. \end{aligned} \tag{105}$$

Finally, when $\epsilon \longrightarrow 0$,

$$\begin{aligned} &\mathbf{B}^* q_\varepsilon \rightharpoonup u, \\ &\left\| y_\varepsilon(T,) \right\|_H \longrightarrow \| y(T,) \|_H = 0. \end{aligned} \tag{106}$$

4. Applications

Assumptions (H_3) and (H_4) are not necessary in the case of the linear system.

4.1. Nonlinear Population Dynamics System Structuring by Age and Spatial Diffusion. We consider the following nonlinear population dynamics structuring by age and spatial position:

$$\begin{cases} \dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}-\mathbf{A}y+\mu_0(a)y=1_{(a_1,a_2)}\mathbf{B}u, & \text{in } Q=\Omega\times(0,A)\times(0,T), \\ \text{Boundary condition,} & \text{on } \Sigma=\partial\Omega\times(0,A)\times(0,T), \\ y(x,0,s,t)=\Phi\left(\int_0^A \beta y \mathrm{d}a\right)\int_0^A \beta y \mathrm{d}a, & \text{in } \Omega\times(0,T), \\ y(x,a,s,0)=y_0, & \text{in } \Omega\times(0,A), \end{cases} \tag{107}$$

where $\mathbf{A}$ is a operator define in the domain $\Omega\subset\mathbf{R}^N$, $N\in\mathbb{N}$, and $\mathbf{B}=1_\omega$.

For

$$\mathbf{A}\psi=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\sigma_{i,j}\frac{\partial\psi}{\partial x_j}\right), \quad \text{with } \sigma_{ij} = \sigma_{ji}\in C^2(\overline{\Omega}) \text{ for } 1\le i,\ j\le N, \tag{108}$$

we assume there exists a constant $C>0$ such that

$$\sum_{i,j=1}^{N}\sigma_{i,j}(x)\xi_i\xi_j>C\|\xi\|^2\left(x\in\overline{\Omega},\xi\in\mathbb{R}^N\right),$$
$$D(\mathbf{A})=\left\{\psi\in H^2(\Omega)\quad \text{such that } \mathbf{A}\psi\in L^2(\Omega) \text{ and } \frac{\partial\psi}{\partial\nu_{\mathbf{A}}}=\sum_{i=1}^{N}\sigma_{i,j}\frac{\partial\psi}{\partial x_i}\cdot n_i=0\right\}. \tag{109}$$

The pair $(\mathbf{A},\mathbf{B})$ is observable for every time $T>0$. Moreover, the operator $\mathbf{A}$ verifies hypotheses $(H3)$, where $V=H^1(\Omega), H=L^2(\Omega)$. Then, applying Theorem 2, the system (107) is null controllable for every time $T>a_1+A-a_2$.

4.2. Nonlinear Population Dynamics Models with Degenerate Diffusion. We consider the following nonlinear population dynamics with degenerate diffusion (see [5]):

$$\begin{cases} \dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}-\partial_x(k(x)\partial_x y)+\mu_0(a)y=1_{(a_1,a_2)}\mathbf{B}u, & \text{in } Q=\Omega\times(0,A)\times(0,T), \\ y(0,a,t)=0 \text{ or } ay_x(0,a,t)=0, & \text{in } (0,A)\times(0,T), \\ y(1,a,t)=0, & \text{in } (0,A)\times(0,T), \\ y(x,0,s,t)=\Phi\left(\int_0^A \beta y \mathrm{d}a\right)\int_0^A \beta y \mathrm{d}a, & \text{in } \Omega\times(0,T), \\ y(x,a,s,0)=y_0, & \text{in } \Omega\times(0,A), \end{cases} \tag{110}$$

where k is the nonnegative function in $\Omega = (0,1)$ and degenerates. Moreover, $\mathbf{B} = \chi_{(l_1,l_2)}$.

Let us set the state and the control space as follows:

$$V = H_k^1(\Omega) = \left\{ \frac{\varphi \in L^2(\Omega) \text{ absolutely continuous in } [0,1]}{\int_0^1 k(x)(\partial_x\varphi)^2 dx} < +\infty, \quad \varphi(0) = \varphi(1) = 0 \right\},$$
$$H_k^2(\Omega) = \left\{ \frac{\varphi \in H_k^1(\Omega)}{k\varphi_x \in H^1(\Omega)} \right\}, \tag{111}$$

equipped with the following norm

$$\begin{aligned} \|\varphi\|_{H_k^1}^2 &= \|\varphi\|_{L^2(\Omega)}^2 + \left\|\sqrt{k}\,\partial_x\varphi\right\|_{L^2(\Omega)}^2, \\ \|\varphi\|_{H_k^2}^2 &= \|\varphi\|_{H_k^1(\Omega)}^2 + \left\|\partial_x(k\partial_x\varphi)\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{112}$$

We consider the unbounded operator A on $H = L^2(\Omega)$ defined by

$$\begin{aligned} \mathbf{D}(\mathbf{A}) &= H_k^2(\Omega), \\ \mathbf{A}\psi &= \partial_x(k(x)\partial_x\psi). \end{aligned} \tag{113}$$

Proposition 5 (see [14]. *The operator* $\mathbf{A}$: $\mathbf{D}(\mathbf{A}) \longrightarrow L^2(\Omega)$ *is closed, self-adjoint, and negative with dense domain.*

Hence, $\mathbf{A}$ is the infinitesimal generator of a strongly continuous semigroup $e^{t\mathbf{A}}$ on $L^2(\Omega)$.

The function $k \in C^0([0,1]) \cap C^1(0,1)$ is such that it satisfies $k(0) = 0$ and $k > 0$ in Ω, and there exists $M_1 \in [0,1)$ such that

$$xk'(x) \le M_1 k(x), \quad \text{for all } x \in [0,1]. \tag{114}$$

Under the above assumptions, the pair $(\mathbf{A}, \mathbf{B})$ is observable for every time $T > 0$ (see [15]).

Moreover, the operator $\mathbf{A}$ verifies hypotheses (H_3), where $V = H_k^1(\Omega)$ and $H = L^2(\Omega)$.

Now, multiplying $\partial_x(k(x)\partial_x\varphi)$ by ψ and integrating by part over $(0,1)$, where φ and $\psi \in H_k^1(\Omega)$, we obtain

$$\left|\int_0^1 \partial_x(k(x)\partial_x\varphi)\psi dx\right| = \left|\int_0^1 k(x)\varphi_x\psi_x dx\right|. \tag{115}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\left|\int_0^1 \partial_x(k(x)\partial_x\varphi)\psi dx\right| \le C\left\|k(x)\partial_x\varphi\right\|_{L^2(\Omega)}\left\|k(x)\partial_x\psi\right\|_{L^2(\Omega)}. \tag{116}$$

Then,

$$\left|\int_0^1 \partial_x(k(x)\partial_x\varphi)\psi\, dx\right| \le C\|\varphi\|_{H_k^1(\Omega)}\|\psi\|_{H_k^1(\Omega)}. \tag{117}$$

Similarly, we have

$$-\int_0^1 \partial_x(k(x)\partial_x\varphi)\varphi dx = \int_0^1 k(x)(\partial_x\varphi)^2 dx. \tag{118}$$

Then, for all $\gamma \ge 1$ and $\alpha < 1$, we have

$$-\int_0^1 \partial_x(k(x)\partial_x\varphi)\psi dx + \gamma\int_0^1 \varphi^2 dx = \int_0^1 k(x)(\partial_x\varphi)^2 dx + \gamma\int_0^1 \varphi^2 dx \ge \alpha\|\varphi\|_{H_k^1(\Omega)}^2. \tag{119}$$

Then, applying Theorem 2, system (110) is null controllable for every time $T > a_1 + A - a_2$.

4.3. Nonlinear Population Dynamics Models with Degenerate Diffusion (the Nondivergence Form). We consider the

following nonlinear population dynamics with degenerate diffusion (see [5]):

$$\begin{cases} \dfrac{\partial y}{\partial t}+\dfrac{\partial y}{\partial a}-k(x)\partial^2_{xx}y+\mu_0(a)y=1_{(a_1,a_2)}\mathbf{B}u, & \text{in } Q=\Omega\times(0,A)\times(0,T),\\ y(0,a,t)=y(1,a,t)=0, & (0,A)\times(0,T),\\ y(x,0,s,t)=\Phi\left(\int_0^A \beta y\mathrm{d}a\right)\int_0^A \beta y\mathrm{d}a, & \text{in } \Omega\times(0,T),\\ y(x,a,s,0)=y_0, & \text{in } \Omega\times(0,A), \end{cases} \tag{120}$$

where k is the nonnegative function in $\Omega=(0,1)$ and degenerates. Moreover, $\mathbf{B}=\chi_{(l_1,l_2)}$.

Let us set the state and the control space as follows:

$$\begin{aligned} H&=L^2_{1/k}(\Omega)=\left\{\varphi\in L^2(\Omega)\mid \int_0^1\frac{\varphi^2}{k}\mathrm{d}x<n+q\infty\right\},\\ V&=H^1_{1/k}(\Omega)=L^2_{1/k}(\Omega)\cap H^1_0(\Omega),\\ H^2_{1/k}(\Omega)&=\left\{\varphi\in H^1_{1/k}(\Omega)\mid \quad tkn\varphi^2_{xx}q\in hL^2_{1/k}(\Omega)\right\}, \end{aligned} \tag{121}$$

with the following norms:

$$\begin{aligned} \|\varphi\|^2_{H^1_k}&=\left\|\frac{1}{\sqrt{k}}\varphi\right\|^2_{L^2(\Omega)}+\left\|\partial_x\varphi\right\|^2_{L^2(\Omega)},\\ \|\varphi\|^2_{H^2_k}&=\|\varphi\|^2_{H^1_k(\Omega)}+\left\|k\partial^2_{xx}\varphi\right\|^2_{L^2(\Omega)}. \end{aligned} \tag{122}$$

We consider the unbounded operator A on $H=L^2_{1/k}(\Omega)$ defined by

$$\begin{aligned} \mathbf{D}(\mathbf{A})&=H^2_{1/k}(\Omega),\\ \mathbf{A}\psi&=k(x)\partial^2_{xx}\psi. \end{aligned} \tag{123}$$

Proposition 6 (see [16]). *The operator* $\mathbf{A}$: $\mathbf{D}(\mathbf{A})\longrightarrow L^2(\Omega)$ *is closed, self-adjoint, and negative with dense domain.*

Hence, $\mathbf{A}$ is the infinitesimal generator of a strongly continuous semigroup $e^{t\mathbf{A}}$ on $L^2_{1/k}(\Omega)$.

The function $k\in C^0([0,1])\cap C^3(0,1)$ is such that

> The function $(xk_x/k)\in L^\infty(0,\varepsilon)$, and there exists $M_1\in(0,2)$ and $R_1>0$ such that $(xk_x/k)\le M_1$ and $|\partial_{xx}(xk_x/k)|\le R_1(1/k(x))$ for all $x\in(0,\epsilon)$.
>
> The function $((x-1)k_x/k)\in L^\infty(1-\varepsilon,1)$, and there exists $M_2\in(0,2)$ and $R_2>0$ such that $((x-1)k_x/k)\le M_2$ and $|\partial_{xx}((x-1)k_x/k)|\le C(1/k(x))$ for all $x\in(1-\varepsilon,1)$.

Under the above assumptions, the pair $(\mathbf{A},\mathbf{B})$ is observable for every time $T>0$ (see [16]).

Moreover, the operator $\mathbf{A}$ verifies assumption (H_3), where $V=H^1_{1/k}(\Omega)$ and $H=L^2_{1/k}(\Omega)$.

Indeed, let φ and $\psi\in H^1_{1/k}(\Omega)$, and we have

$$\left|\langle k(x)\partial^2_{xx}\varphi,\psi\rangle_{(H^1_{1/k})',H^1_{1/k}}\right|=\left|\int_0^1\varphi_x\psi_x\mathrm{d}x\right|. \tag{124}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\left|\langle k(x)\partial^2_{xx}\varphi,\psi\rangle_{(H^1_{1/k})',H^1_{1/k}}\right|\le\left\|\partial_x\varphi\right\|_{L^2(\Omega)}\left\|\partial_x\psi\right\|_{L^2(\Omega)}. \tag{125}$$

Using the Hardy–Poincaré inequality,

$$\text{for all } \varphi\in H^1_{1/k}(\Omega),\quad \exists C>0 \text{ such that } \int_0^1\frac{\varphi^2}{k}\mathrm{d}x\le C\int_0^1\varphi_x^2\mathrm{d}x, \tag{126}$$

we obtain

$$\left|\langle k(x)\partial^2_{xx}\varphi,\psi\rangle_{(H^1_{1/k})',H^1_{1/k}}\right|\le\|\varphi\|_{H^1_{1/k}(\Omega)}\|\psi\|_{H^1_{1/k}(\Omega)}. \tag{127}$$

Similarly, we have

$$-\langle k(x)\partial^2_{xx}\varphi,\varphi\rangle_{(H^1_{1/k})',H^1_{1/k}}=\int_0^1(\partial_x\varphi)^2\mathrm{d}x. \tag{128}$$

Then, for all $\gamma\ge1$ and $\alpha<1$, we have

$$-\langle k(x)\partial^2_{xx}\varphi,\varphi\rangle_{(H^1_{1/k})',H^1_{1/k}}+\gamma\int_0^1\frac{\varphi^2}{k}\mathrm{d}x=\int_0^1(\partial_x\varphi)^2\mathrm{d}x+\gamma\int_0^1\frac{\varphi^2}{k}\mathrm{d}x\ge\alpha\|\varphi\|^2_{H^1_{1/k}(\Omega)}. \tag{129}$$

Then, applying Theorem 2, we obtain this result.

Proposition 7. *By applying Theorem 2, for* $y_0 \in L^2(0,T; L^2_{1/k}(\Omega))$ *and for every* $T > a_1 + A - a_2$*, there exists* $u \in L^2((0,A)\times(0,T); L^2_{1/k}(\Omega))$ *such that the solution of system (120) verify*

$$y(.T) = 0, \quad \text{a.e in } \Omega \times (0,A). \tag{130}$$

5. Conclusion

Considering a nonlinear infinite dimensional system obtained by grafting an operator $\mathbf{A}$ and an age structure, we show there is a time T dependent on the constraints on the age and the observability minimal time T' of the pair $(\mathbf{A}, \mathbf{B})$ ($\mathbf{B}$ is the control operator) from which the system is null controllable. In application, the spatial diffusion operator and two spatial degenerate diffusion operators are provided.

In the desire to consolidate even more the result obtained in this article, we will think in our future work of the case of controllability with positivity constraint and also to establish algorithms allowing to calculate the control from its characterization.

These outstanding questions are therefore in this order:

Controllability with positivity constraints: we will be interested in controllability with positivity constraint.

Numerical implementation: for a given fertility rate β, the mortality rate 'μ, the initial condition y_0, and a positive parameter $\varepsilon > 0$, how to determine a numerical algorithm allowing to determine the ϵ-approximate null control function h?

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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