An Iterative Method for Solving Split Monotone Variational Inclusion Problems and Finite Family of Variational Inequality Problems in Hilbert Spaces

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The purpose of this paper is to study the convergence analysis of an intermixed algorithm for finding the common element of the set of solutions of split monotone variational inclusion problem (SMIV) and the set of a finite family of variational inequality problems. Under the suitable assumption, a strong convergence theorem has been proved in the framework of a real Hilbert space. In addition, by using our result, we obtain some additional results involving split convex minimization problems (SCMPs) and split feasibility problems (SFPs). Also, we give some numerical examples for supporting our main theorem.

1. Introduction

Let $H_1$ and $H_2$ be real Hilbert spaces whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, and let $C$, $Q$ be nonempty closed convex subsets of $H_1$ and $H_2$, respectively. For a mapping $S: C \to C$, we denoted by $F(S)$ the set of fixed points of $S$ (i.e., $F(S) = \{x \in C: Sx = x\}$). Let $A: C \to H$ be a nonlinear mapping. The variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C,$$  \hspace{1cm} (1)

and the solution set of problem (1) is denoted by $VI(C, A)$. It is known that the variational inequality, as a strong and great tool, has already been investigated for an extensive class of optimization problems in economics and equilibrium problems arising in physics and many other branches of pure and applied sciences. Recall that a mapping $A: C \to C$ is said to be $\alpha$-inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$  \hspace{1cm} (2)

A multivalued mapping $M: H_1 \to 2^{H_1}$ is called monotone if for all $x, y \in H_1$, $\langle x - y, u - v \rangle \geq 0$, for any $u \in Mx$ and $v \in My$. A monotone mapping $M: H_1 \to 2^{H_1}$ is maximal if the graph $G(M)$ for $M$ is not properly contained in the graph of any other monotone mapping. It is generally known that $M$ is maximal if and only if for $(x, u) \in H_1 \times H_1$, $(x - y, u - v) \geq 0$ for all $(y, v) \in G(M)$ implies $u \in Mx$. Let $M: H_1 \to 2^{H_1}$ be a multivalued maximal monotone mapping. The resolvent mapping $J_M^\lambda: H_1 \to H_1$ associated with $M$ is defined by

$$J_M^\lambda(x) = (I + \lambda M)^{-1}(x), \quad \forall x \in H_1, \lambda > 0,$$  \hspace{1cm} (3)

where $I$ stands for the identity operator on $H_1$. We note that for all $\lambda > 0$, the resolvent $J_M^\lambda$ is single-valued, nonexpansive, and firmly nonexpansive.
In 2011, Moudafi [1] introduced the following split monotone variational inclusion problem (SMVI):

\[ \text{find } x^* \in H_1 \text{ such that } \theta \in A_1(x^*) + M_1(x^*) \]  

and such that \( y^* = Tx^* \in H_2 \) solves \( \theta \in A_2(y^*) + M_2(y^*) \). 

where \( \theta \) is the zero vector in \( H_1 \) and \( H_2, M_1: H_1 \rightarrow 2^{H_1} \) and \( M_2: H_2 \rightarrow 2^{H_2} \) are multivalued mappings on \( H_1 \) and \( H_2 \), \( A_1: H_1 \rightarrow H_1 \) and \( A_2: H_2 \rightarrow H_2 \) are two given single-valued mappings, and \( T: H_1 \rightarrow H_2 \) is a bounded linear operator with adjoint \( T^* \) of \( T \). We note that if (4) and (5) are considered separately, we have that (4) is a variational inclusion problem with its solution set \( \text{VI}(H_1, A_1, M_1) \) and (5) is a variational inclusion problem with its solution set \( \text{VI}(H_1, A_2, M_2) \). We denoted the set of all solutions of (SMVI) by \( \Omega = \{ x^* \in \text{VI}(H_1, A_1, M_1); x^* \in \text{VI}(H_2, A_2, M_2) \} \).

It is worth noticing that by taking \( M_1 = N_C \) and \( M_2 = N_Q \) normal cones to closed convex sets \( C \) and \( Q \) then (SMVI) (4) and (5) reduce to the split variational inequality problem (SVIP) that was introduced by Censor et al. [2]. In [1], they mentioned that (SMVI) (4) and (5) contain many special cases, such as split minimization problem (SMP), split minimax problem (SMMC), and split equilibrium problem (SEP). Some related works can be found in [1, 3–10].

For solving (SMVI) (4) and (5), Modafi [1] proposed the following algorithm.

**Algorithm 1.** Let \( \lambda > 0 \), \( x_0 \in H_1 \), and the sequence \( \{ x_n \} \) be generated by

\[ x_{n+1} = \lambda (1 - \lambda f)(x_n + \gamma T^* (j_{H_1} (I - \lambda g) - I)Tx_n), \quad n \in \mathbb{N}, \tag{6} \]

where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are two sequences of the real number in \((0, 1)\), \( T, S: C \rightarrow C \) are \( \lambda \)-strictly pseudo-contractions, \( f: C \rightarrow H \) is a \( \rho_1 \)-contraction, \( g: C \rightarrow H \) is a \( \rho_2 \)-contraction, and \( k \in (0, 1 - \lambda) \) is a constant.

Under some control conditions, they proved that the sequence \( \{ x_n \} \) converges strongly to \( P_{F(T)} f(y^*) \) and \( \{ y_n \} \) converges strongly to \( P_{F(S)} f(x^*) \), respectively, where \( x^* \in F(T) \), \( y^* \in F(S) \), and \( P_{F(T)} \) and \( P_{F(S)} \) are the metric projection of \( H \) onto \( F(T) \) and \( F(S) \), respectively. After that, many authors have developed and used this algorithm to solve the fixed-point problems of many nonlinear operators in real Hilbert spaces (see for example [21–27]). Question: can we prove the strong convergence theorem of two sequences of split monotone variational inclusion problems and fixed-point problems of nonlinear mappings in real Hilbert spaces?

Theorem 1 (see [1]). Let \( H_1, H_2 \) be real Hilbert spaces. Let \( T: H_1 \rightarrow H_2 \) be a bounded linear operator with adjoint \( T^* \).

For \( i = 1, 2 \), let \( A_i: H_i \rightarrow H_i \) be \( \alpha_i \)-inverse strongly monotone with \( \alpha = \min[\alpha_1, \alpha_2] \) and let \( M_i: H_i \rightarrow 2^{H_i} \) be two maximal monotone operators. Then, the sequence generated by (6) converges weakly to an element \( x^* \in \Omega \) provided that \( \Omega \neq \emptyset \), \( \lambda \in (0, 2\alpha) \), and \( \gamma \in (1, 1/L) \) with \( L \) being the spectral radius of the operator \( T^*T \).

Since then, because of a lot of applications of (SMVI), it receives much attention from many authors. They presented many approximation methods for solving (SMVI) (4) and (5). Also the iterative methods for solving (SMVIP) (4) and (5) and fixed-point problems of some nonlinear mappings have been investigated (see [11–19]).

On the other hand, Yao et al. [20] presented an intermixed Algorithm 1.3 for two strict pseudo-contractions in real Hilbert spaces. They also showed that the suggested algorithms converge strongly to the fixed points of two strict pseudo-contractions, independently. As a special case, they can find the common fixed points of two strict pseudo-contractions in Hilbert spaces (i.e., a mapping \( S: C \rightarrow C \) is said to be \( \kappa \)-strictly pseudo-contractive if there exists a constant \( \kappa \in (0, 1) \) such that \( \| Sx - Sy \|^2 \leq \| x - y \|^2 + k \| (I - S)x - (I - S)y \| \) for all \( x, y \in C \).

Algorithm 2. For arbitrarily given \( x_0, y_0 \in C \), let the sequences \( \{ x_n \} \) and \( \{ y_n \} \) be generated iteratively by

\[
\begin{align*}
x_{n+1} & = (1 - \beta_n)x_n + \beta_n P_{C} \{ \alpha_n f(y_n) \} + (1 - k - \alpha_n)x_n + kTx_n, \quad n \geq 0, \\
y_{n+1} & = (1 - \beta_n)y_n + \beta_n P_{C} \{ \alpha_n f(x_n) \} + (1 - k - \alpha_n)y_n + kSy_n, \quad n \geq 0, \tag{7}
\end{align*}
\]

The purpose of this paper is to modify an intermixed algorithm to answer the question above and prove a strong convergence theorem of two sequences for finding a common element of the set of solutions of (SMVI) (4) and (5) and the set of solutions of a finite family of variational inequality problems in real Hilbert spaces. Furthermore, by applying our main result, we obtain some additional results involving split convex minimization problems (SCMPs) and split feasibility problems (SFPs). Finally, we give some numerical examples for supporting our main theorem.

2. Preliminaries

Let \( H \) be a real Hilbert space and \( C \) be a nonempty closed convex subset of \( H \). We denote the strong convergence of \( \{ x_n \} \) to \( x \) and the weak convergence of \( \{ x_n \} \) to \( x \) by notations \( x_n \rightharpoonup x \) as \( n \to \infty \) and \( x_n \rightharpoonup x \) as \( n \to \infty \),
respectively. For each \( x, y, z \in H \) and \( \alpha, \beta, \gamma \in [0, 1] \) with \( \alpha + \beta + \gamma = 1 \), we have

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,
\]
\[
\|ax + \beta y + \gamma z\|^2 = a\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - a\beta\|x - y\|^2 - a\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.
\] (8)

### Definition 1
Let \( H \) be a real Hilbert space and \( C \) be a closed convex subset of \( H \). Let \( S: C \to H \) be a mapping. Then, \( S \) is said to be

1. **Monotone**, if \( \langle Sx - Sy, x - y \rangle \geq 0, \forall x, y \in H \)
2. **Firmly nonexpansive**, if \( \langle Sx - Sy, x - y \rangle \geq \|Sx - Sy\|^2, \forall x, y \in H \)
3. **Lipschitz continuous**, if there exists a constant \( L > 0 \) such that \( \|Sx - Sy\| \leq L\|x - y\|, \forall x, y \in H \)
4. **Nonexpansive**, if \( \|Sx - Sy\| \leq \|x - y\|, \forall x, y \in H \)

It is well known that if \( S \) is \( \alpha \)-inverse strongly monotone, then it is \( 1/\alpha \)-Lipschitz continuous and every nonexpansive mapping \( S \) is 1-Lipschitz continuous. We note that if \( S: H \to H \) is a nonexpansive mapping, then it satisfies the following inequality (see Theorem 3 in [28] and Theorem 1 in [29]):

\[
\langle Sy - Sx, (I - S)x - (I - S)y \rangle \leq \frac{1}{2}\| (I - S)x - (I - S)y \|^2,
\]
\( \forall x, y \in H. \) (9)

Particularly, for every \( x \in H \) and \( y \in F(S) \), we have

\[
\langle y - Sx, (I - S)x \rangle \leq \frac{1}{2}\| (I - S)x \|^2.
\] (10)

For every \( x \in H \), there is a unique nearest point \( P_Cx \) in \( C \) such that

\[
\|x - P_Cx\| \leq \|x - y\|, \text{ } \forall y \in C.
\] (11)

Such an operator \( P_C \) is called the **metric projection of \( H \) onto \( C \)**.

### Lemma 1 (see [30])
For a given \( z \in H \) and \( u \in C \),

\[
u = P_Cz \iff \langle u - z, v - u \rangle \geq 0, \text{ } \forall v \in C.
\] (12)

Furthermore, \( P_C \) is a firmly nonexpansive mapping of \( H \) onto \( C \) and satisfies

\[
\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y \rangle, \text{ } \forall x, y \in H.
\] (13)

Moreover, we also have the following lemma.

### Lemma 2 (see [31])
Let \( H \) be a real Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \), and let \( A \) be a mapping of \( C \) into \( H \). Let \( u \in C \). Then, for \( \lambda > 0 \),

\[
u \in VI(C, A) \iff u = P_C(\lambda - \lambda A)u,
\] (14)

where \( P_C \) is the metric projection of \( H \) onto \( C \).

### Lemma 3
Let \( C \) be a nonempty closed and convex subset of a real Hilbert space \( H \). For every \( i = 1, 2, \ldots, N \), let \( A_i: C \to H \) be the \( \alpha_i \)-inverse strongly monotone with \( \alpha = \min_{i=1,2,\ldots,N} \{\alpha_i\} \). If \( \cap_{i=1}^N VI(C, A_i) \neq \emptyset \), then

\[
VI \left( C, \sum_{i=1}^N a_i A_i \right) = \bigcap_{i=1}^N VI(C, A_i),
\] (15)

where \( 0 < a_i < 1 \) for all \( i = 1, 2, \ldots, N \) and \( \sum_{i=1}^N a_i = 1 \). Moreover, \( I - \lambda \sum_{i=1}^N a_i A_i \) is a nonexpansive mapping for all \( \lambda \in (0, 2\alpha) \).

**Proof.** By Lemma 4.3 of [32], we have that \( VI(C, \sum_{i=1}^N a_i A_i) = \bigcap_{i=1}^N VI(C, A_i) \). Let \( \lambda \in (0, 2\alpha) \) and let \( x, y \in C \). As the same argument as in the proof of Lemma 8 in [16], we have \( I - \lambda \sum_{i=1}^N a_i A_i \) is nonexpansive.

### Lemma 4 (see [33])
Let \( H \) be a real Hilbert space, \( A: H \to H \) be a single-valued nonlinear mapping, and \( M: H \to H^* \) be a set-valued mapping. Then, a point \( u \in H \) is a solution of variational inclusion problem if and only if

\[
u = J_{\lambda}^M (I - \lambda A)u, \forall \lambda > 0, \text{ i.e.,}
\]

\[
VI(H, A, M) = F(J_{\lambda}^M (I - \lambda A)), \text{ } \forall \lambda > 0.
\] (16)

Furthermore, if \( A \) is \( \alpha \)-inverse strongly monotone and \( \lambda \in (0, 2\alpha) \), then \( VI(H, A, M) \) is a closed convex subset of \( H \).

### Lemma 5 (see [33])
Let \( H \) be a real Hilbert space, \( A: H \to H \) be a single-valued, nonexpansive, and 1-inverse strongly monotone for all \( \lambda > 0 \).

The following two lemmas are the particular case of the above Lemmas 7 and 8 in [16].

### Lemma 6 (see [16])
For every \( i = 1, 2, \ldots, N \), let \( H_i \) be real Hilbert spaces, let \( M_i: H_i \to H_i \) be a multivalued maximal monotone mapping, and let \( A_i: H_i \to H_i \) be an \( \alpha_i \)-inverse strongly monotone mapping. Let \( T_i: H_i \to H_i \) be a bounded linear operator with adjoint \( T_i^* \) of \( T_i \), and let \( G: H_i \to H_i \) be a mapping defined by \( G(x) = J_{\lambda_i^1}^N (I - \lambda_i A_i) (x - yT_i^* (I - J_{\lambda_i^1}^N (I - \lambda_i A_i)) Tx) \), for all \( x \in H_i \). Then, \( \| (G - G_y)^\perp \| \leq \|x - y\| \leq \gamma \|x - y\| L \) for all \( x, y \in H_i \), where \( L \) is the spectral radius of the operator \( T_i T_i^* \).

Furthermore, if \( \gamma < 1 \), then \( G \) is a nonexpansive mapping.
Lemma 7 (see [16]). Let $H_1$ and $H_2$ be Hilbert spaces. For $i = 1, 2$, let $M_i: H_i \rightarrow 2^{H_i}$ be a multivalued maximal monotone mapping and let $A_i: H_i \rightarrow H_i$ be an $\alpha_i$-inverse strongly monotone mapping. Let $T: H_1 \rightarrow H_2$ be a bounded linear operator with adjoint $T^*$. Assume that $\Omega \neq \emptyset$. Then, $x^* \in \Omega$ if and only if $x^* = G(x^*)$, where $G: H_1 \rightarrow H_1$ is a mapping defined by
\[
G(x) = f_{\lambda_i}^{M_i}(I - \lambda_i A_i)(x - \gamma T^*(I - f_{\lambda_i}^{M_i}(I - \lambda_i A_i))Tx),
\]
for all $x \in H_1$, $\lambda_i \in (0, 2\alpha_i)$, $\lambda_2 \in (0, 2\alpha_2)$, and $0 < \gamma < 1/L$, where $L$ is the spectral radius of the operator $T^*T$.

Next, we give an example to support Lemma 7.

Example 1. Let $\mathbb{R}$ be a set of real numbers and $H_1 = H_2 = \mathbb{R}^2$, and let $\langle \cdot, \cdot \rangle: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be inner product defined by $\langle x, y \rangle = x \cdot y = x_1y_1 + x_2y_2$, for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and the usual norm $\| \cdot \|: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $\|x\| = \sqrt{x^2_1 + x^2_2}$, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let $T: H_1 \rightarrow H_2$ be defined by $Tx = (2x_1, 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $T^*: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $T^*z = (2z_1, 2z_2)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let $M_1, M_2: \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$ be defined by $M_1 x = [(3x_1 - 5, 3x_2 - 5)]$ and $M_2 x = [(x_1/3 - 2, x_2/3 - 2)]$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A_1, A_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $A_1 x = ((x_1 - 2)/2, (x_2 - 2)/2)$ and $A_2 x = ((x_1 - 2)/3, (x_2 - 2)/3)$, respectively, for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then, $(2, 2)$ is a fixed point of $G$. That is, $(2, 2) \in F(G)$.

Proof. It is obvious to see that $\Omega = \{(2, 2)\}$, $A_2$ is $2$ inverse strongly monotone, and $A_2$ is $3$ inverse strongly monotone. Choose $\lambda_1 = 1/3$. Since $M_1 x = [(3x_1 - 5, 3x_2 - 5)]$ and the resolvent of $M_1, f_{\lambda_1}^{M_1} x = (I + \lambda_1 M_1)^{-1} x$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, we obtain that $f_{\lambda_1}^{M_1} x = \frac{x + 5}{6}$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. (18)

Choose $\lambda_2 = 1$. Since $M_2 x = [(x_1/3 - 2, x_2/3 - 2)]$ and the resolvent of $M_2, f_{\lambda_2}^{M_2} x = (I + \lambda_2 M_2)^{-1} x$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, we obtain that $f_{\lambda_2}^{M_2} x = \frac{3x + 3}{4}$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. (19)

Since the spectral radius of the operator $T^*T$ is 4, we choose $\gamma = 0.1$. Then, from (18) and (19), we get that
\[
\begin{align*}
x_{n+1} &= \delta_n x_n + \sigma_n P_C \left( I - \mu_n \sum_{i=1}^{N} \alpha_i^2 B_i^T B_i \right) x_n + \eta_n \left( \alpha_n f(y_n) + (1 - \alpha_n)G^x x_n \right), \\
y_{n+1} &= \delta_n y_n + \sigma_n P_C \left( I - \mu_n \sum_{i=1}^{N} \alpha_i^2 B_i^T B_i \right) y_n + \eta_n \left( \alpha_n g(y_n) + (1 - \alpha_n)G^y y_n \right),
\end{align*}
\]

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then, by Lemma 7, we have that $(2, 2) \in F(G)$. \hfill $\Box$

Lemma 8 (see [34]). Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying $s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 0$ where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
\[
\begin{align*}
(1) & \sum_{n=0}^{\infty} \alpha_n = \infty, \\
(2) & \limsup_{n \rightarrow \infty} \alpha_n / \delta_n \leq 0 \text{ or } \sum_{n=0}^{\infty} |\delta_n| < \infty. \text{ Then } \lim_{n \rightarrow \infty} s_n = 0.
\end{align*}
\]

3. Main Results

In this section, we introduce an iterative algorithm of two sequences which depend on each other by using the intermixed method. Then, we prove a strong convergence theorem for solving two split monotone variational inclusion problems and a finite family of variational inequality problems.

Theorem 2. Let $H_1$ and $H_2$ be Hilbert spaces, and let $C$ be a nonempty closed convex subset of $H_1$. Let $T: H_1 \rightarrow H_2$ be a bounded linear operator, and let $f, g: H_1 \rightarrow H_1$ be $\rho_f, \rho_g$-contraction mappings with $\rho = \max\{\rho_f, \rho_g\}$. For $i = 1, 2$, let $M_i^\gamma, M_i^{\rho_i}: H_i \rightarrow 2^{H_i}$ be multivalued maximal monotone mappings and let $A_i, A_i^\gamma: H_i \rightarrow H_i$ be $\alpha_i, \alpha_i^\gamma$ inverse strongly monotone mappings, respectively. For $i = 1, 2, \ldots, N$, let $B_i, B_i^\gamma: H_1 \rightarrow H_1$ be $\beta_i, \beta_i^\gamma$ inverse strongly monotone mappings, respectively. Let $G_i^x, G_i^y: H_1 \rightarrow H_1$ be defined by $G_i^x x = f_{\lambda_i}^{M_i^\gamma}(I - \lambda_i^\gamma A_i^\gamma)(x - \gamma T^*(I - f_{\lambda_i}^{M_i^\gamma}(I - \lambda_i^\gamma A_i^\gamma))Tx), \forall x \in H_1$, and $G_i^y y = f_{\lambda_i}^{M_i^\gamma}(I - \lambda_i^\gamma A_i^\gamma)(y - \gamma T^*(I - f_{\lambda_i}^{M_i^\gamma}(I - \lambda_i^\gamma A_i^\gamma))Ty), \forall y \in H_1$, respectively, where $\lambda_i^\gamma \in (0, 2\alpha_i^\gamma)$, $\lambda_i^\gamma \in (0, 2\alpha_i^\gamma)$, and $0 < \gamma, \gamma < 1/L$ with $L$ being a spectral radius of $T^*T$. Assume that $\mathcal{F}^x = \Omega \cap (\cap_{i=1}^{N} \text{VI}(C, B_i^\gamma)) \neq \emptyset$ and $\mathcal{F}^y = \Omega \cap (\cap_{i=1}^{N} \text{VI}(C, B_i^\gamma)) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1, y_1 \in H_1$ and
for all $n \geq 1$ where $\{\delta_n, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0, 1]$ with $\delta_n + \sigma_n + \eta_n = 1$, $\{a_1^\delta, a_2^\delta, \ldots, a_N^\delta\}, \{a_1^\sigma, a_2^\sigma, \ldots, a_N^\sigma\} \subseteq (0, 1)$, and $\{\mu_n^\delta, \mu_n^\sigma\} \subseteq (0, \infty)$. Assume the following condition holds:

1. $\sum_{n=1}^{\infty} \mu_n^\delta < \infty, \sum_{n=1}^{\infty} \mu_n^\sigma < \infty,$ and $0 < a < \mu_n^\delta \leq 2x, 0 < b \leq \mu_n^\sigma \leq 2y,$ for some $a, b \in \mathbb{R}$. 
2. $\sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \alpha_n = 0.$ 
3. $\sum_{i=1}^{N} a_i^\delta = \sum_{i=1}^{N} a_i^\sigma = 1.$ 
4. $0 < \frac{\pi}{\alpha} \leq \delta_n, \sigma_n, \eta_n \leq \frac{b}{1}$, for all $n \in \mathbb{N}$, for some $\frac{\pi}{\alpha}, \frac{b}{1} > 0.$

5. $\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| \sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|$, and $\sum_{n=1}^{\infty} |\eta_{n+1} - \eta_n| < \infty$. Then, $\{x_n\}$ converges strongly to $\bar{x} = P_{\mathcal{F}} f(\bar{y})$ and $\{y_n\}$ converges strongly to $\bar{y} = P_{\mathcal{F}} g(\bar{x}).$

Proof. We divided the proof into five steps.

**Step 1.** We will show that $\{x_n\}$ and $\{y_n\}$ are bounded. Let $x^* \in \mathcal{F}^x$ and $y^* \in \mathcal{F}^y$. Then, from Lemma 7 and Lemma 6, we get

$$\|G_n x_n - x^*\| = \|M_n^x (I - \lambda_n^x A_n^x)(x_n - y^* T^* (I - \lambda_n^x A_n^x) T x_n) - x^*\|,$$

$$\leq \|x_n - x^*\|. \tag{22}$$

From (21), Lemma 3, and (22), we have

$$\|x_{n+1} - x^*\| = \|\delta_n x_n + \sigma_n P_C \left(1 - \mu_n^\delta \sum_{i=1}^{N} a_i^\delta B_i \right) x_n + \eta_n f(y_n) + (1 - \alpha_n) G_n x_n - x^*\|$$

$$\leq \delta_n \|x_n - x^*\| + \sigma_n \|P_C \left(1 - \mu_n^\delta \sum_{i=1}^{N} a_i^\delta B_i \right) x_n - x^*\| + \eta_n \|f(y_n) + (1 - \alpha_n) G_n x_n - x^*\|$$

$$\leq \delta_n \|x_n - x^*\| + \sigma_n \|x_n - x^*\| + \eta_n \|f(y_n) - x^*\| + (1 - \alpha_n) \|G_n x_n - x^*\|$$

$$\leq (1 - \eta_n) \|x_n - x^*\| + \eta_n \|f(y_n) - x^*\| + (1 - \alpha_n) \|G_n x_n - x^*\|$$

$$\leq (1 - \eta_n) \|x_n - x^*\| + \eta_n \|f(y_n) - f(x^*) + f(x^*) - x^*\| + (1 - \alpha_n) \|x_n - x^*\|$$

$$\leq (1 - \eta_n) \|x_n - x^*\| + \eta_n \|f(y_n) - f(x^*)\| + (1 - \alpha_n) \|x_n - x^*\|$$

$$\leq (1 - \eta_n) \|x_n - x^*\| + \eta_n \|f(y_n) - f(x^*)\| + (1 - \alpha_n) \|x_n - x^*\|$$

Similarly, from definition of $y_n$, we have

$$\|y_{n+1} - y^*\| \leq (1 - \eta_n \alpha_n) \|y_n - y^*\| + \eta_n \alpha_n \|x_n - x^*\| + \eta_n \alpha_n \|g(x^*) - y^*\|. \tag{24}$$

Hence, from (23) and (24), we obtain

$$\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \leq (1 - \eta_n \alpha_n) \left(\|x_n - x^*\| + \|y_n - y^*\|\right) + \eta_n \alpha_n \left(\|x_n - x^*\| + \|y_n - y^*\|\right)$$

$$+ \eta_n \alpha_n \left(\|f(y^*) - x^*\| + \|g(x^*) - y^*\|\right)$$

$$= (1 - (\eta_n \alpha_n)) \left(\|x_n - x^*\| + \|y_n - y^*\|\right) + \eta_n \alpha_n \left(\|f(y^*) - x^*\| + \|g(x^*) - y^*\|\right). \tag{25}$$

By induction, we have
\[ \|x_n - x^*\| + \|y_n - y^*\| \leq \max \left\{ \|x_1 - x^*\| + \|y_1 - y^*\|, \frac{\|f(y^*) - x^*\| + \|g(x^*) - y^*\|}{1 - \rho} \right\}, \quad (26) \]

for every \( n \in \mathbb{N} \). Thus, \( \{x_n\} \) and \( \{y_n\} \) are bounded.

**Step 2.** We will show that \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} \|y_{n+1} - y_n\| = 0 \). Put
\[ u_n = P_C \left( I - \mu_n \sum_{i=1}^N a_i^x B_i^x \right) x_n, \]
\[ v_n = P_C \left( I - \mu_n \sum_{i=1}^N a_i^y B_i^y \right) y_n, \]
\( z_n = \alpha_n f(y_n) + (1 - \alpha_n) \tilde{G}^{x} x_n \), and \( w_n = \alpha_n f(x_n) + (1 - \alpha_n) \tilde{G}^{y} y_n \), for all \( n \geq 1 \). From Lemma 6, we have

\[ \|z_n - z_{n-1}\| = \left\| \alpha_n f(y_n) + (1 - \alpha_n) \tilde{G}^{x} x_n - \left( \alpha_{n-1} f(y_{n-1}) + (1 - \alpha_{n-1}) \tilde{G}^{x} x_{n-1} \right) \right\|, \]
\[ \leq \alpha_n \|f(y_n) - f(y_{n-1})\| + (1 - \alpha_n) \|\tilde{G}^{x} x_n - \tilde{G}^{x} x_{n-1}\| + \|\alpha_n - \alpha_{n-1}\| \|f(y_{n-1})\| + \|\alpha_n - \alpha_{n-1}\| \|\tilde{G}^{x} x_{n-1}\| \]
\[ \leq \alpha_n \rho_f \|y_n - y_{n-1}\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + \|\alpha_n - \alpha_{n-1}\| \|f(y_{n-1})\| + \|\alpha_n - \alpha_{n-1}\| \|\tilde{G}^{x} x_{n-1}\| \]
\[ \leq \alpha_n \rho_f \|y_n - y_{n-1}\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + \|\alpha_n - \alpha_{n-1}\| \|f(y_{n-1})\| + \|\tilde{G}^{x} x_{n-1}\|. \quad (27) \]

By applying Lemma 3, we get that

\[ \|u_n - u_{n-1}\| = \left\| P_C \left( I - \mu_n \sum_{i=1}^N a_i^x B_i^x \right) x_n - P_C \left( I - \mu_{n-1} \sum_{i=1}^N a_i^x B_i^x \right) x_{n-1} \right\|, \]
\[ \leq \|x_n - x_{n-1}\| + \left\| P_C \left( I - \mu_n \sum_{i=1}^N a_i^x B_i^x \right) x_n - P_C \left( I - \mu_{n-1} \sum_{i=1}^N a_i^x B_i^x \right) x_n \right\| \|a_i^x\| B_i^x \|x_{n-1}\| \]
\[ \leq \|x_n - x_{n-1}\| + \left\| \mu_n - \mu_{n-1} \right\| \sum_{i=1}^N \|a_i^x\| B_i^x \|x_{n-1}\|. \quad (28) \]

From the definition of \( \{x_n\} \), (27), and (28), we have

\[ \|x_{n+1} - x_n\| = \|\delta_{n} x_n + \sigma_n u_n + \eta_n z_n - (\delta_{n-1} x_{n-1} + \sigma_{n-1} u_{n-1} + \eta_{n-1} z_{n-1})\|, \]
\[ \leq \|\delta_{n} x_n - x_{n-1}\| + \|\delta_{n-1} x_{n-1}\| + \|\sigma_n u_n - u_{n-1}\| + \|\sigma_{n-1} u_{n-1}\| + \|\sigma_n - \sigma_{n-1}\| \|u_{n-1}\| + \|\eta_n z_n - z_{n-1}\| + \|\eta_n - \eta_{n-1}\| \|z_{n-1}\| \]
\[ \leq \|\delta_{n} x_n - x_{n-1}\| + \|\sigma_n - \sigma_{n-1}\| \|x_{n-1}\| + \|\sigma_n - \sigma_{n-1}\| \|u_{n-1}\| + \|\sigma_n - \sigma_{n-1}\| \|u_{n-1}\| + \|\eta_n z_{n-1}\| \]
\[ + \| \alpha_n \rho_f y_n - y_{n-1}\| + (1 - \alpha_n) \|x_n - x_{n-1}\| + \|\alpha_n - \alpha_{n-1}\| \|f(y_{n-1})\| + \|\tilde{G}^{x} x_{n-1}\| \]
\[ + \|\eta_n - \eta_{n-1}\| \|z_{n-1}\| \]
\[ = (1 - \eta_n \alpha_n) \|x_n - x_{n-1}\| + \eta_n \alpha_n \rho_f \|y_n - y_{n-1}\| + \|\sigma_n - \sigma_{n-1}\| \|x_{n-1}\| + \|\eta_n \alpha_n - \alpha_{n-1}\| \|f(y_{n-1})\| + \|\tilde{G}^{x} x_{n-1}\| \]
\[ + \|\eta_n - \eta_{n-1}\| \|z_{n-1}\|. \quad (29) \]

By the same argument as in (27) and (29), we also have
\[
\|\omega_n - \omega_{n-1}\| \leq \alpha_n \rho \|x_n - x_{n-1}\| + (1 - \alpha_n) \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left( \|g(x_{n-1})\| + \|\bar{G}^y y_{n-1}\| \right).
\]

\[
\|y_{n+1} - y_n\| \leq (1 - \eta_n \alpha_n) \|y_n - y_{n-1}\| + \eta_n \alpha_n \rho \|x_n - x_{n-1}\| + \sigma_n \left| \mu_n - \mu_{n-1} \right| \sum_{i=1}^{N} \alpha_i^2 \|B_i^y y_{n-1}\| + |\delta_n - \delta_{n-1}| \|y_{n-1}\| + |\sigma_n - \sigma_{n-1}| \left( \|u_{n-1}\| + \|v_{n-1}\| \right)
\]

\[
+ |\eta_n - \eta_{n-1}| \left( \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| \right) + \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| \right) \sum_{i=1}^{N} \alpha_i^2 \|B_i^y y_{n-1}\| + \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| \right).
\]

From (29) and (31), we obtain that

\[
\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \leq (1 - \rho) \eta_n \alpha_n \left( \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\| \right) + \sigma_n \left| \mu_n - \mu_{n-1} \right| \sum_{i=1}^{N} \alpha_i^2 \|B_i^y y_{n-1}\| + |\delta_n - \delta_{n-1}| \|x_{n-1}\| + |\eta_n - \eta_{n-1}| \left( \|x_{n-1}\| + \|y_{n-1}\| \right)
\]

From (32), conditions (1), (2), and (5), and Lemma 8, we obtain that

\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0,
\]

\[
\lim_{n \to \infty} \|y_{n+1} - y_n\| = 0.
\]
Then, we have
\[
\left\| x_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| \to 0 \text{ as } n \to \infty. \tag{37}
\]

Observe that
\[
x_{n+1} - x_n = a_n \left( P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - x_n \right) + \eta_n (x_n - x_n).
\tag{38}
\]

From (33) and (37), we have
\[
\| z_n - x_n \| \to 0 \text{ as } n \to \infty. \tag{39}
\]

By the same argument as above, we also have that
\[
\| w_n - x_n \| \to 0 \text{ as } n \to \infty. \tag{40}
\]

Note that

\[
\left\| z_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\| \leq \left\| z_n - x_n \right\| + \left\| x_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\|
+ \left\| P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\|
\leq \left\| z_n - x_n \right\| + \left\| x_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\| + \left\| x_n - z_n \right\|
= 2 \left\| z_n - x_n \right\| + \left\| x_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) x_n \right\|.
\tag{41}
\]

By (37) and (39), we get that
\[
\left\| z_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^x B_i^x \right) z_n \right\| \to 0 \text{ as } n \to \infty. \tag{42}
\]

By the same argument as (41), we also obtain
\[
\left\| w_n - P_C \left( I - \mu_n^+ \sum_{i=1}^{N} a_i^y B_i^y \right) w_n \right\| \to 0 \text{ as } n \to \infty. \tag{43}
\]

Consider
\[
\left\| x_{n+1} - x_n \right\| \leq \left\| x_n - x_n \right\| + \left\| x_n - z_n \right\| + \left\| z_n - \tilde{G}^x x_n \right\|
\]

It follows from (33) and (45) that
\[
\left\| x_n - \tilde{G}^x x_n \right\| \to 0 \text{ as } n \to \infty. \tag{47}
\]

Consider
\[
\left\| z_n - \tilde{G}^x z_n \right\| \leq \left\| z_n - x_n \right\| + \left\| x_n - \tilde{G}^x x_n \right\| + \left\| \tilde{G}^x x_n - \tilde{G}^x z_n \right\|
\leq 2 \left\| z_n - x_n \right\| + \left\| x_n - \tilde{G}^x x_n \right\|
\tag{48}
\]

From (39) and (47), we obtain
\[
\left\| z_n - \tilde{G}^x z_n \right\| \to 0 \text{ as } n \to \infty. \tag{49}
\]

Step 4. We will show that \( \limsup_{n \to \infty} \langle f (\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle \leq 0 \) and \( \limsup_{n \to \infty} \langle g (\tilde{x}) - \tilde{y}, z_n - \tilde{y} \rangle \leq 0 \), where \( \tilde{x} = P_{\tilde{F}} f (\tilde{y}) \) and \( \tilde{y} = P_{\tilde{G}} f (\tilde{x}) \). First, we take a subsequence \( \{ z_n \} \) of \( \{ z_n \} \) such that
\[
\limsup_{n \to \infty} \langle f (\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle = \lim_{k \to \infty} \langle f (\tilde{y}) - \tilde{x}, z_n - \tilde{x} \rangle.
\tag{51}
\]
Since \( \{x_n\} \) is bounded, there exists a subsequence \( x_{n_k} \) of \( \{x_n\} \) such that \( x_{n_k} \to q_1 \) as \( k \to \infty \). From (39), we get that \( z_{n_k} \to q_1 \). Next, we need to show that \( q_1 \in \mathcal{F}^x = \mathcal{F}^x \cap (\cap_{i=1}^N VI(C_i, B_i^r)) \). Assume that \( q_1 \notin \mathcal{F}^x \). By Lemma 7, we get that \( q_1 \notin G \). Applying Opial’s condition and (49), we get that

\[
\liminf_{k \to \infty} \| z_{n_k} - q_1 \| < \liminf_{k \to \infty} \| z_{n_k} - G^x q_1 \|
\leq \liminf_{n \to \infty} \| z_{n_k} - G^x z_{n_k} \| + \liminf_{k \to \infty} \| G^x z_{n_k} - G^x q_1 \|
\leq \liminf_{k \to \infty} \| z_{n_k} - q_1 \|.
\]

This is a contradiction. Thus, \( q_1 \in \mathcal{F}^x \).

Assume that \( q_1 \notin \cap_{i=1}^N VI(C, B_i^r) \). Then, from Lemma 3 and Lemma 2, we have \( q_1 \notin F(P_C (I - \mu_n^x \sum_{i=1}^N a_i B_i^r)) \). From Opial’s condition and (42), we obtain

\[
\liminf_{k \to \infty} \| z_{n_k} - q_1 \| < \liminf_{k \to \infty} \| z_{n_k} - P_C \left( I - \mu_n^x \sum_{i=1}^N a_i^z B_i^z \right) q_1 \|
\leq \liminf_{k \to \infty} \| z_{n_k} - P_C \left( I - \mu_n^x \sum_{i=1}^N a_i^z B_i^z \right) z_{n_k} \| + \liminf_{k \to \infty} \| P_C \left( I - \mu_n^x \sum_{i=1}^N a_i^z B_i^z \right) z_{n_k} - P_C \left( I - \mu_n^x \sum_{i=1}^N a_i^z B_i^z \right) q_1 \|
\leq \liminf_{k \to \infty} \| z_{n_k} - q_1 \|.
\]

This is a contradiction. Thus, \( q_1 \in \cap_{i=1}^N VI(C, B_i^r) \), and so,

\[
q_1 \in \mathcal{F}^x = \mathcal{F}^x \cap \left( \cap_{i=1}^N VI(C, B_i^r) \right).
\]

However, \( z_{n_k} \to q_1 \). From (54) and Lemma 1, we can derive that

\[
\limsup_{n \to \infty} \langle f(y) - \bar{x}, z_n - \bar{x} \rangle = \lim_{k \to \infty} \langle f(y) - \bar{x}, z_{n_k} - \bar{x} \rangle,
\]

\[
= \langle f(y) - \bar{x}, q_1 - \bar{x} \rangle \leq 0.
\]

(55)

\[
\| z_n - \bar{x} \|^2 = \langle \alpha_n (f(y) - \bar{x}) + (1 - \alpha_n) (G^x x_n - \bar{x}), z_n - \bar{x} \rangle,
\]

\[
= \alpha_n \langle f(y) - \bar{x}, z_n - \bar{x} \rangle + (1 - \alpha_n) \langle G^x x_n - \bar{x}, z_n - \bar{x} \rangle \leq \alpha_n \| y_n - \bar{y} \| \| z_n - \bar{x} \| + \alpha_n \langle f(y) - \bar{x}, z_n - \bar{x} \rangle + (1 - \alpha_n) \| x_n - \bar{x} \| \| z_n - \bar{x} \| \leq \alpha_n \rho \| y_n - \bar{y} \| \| z_n - \bar{x} \| + \alpha_n \| f(y) - \bar{x}, z_n - \bar{x} \rangle + (1 - \alpha_n) \| x_n - \bar{x} \| \| z_n - \bar{x} \|
\leq \alpha_n \rho \left[ \| y_n - \bar{y} \|^2 + \| z_n - \bar{x} \|^2 \right] + \alpha_n \| f(y) - \bar{x}, z_n - \bar{x} \rangle + (1 - \alpha_n) \left[ \| x_n - \bar{x} \|^2 + \| z_n - \bar{x} \|^2 \right],
\]

which implies that
\[ \left\| z_n - \bar{x} \right\|^2 \leq \frac{\alpha_n \rho}{1 + \alpha_n (1 - \rho)} \left\| y_n - \bar{y} \right\|^2 + \left( 1 - \alpha_n \right) \left\| x_n - \bar{x} \right\|^2 + \frac{2 \alpha_n}{1 + \alpha_n (1 - \rho)} \langle f (\bar{y}) - \bar{x}, z_n - \bar{x} \rangle. \] (58)

From the definition of \( \{x_n\} \) and (58), we get

\[ \left\| x_{n+1} - \bar{x} \right\|^2 \leq \delta_n \left\| x_n - \bar{x} \right\|^2 + \sigma_n \left\| P_C \left( I - \eta_n \sum_{i=1}^{N} a_i^x B_i^x \right) x_n - \bar{x} \right\|^2 + \eta_n \left\| z_n - \bar{x} \right\|^2, \]

\[ \leq (1 - \eta_n) \left\| x_n - \bar{x} \right\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1 - \rho)} \left\| y_n - \bar{y} \right\|^2 + \left( 1 - \alpha_n \right) \eta_n \left\| x_n - \bar{x} \right\|^2 + \frac{2 \alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \langle f (\bar{y}) - \bar{x}, z_n - \bar{x} \rangle \tag{59} \]

Applying the same argument as in (58) and (59), we get

\[ \left\| y_{n+1} - \bar{y} \right\|^2 \leq \left( 1 - \frac{\alpha_n \eta_n (2 - \rho)}{1 + \alpha_n (1 - \rho)} \right) \left\| y_n - \bar{y} \right\|^2 + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1 - \rho)} \left\| x_n - \bar{x} \right\|^2 + \frac{2 \alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \langle g (\bar{x}) - \bar{y}, z_n - \bar{y} \rangle. \] (60)

From (58) and (59), we have

\[ \left\| x_{n+1} - \bar{x} \right\|^2 + \left\| y_{n+1} - \bar{y} \right\|^2 \leq \left( 1 - \frac{\alpha_n \eta_n (2 - \rho)}{1 + \alpha_n (1 - \rho)} \right) \left( \left\| x_n - \bar{x} \right\|^2 + \left\| y_n - \bar{y} \right\|^2 \right) + \frac{\alpha_n \eta_n \rho}{1 + \alpha_n (1 - \rho)} \left[ \left\| x_n - \bar{x} \right\|^2 + \left\| y_n - \bar{y} \right\|^2 \right] + \frac{2 \alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \left[ \langle f (\bar{y}) - \bar{x}, z_n - \bar{x} \rangle + \langle g (\bar{x}) - \bar{y}, z_n - \bar{y} \rangle \right], \]

\[ \leq \left( 1 - \frac{2 \alpha_n \eta_n (1 - \rho)}{1 + \alpha_n (1 - \rho)} \right) \left( \left\| x_n - \bar{x} \right\|^2 + \left\| y_n - \bar{y} \right\|^2 \right) + \frac{2 \alpha_n \eta_n}{1 + \alpha_n (1 - \rho)} \left[ \langle f (\bar{y}) - \bar{x}, z_n - \bar{x} \rangle + \langle g (\bar{x}) - \bar{y}, z_n - \bar{y} \rangle \right]. \] (61)

According to condition (2) and (4), (61), and Lemma 8, we can conclude that \( \{x_n\} \) and \( \{y_n\} \) converge strongly to \( \bar{x} = P_{g^\perp} f (\bar{y}) \) and \( \bar{y} = P_{g^\perp} g (\bar{x}) \), respectively. Furthermore, from (39) and (40), we get that \( \{z_n\} \) and \( \{w_n\} \) converge strongly to \( \bar{x} = P_{g^\perp} f (\bar{y}) \) and \( \bar{y} = P_{g^\perp} g (\bar{x}) \), respectively. This completes the proof. □

One of the great special cases of the SMVIP is the split variational inclusion problem that has a wide variety of application backgrounds, such as split minimization problems and split feasibility problems.

If we set \( A_i^x = 0 \) and \( A_i^y = 0 \) in Theorem 2, for all \( i = 1, 2 \), then we get the strong convergence theorem for the split variational inclusion problem and the finite families of the variational inequality problems as follows:

**Corollary 1.** Let \( H_1 \) and \( H_2 \) be Hilbert spaces, and let \( C \) be a nonempty closed convex subset of \( H_1 \). Let \( T: H_1 \rightarrow H_2 \) be a bounded linear operator, and let \( f, g: H_1 \rightarrow H_1 \) be \( \rho_f, \rho_g \)-contraction mappings with \( \rho = \max \{ \rho_f, \rho_g \} \). For every \( i = 1, 2 \), let \( M_i^x, M_i^y : H_i \rightarrow 2^{H_i} \) be multivalued maximal monotone mappings. For \( i = 1, 2, \ldots, N \), let \( B_i^x, B_i^y \):
\[ H_1 \rightarrow H_1 \text{ be } \beta^{\rho}_1, \phi^{\rho}_1 \text{-inverse strongly monotone with } \begin{aligned} \overline{\partial}_x = \min_{i=1,2,..,N} (\beta^{\rho}_1) \text{ and } \overline{\partial}_y = \min_{i=1,2,..,N} (\beta^{\rho}_1). \end{aligned} \text{ Let } \delta^y = \{x^* \in H_1 : 0 \in M^*_1 x^*, \overline{x} = H^* x \in H_2 : 0 \in M^*_2 \overline{x}\} \text{ and } \delta^y = \{y^* \in H_1 : 0 \in M^*_1 y^*, \overline{y} = H^* y \in H_2 : 0 \in M^*_2 \overline{y}\}. \text{ Assume that} \]

\[
\begin{align*}
    x_{n+1} &= \delta_n x_n + \sigma_n P_C \left( I - \mu^\delta_n \sum_{i=1}^N a^*_i b^*_i \right) x_n + \eta_n \left( a_n f(\gamma_n) + (1 - a_n) \lambda^{M^*_i}_i \right) \left( x - y^* T^* \left( I - \lambda^{M^*_i}_i \right) T x_n \right), \\
    y_{n+1} &= \delta_n y_n + \sigma_n P_C \left( I - \mu^\delta_n \sum_{i=1}^N a^*_i b^*_i \right) y_n + \eta_n \left( a_n g(x_n) + (1 - a_n) \lambda^{M^*_i}_i \right) \left( y - y^* T^* \left( I - \lambda^{M^*_i}_i \right) T y_n \right),
\end{align*}
\]  

\hspace{1cm} (62)

for all \( n \geq 1 \), where \( \{\delta_n, \{a_n\}, \{\eta_n\}, \{a_n\} \leq [0, 1] \) with \( \delta_n + \sigma_n + \eta_n = 1, \{a^*_i, a^*_i, \ldots, a^*_i\}, \{a^*_i, a^*_i, \ldots, a^*_i\} \in (0, 1), \lambda^*_i, \lambda^*_i \in (0, \infty) \) for all \( i = 1, 2 \), and \( 0 < y^*, y^* < 1/L \) with \( L \) being a spectral radius of \( T^* T \). Assume the following conditions:

\( (1) \sum_{n=1}^{\infty} \mu^\delta_n < \infty, \sum_{n=1}^{\infty} \mu^\delta_n < \infty, \) and \( 0 < a < \mu^\delta_n < 2\overline{\beta}_x, \)

\( 0 < b < \mu^\delta_n < 2\overline{\beta}_y, \) for some \( a, b \in \mathbb{R}. \)

\( (2) \sum_{n=1}^{\infty} a_n = \infty, \lim_{n \to \infty} \sigma_n = 0. \)

\( (3) \sum_{n=1}^{\infty} a^*_n = \sum_{n=1}^{\infty} a^*_n = 1. \)

\( (4) 0 < \beta \leq \delta_n, \sigma_n, \eta_n \leq \frac{1}{10}, \) for all \( n \in \mathbb{N}, \) for some \( \beta, \sigma, \eta > 0. \)

\( (5) \sum_{n=1}^{\infty} |\delta_n| + \sum_{n=1}^{\infty} |\sigma_n| - \sigma_n, \) and \( \sum_{n=1}^{\infty} |a^*_n| = \infty. \) Then, \( \{x_n\} \) converges strongly to \( \overline{x} = P_{x^*} f(\overline{y}) \) and \( \{y_n\} \) converges strongly to \( \overline{y} = P_{x^*} g(\overline{x}) \).

Next, we consider the following split convex minimization problem (SCMP): find

\[ x^* \in H_1, \] such that \( \psi_1(x^*) + \psi_2(x^*) = \min_{x \in H_1} [\psi_1(x) + \psi_2(x)] \]

and such that \( y^* = T x^* \in H_2 \) solves

\[ \psi_2(y^*) = \min_{y \in H_2} [\psi_2(y) + \psi_2(y)], \]

where \( T : H_1 \rightarrow H_2 \) is a bounded linear operator with adjoint \( T^* \), \( \psi_i \) and \( \psi_i \) defined as above, for \( i = 1, 2 \). We denoted the set of all solutions of (64) and (65) by \( \Theta \). That is, \( \Theta = \{x^* \} \) which solves (64): \( T x^* \) solves (65).

If we set \( A^*_1 = \nabla \psi_1^* A^*_1 = \nabla \psi_1^* , \) and \( M^*_i = \partial \psi_1^* M^*_i = \partial \psi_1^* \), for \( i = 1, 2 \), in Theorem 2, then we get the strong convergence theorem for finding the common solution of the split convex minimization problems and the finite families of the variational inequality problems as follows.

**Theorem 3.** Let \( H_1 \) and \( H_2 \) be Hilbert spaces, and \( \psi \) be a convex and differentiable function and \( \psi : H \rightarrow (\mathbb{R}, \mathbb{R}) \) be a proper convex and lower semicontinuous function. It is well known that if \( \psi \) is \( 1/\alpha \)-Lipschitz continuous, then it is \( \alpha \)-inverse strongly monotone, and sometimes \( \psi \) is the gradient of \( \phi \) (see [10]). It is also known that the subdifferential \( \partial \psi \) of \( \psi \) is maximal monotone (see [35]). Moreover,

\[ \psi(x^*) + \psi(x^*) = \min_{x \in H} \{\psi(x) + \psi(x)\} \in \mathbb{R} \psi \phi(x^*) + \partial \phi(x^*). \]

\hspace{1cm} (63)
\[ VI(C, B_1^n) \neq \emptyset \quad \text{and} \quad \mathcal{F}^y = \Theta^y \cap (\cap_{i=1}^N VI(C, B_1^n)) \neq \emptyset. \] Let \( \{x_n\} \) and \( \{y_n\} \) be sequences generated by \( x_1, y_1 \in H_1 \) and

\[
\begin{align*}
    x_{n+1} &= \delta_n x_n + \sigma_n P_C \left( 1 - \mu_n \sum_{i=1}^N a_i^x B_1^y \right) x_n + \eta_n (a_n f (y_n) + (1 - \alpha_n) G^x x_n), \\
y_{n+1} &= \delta_n y_n + \sigma_n P_C \left( 1 - \mu_n \sum_{i=1}^N a_i^y B_1^y \right) y_n + \eta_n (a_n g(x_n) + (1 - \alpha_n) G^y y_n),
\end{align*}
\] (66)

for all \( n \geq 1 \) where \( \{\delta_n\}, \{\sigma_n\}, \{\eta_n\}, \{\alpha_n\} \subseteq [0, 1] \) with \( \delta_n + \sigma_n + \eta_n = 1 \), \( \{a_1^x, a_2^x, \ldots, a_N^x\} \), \( \{a_1^y, a_2^y, \ldots, a_N^y\} \) \( \subset (0, 1) \), and \( \{\mu_n^x\}, \{\mu_n^y\} \subset (0, \infty) \). Assume the following condition holds:

1. \( \sum_{n=1}^{\infty} \mu_n^x < \infty, \sum_{n=1}^{\infty} \mu_n^y < \infty \) and \( 0 < a < \mu_n^x \leq 2\rho_x \), \( 0 < b < \mu_n^y \leq 2\rho_y \) for some \( a, b \in \mathbb{R} \).

2. \( \sum_{n=1}^{\infty} \sigma_n = c \), \( \lim_{n \to \infty} \eta_n = 0 \).

3. \( \sum_{n=1}^{\infty} a_n^x = \sum_{n=1}^{\infty} a_n^y = 1 \).

4. \( 0 < a \leq \delta_n, \sigma_n, \eta_n \leq \beta < 1 \), for all \( n \in \mathbb{N} \), for some \( \alpha, \beta > 0 \).

5. \( \sum_{n=1}^{\infty} |\delta_n - \delta|_n, \sum_{n=1}^{\infty} |\sigma_n - \sigma|_n, \sum_{n=1}^{\infty} |\alpha_n - \alpha|_n < \infty \). Then, \( \{x_n\} \) converges strongly to \( \bar{x} = P_{\mathcal{F}^x} f(\bar{y}) \) and \( \{y_n\} \) converges strongly to \( \bar{y} = P_{\mathcal{F}^y} g(\bar{x}) \).

4.2. The Split Feasibility Problem. Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces. Let \( C \) and \( Q \) be the nonempty closed convex subset of \( H_1 \) and \( H_2 \), respectively. The split feasibility problem (SFP) is to find

\[ \text{a point } x \in C, \quad \text{such that } Ax \in Q. \] (67)

The set of all solutions (SFP) is denoted by \( \Psi = \{x \in C : Ax \in Q\} \). This problem was introduced by Censor and Elfving [8] in 1994. The split feasibility problem was investigated extensively as a widely important tool in many fields such as signal processing, intensity-modulated radiation therapy problems, and computer tomography (see [36–38] and the references therein).

Let \( H \) be a real Hilbert space, and let \( h \) be a proper lower semicontinuous convex function of \( H \) into \( (-\infty, +\infty] \). The subdifferential \( \partial h \) of \( h \) is defined by

Theorem 4. Let \( H_1 \) and \( H_2 \) be Hilbert spaces, and let \( C \) and \( Q \) be the nonempty closed convex subset of \( H_1 \) and \( H_2 \), respectively. Let \( T : H_1 \rightarrow H_2 \) be a bounded linear operator with adjoint \( T^* \), and let \( f, g : H_1 \rightarrow H_1 \) be \( \rho_f \), \( \rho_g \)-contraction mappings with \( \rho = \max \{\rho_f, \rho_g\} \). For \( i = 1, 2, \ldots, N \), let \( B_1^n, B_2^n : H_1 \rightarrow H_1 \) be \( \beta_i^1, \beta_i^2 \)-inverse strongly monotone with \( \beta_i = \min_{i=1,2} \min_{j=1,2,\ldots,N} \{|\beta_i^j|\} \) and \( \beta_i^1 = \min_{i=1,2,\ldots,N} \{|\beta_i^1|\} \). Assume that \( \mathcal{F}^x = \Psi^x \cap (\cap_{i=1}^N VI(C, B_1^n)) \neq \emptyset \) and \( \mathcal{F}^y = \Psi^y \cap (\cap_{i=1}^N VI(C, B_2^n)) \neq \emptyset \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences generated by \( x_1, y_1 \in H_1 \) and

\[
\begin{align*}
    x_{n+1} &= \delta_n x_n + \sigma_n P_C \left( 1 - \mu_n \sum_{i=1}^N a_i^x B_1^y \right) x_n + \eta_n (a_n f (y_n) + (1 - \alpha_n) P_C (x - y^* T^* (1 - P_Q) T x_n)), \\
y_{n+1} &= \delta_n y_n + \sigma_n P_C \left( 1 - \mu_n \sum_{i=1}^N a_i^y B_2^y \right) y_n + \eta_n (a_n g(x_n) + (1 - \alpha_n) P_C (y - y^* T^* (1 - P_Q) T y_n)),
\end{align*}
\] (68)
for all $n \geq 1$, where \( \{d_n, \sigma_n, \eta_n, \alpha_n\} \subseteq [0, 1] \) with \( d_n + \sigma_n + \eta_n = 1 \), \( \{a_1^y, a_2^y, \ldots, a_N^y\}, \{a_1^x, a_2^x, \ldots, a_N^x\} \subseteq (0, 1) \), \( \{\mu_n^x\}, \{\mu_n^y\} \subseteq (0, \infty) \), \( \lambda_1^x, \lambda_2^x \in (0, \infty) \) for all $i = 1, 2$, and \( 0 < \gamma^x, \gamma^y < 1/L $ with $L$ being a spectral radius of $T^* T$. Assume the following condition holds:

\[
(1) \sum_{n=1}^{\infty} \mu_n^y < \infty, \sum_{n=1}^{\infty} \mu_n^x < \infty, \text{ and } 0 < a < \mu_n^x \leq 2 \beta_2, 0 < b < \mu_n^y \leq 2 \beta_1, \text{ for some } a, b \in \mathbb{R}.
\]

\[
(2) \sum_{n=1}^{\infty} a_n = \infty, \lim_{n \to \infty} a_n = 0.
\]

\[
(3) \sum_{n=1}^{N} a_i^x = \sum_{n=1}^{N} a_i^y = 1.
\]

\[
(4) 0 < \bar{a} \leq \delta_n, \sigma_n, \eta_n \leq \bar{b} < 1, \text{ for all } n \in \mathbb{N}, \text{ for some } \bar{a}, \bar{b} > 0.
\]

\[
(5) \sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|, \sum_{n=1}^{\infty} |\sigma_{n+1} - \sigma_n|, \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \text{ Then, } \{x_n\} \text{ converges strongly to } \bar{x} = P_{\mathcal{F}} f(\bar{y}) \text{ and } \{y_n\} \text{ converges strongly to } \bar{y} = P_{\mathcal{G}} g(\bar{x}).
\]

Proof. Set \( M_i^x = \partial d_i, M_i^y = \partial d_i, M_i^x = \partial \sigma_i, M_i^y = \partial \sigma_i, \) and \( A_i^x = 0 \) and \( A_i^y = 0 \) in Theorem 2. Then, we get the result. \hfill \Box

The split feasibility problem is a significant part of the split monotone variational inclusion problem. It is extensively used to solve practical problems in numerous situations. Many excellent results have been obtained. In what follows, an example of a signal recovery problem is introduced.

Example 2. In signal recovery, compressed sensing can be modeled as the following under-determined linear equation system:

\[
y = Ax + \delta,
\]

where \( x \in \mathbb{R}^N \) is a vector with \( m \) non-zero components to be recovered, \( y \in \mathbb{R}^M \) is the observed or measured data with noisy \( \delta \), and \( A : \mathbb{R}^N \rightarrow \mathbb{R}^M (M < N) \) is a bounded linear observation operator. An essential point of this problem is that the signal \( x \) is sparse; that is, the number of nonzero elements in the signal \( x \) is much smaller than the dimension of the signal \( x \). To solve this situation, a classical model, convex constraint minimization problem, is used to describe the above problem. It is known that problem (69) can be seen as solving the following LASSO problem [40]:

\[
\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - Ax\|_2^2 \quad \text{subject to } \|x\|_1 \leq t,
\]

where \( t > 0 \) is a given constant and \( \| \cdot \|_1 \) is \( \ell_1 \) norm. In particular, LASSO problem (70) is equivalent to the split feasibility problem (SFP) (67) when \( C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\} \) and \( Q = \{y\} \).

5. Numerical Examples

In this section, we give some examples for supporting Theorem 2. In example 3, we give the computer programing to support our main result.

Example 3. Let \( \mathbb{R} \) be a set of real number and \( H_1 = H_2 = \mathbb{R}^2 \). Let \( C = [-20, 20] \times [-20, 20] \), and let \( \langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R} \) be inner product defined by \( \langle x, y \rangle = x \cdot y = x_1 y_1 + x_2 y_2 \), for all \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( y = (y_1, y_2) \in \mathbb{R}^2 \) and the usual norm \( \| \cdot \| : \mathbb{R}^2 \rightarrow \mathbb{R} \) given by \( \|x\| = \sqrt{x_1^2 + x_2^2} \), for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( T x = (2x_1, 2x_2) \) for all \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( T^* : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( T^* z = (2z_1, 2z_2) \) for all \( z = (z_1, z_2) \in \mathbb{R}^2 \). Let \( M_1^x, M_1^y, M_2^x, M_2^y : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( M_1^x = \{(x_1, 2x_2 - 2)\}, M_1^y = \{(x_2, 2x_1 - 2)\}, M_2^x = \{(x_1, 3x_2 - 4)\}, M_2^y = \{(x_2, 3x_1 - 4)\} \), respectively, for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). Let the mapping \( A_1^x, A_1^y, A_2^x, A_2^y : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( A_1^x = ((x_1 - 1)/3), (x_2 - 1)/3)\) and \( A_1^y = ((x_1 + 2)/3, (x_2 + 2)/3)\), respectively, for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). For every \( i = 1, 2, \ldots, N \), let the mappings \( B_i^x, B_i^y : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( B_i^x = ((x_1 - 3i)/3i, (x_2 - 1)/3)\) and \( B_i^y = ((x_1 + 2)/5i, (x_2 + 2)/5i)\), respectively, for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). Let the mappings \( f, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by \( f(x) = (x_1/7, x_1/7) \) and \( g(x) = (x_1/9, x_1/9) \), respectively, for all \( x = (x_1, x_2) \in \mathbb{R}^2 \). Choose \( \gamma^x = 0.1, \gamma^y = 0.1, \lambda_1^x = 2, \lambda_2^x = 1.2, \lambda_1^y = 0.1, \) and \( \lambda_2^y = 1.9 \). Setting \( \{d_1\} = \{m(9n + 3)\}, \{\alpha_1\} = \{(4n + 2)/3\} \} (9n + 3)\}, \{\eta_1\} = \{(4n + (7/3)(9n + 3)\}, \{\alpha_n\} = \{1/20n\}, \{\mu_n^x\} = \{1/7n^2\}, \) and \( \{\mu_n^y\} = \{1/5n^2\} \). Let \( x_1 = (x_1^1, x_1^2) \) and \( y_1 = (y_1^1, y_1^2) \in \mathbb{R}^2 \), and let the sequences \( \{x_n\} \) and \( \{y_n\} \) be generated by (21) as follows:
Table 1: Values of \{x_n\} and \{y_n\} with initial values \(x_1 = (-10, 10)\), \(y_1 = (-10, 10)\), and \(n = N = 50\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(x_n = (x_n^1, x_n^2))</th>
<th>(y_n = (y_n^1, y_n^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((-10.000000, 0.000000))</td>
<td>((-10.000000, 0.000000))</td>
</tr>
<tr>
<td>2</td>
<td>((-4.093342, 5.126154))</td>
<td>((-5.481385, 3.338182))</td>
</tr>
<tr>
<td>3</td>
<td>((-1.544910, 3.037093))</td>
<td>((-3.639877, 0.575101))</td>
</tr>
<tr>
<td>4</td>
<td>((-0.307347, 2.029037))</td>
<td>((-2.789106, -0.719370))</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>30</td>
<td>((0.998745, 0.996821))</td>
<td>((-1.996611, -1.998243))</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>47</td>
<td>((0.998973, 0.998235))</td>
<td>((-1.998394, -1.999022))</td>
</tr>
<tr>
<td>48</td>
<td>((0.999886, 0.998280))</td>
<td>((-1.999454, -1.999055))</td>
</tr>
<tr>
<td>49</td>
<td>((0.999000, 0.998324))</td>
<td>((-1.999511, -1.999087))</td>
</tr>
<tr>
<td>50</td>
<td>((0.999013, 0.998365))</td>
<td>((-1.999566, -1.999118))</td>
</tr>
</tbody>
</table>

\[
x_{n+1} = \frac{n}{9n + 3} x_n + \frac{4n + 2/3}{9n + 3} D \left( I - \frac{1}{2N} \sum_{i=1}^{N} \left( \frac{2}{3} + \frac{1}{N} \right) B_i^x \right) x_n + \frac{4n + 7/3}{9n + 3} \left( \frac{1}{20n} f(y_n) + \frac{20n - 1}{20n} G y_n \right),
\]
\[
y_{n+1} = \frac{n}{9n + 3} y_n + \frac{4n + 2/3}{9n + 3} D \left( I - \frac{1}{2N} \sum_{i=1}^{N} \left( \frac{2}{3} + \frac{1}{N} \right) B_i^y \right) y_n + \frac{4n + 7/3}{9n + 3} \left( \frac{1}{20n} g(x_n) + \frac{20n - 1}{20n} G y_n \right),
\]

for all \(n \geq 1\), where \(x_n = (x_n^1, x_n^2)\) and \(y_n = (y_n^1, y_n^2)\). By the definition of \(M_i^x, M_i^y, A_i^x,\) and \(A_i^y,\) for all \(i = 1, 2, B_i^x\) and \(B_i^y\), for all \(i = 1, 2, \ldots, N,\) and \(f\) and \(g\), we have that \((1, 1) \in \Omega^x \cap \left( \cap_{i=1}^{N} VI(C, B_i^y) \right)\) and \((-2, -2) \in \Omega^y \cap \left( \cap_{i=1}^{N} VI(C, B_i^y) \right)\). Also, it is easy to see that all parameters satisfy all conditions in Theorem 2. Then, by Theorem 2, we can conclude that the sequence \(\{x_n\}\) converges strongly to \((1,1)\) and \(\{y_n\}\) converges strongly to \((-2,-2)\).

Table 1 and Figure 1 show the numerical results of \(\{x_n\}\) and \(\{y_n\}\) where \(x_1 = (-10, 10),\) \(y_1 = (-10, 10),\) and \(N = 50\).

Next, in Example 4, we only show an example in infinite-dimensional Hilbert space for supporting Theorem 2. We omit the computer programming.

**Example 4.** Let \(H_1 = H_2 = C = \ell_2\) be the linear space whose elements consist of all 2-summable sequence \((x_1, x_2, \ldots, x_j, \ldots)\) of scalars, i.e.,
\[
\ell_2 = \left\{ x : x = (x_1, x_2, \ldots, x_j, \ldots) \text{ and } \sum_{j=1}^{\infty} \left| x_j \right|^2 < \infty \right\}, \tag{72}
\]
with an inner product \(\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}\) defined by
\[
\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j, \quad x = (x_j)_{j=1}^{\infty} \text{ and } y = (y_j)_{j=1}^{\infty} \in \ell_2,
\]
and a norm \(\| \cdot \| : \ell_2 \to \mathbb{R}\) defined by \(\|x\|_2 = \left( \sum_{j=1}^{\infty} |x_j|^2 \right)^{1/2}\) for all \(x = (x_j)_{j=1}^{\infty} \in \ell_2\). Let \(T : \ell_2 \to \ell_2\) be defined by \(Tx = (x_1/2, x_2/2, \ldots, x_j/2, \ldots)\) for all \(x = (x_j)_{j=1}^{\infty} \in \ell_2\) and \(T^* : \ell_2 \to \ell_2\) be defined by \(Ty = (2x_1, 2x_2, \ldots, 2x_j, \ldots)\) for all \(y = (y_j)_{j=1}^{\infty} \in \ell_2\). Let \(M_1^y, M_1^y, M_2^y, M_2^y \in \ell_2\) be defined by \(M_1^y x = \left\{ (2x_1, 2x_2, \ldots, 2x_j, \ldots) \right\}\), \(M_1^y x = \left\{ (x_1, x_2, 1, \ldots, x_j, 1, \ldots) \right\}\), \(M_2^y x = \left\{ (3x_1, 3x_2, 3x_j, \ldots) \right\}\), and \(M_2^y x = \left\{ (2x_1 - 1, 2x_2 - 1, \ldots, 2x_j - 1, \ldots) \right\}\), respectively, for all \(x = (x_j)_{j=1}^{\infty} \in \ell_2\). Let the mapping \(A_1^x, A_1^y, A_2^x, A_2^y : \ell_2 \to \ell_2\) be defined by \(A_1^y x = \left( (x_1, x_2, x_3, \ldots) \right)\), \(A_1^x x = \left( (x_1, x_2, x_3, \ldots) \right)\), \(A_2^y x = \left( (x_1, x_2, x_3, \ldots) \right)\), and \(A_2^x x = \left( (x_1, x_2, x_3, \ldots) \right)\), respectively, for all \(x = (x_j)_{j=1}^{\infty} \in \ell_2\). Let for every \(i = 1, 2, \ldots, N\), the mappings \(B_i^x, B_i^y : \ell_2 \to \ell_2\) be defined by \(B_i^x x = (x_1/4, x_2/4, \ldots, x_j/4, \ldots)\) and \(B_i^y x = \left( (2x_1 - 1, 2x_2 - 1, \ldots) \right)\) respectively, for all \(x = (x_j)_{j=1}^{\infty} \in \ell_2\).
for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$, and let $a_i^x = (5/6^i + 1/N6^i)$ and $a_i^y = (7/8^i + 1/N8^i)$. Let the mappings $f, g : \ell_2 \to \ell_2$ be defined by $f(x) = (x_i/5, x_i/5, \ldots , x_i/5, \ldots)$, $g(x) = (x_i/4, x_i/4, \ldots , x_i/4, \ldots)$, respectively, for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$.

Let $\lambda_1^x = 1, \lambda_1^y = 1, \lambda_2^x = 0.5$, and $\lambda_2^y = 2$. Since $L = 1/4$, we choose $y^x$ and $y^y = 0.5$. Let $x_1 = (x_1^1, x_1^2, \ldots , x_1^j, \ldots)$ and $y_1 = (y_1^1, y_1^2, \ldots , y_1^j, \ldots) \in \ell_2$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by (2) as follows:

$$
\begin{align*}
x_{n+1} &= \frac{n}{5n+2} x_n + \frac{2n+2/3}{5n+2} P_C \left( I - \frac{1}{3n} \sum_{i=1}^{N} \left( \frac{5}{6} + \frac{1}{N6^i} \right) B_i^x \right) x_n + \frac{2n+4/3}{5n+3} \left( \frac{1}{10n} f(y_n) + \frac{10n-1}{20n} G^x x_n \right), \\
y_{n+1} &= \frac{n}{5n+2} y_n + \frac{2n+2/3}{5n+2} P_C \left( I - \frac{1}{4n^2} \sum_{i=1}^{N} \left( \frac{7}{8} + \frac{1}{N8^i} \right) B_i^y \right) y_n + \frac{2n+4/3}{5n+2} \left( \frac{1}{10n} g(x_n) + \frac{10n-1}{20n} G^y y_n \right),
\end{align*}
$$

(73)

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2, \ldots , x_n^j, \ldots)$ and $y_n = (y_n^1, y_n^2, \ldots , y_n^j, \ldots)$. It is easy to see that $M_i^x$, $M_i^y$, $A_i^x$, and $A_i^y$, $\forall i = 1, 2, B_i^x$ and $B_i^y$, $\forall i = 1, 2, \ldots , N$, $T, f, g$, and all parameters satisfy Theorem 2. Furthermore, we have that $0 \in \Omega^x \cap (\cap_{i=1}^{N} VI(C, B_i^x))$ and $1 \in \Omega^y \cap (\cap_{i=1}^{N} VI(C, B_i^y))$. Then, by Theorem 2, we can conclude that the sequence $\{x_n\}$ converges strongly to 0 and $\{y_n\}$ converges strongly to 1.

### 6. Conclusion

(1) Table 1 and Figure 1 in Example 3 show that the sequence $\{x_n\}$ converges to $(1,1) \in \Omega^x \cap (\cap_{i=1}^{N} VI(C, B_i^x))$ and $\{y_n\}$ converges to $(-2,-2) \in \Omega^y \cap (\cap_{i=1}^{N} VI(C, B_i^y))$

(2) Example 4 is an example in infinite-dimensional Hilbert space for supporting Theorem 2.
(3) Theorem 2 guarantees the convergence of \( \{x_n\} \) and \( \{y_n\} \) in Example 3 and Example 4.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare no conflicts of interest.

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**References**


