Research Article

Radial Radio Number of Hexagonal and Its Derived Networks

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A mapping $\Upsilon: V(G) \longrightarrow N \cup \{0\}$ for a connected graph $G = (V, E)$ is called a radial radio labelling if it satisfies the inequality $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq \text{rad}(G) + 1 \forall x, y \in V(G)$, where $\text{rad}(G)$ is the radius of the graph G . The radial radio number of Υ denoted by $rr(\Upsilon)$ is the maximum number mapped under Υ . The radial radio number of G denoted by $rr(G)$ is equal to $\min \{rr(\Upsilon) \mid \Upsilon \text{ is a radial radio labelling of } G\}$.

1. Introduction

In this twenty-first century, our day-to-day life is pervaded by the communication devices which are functioning with the help of electromagnetic waves. By designating a portion of the electromagnetic spectrum that has wavelengths ranging from 1 mm to 100 kms, or equivalently, frequencies from 300 GHz to 3 kHz are called radio waves which are used in the field of communication engineering. Due to the high cost of the spectrum, maximizing the number of channels in a predefined bandwidth gives a huge profit to the country. Hence, the graph labelling concepts play a vital role in maximizing such an optimization channel assignment problem. This channel assignment problem inspired Hale [1] in 1980 to introduce the graph-theoretic concept using graph labelling. Chartrand et al. [2] were motivated by these concepts in 2001 and introduced a new channel assignment problem called the radio labelling problem which is used to allot the maximum number of channels for the frequency modulation radio stations. A radio labelling of a connected

graph G is an injection $\Upsilon: V(G) \longrightarrow N$ such that $d(x, y) + |\Upsilon(x) - \Upsilon(y)| \geq 1 + \text{diam}(G) \forall x, y \in V(G)$. The radio number of Υ , denoted by $rn(\Upsilon)$, is the maximum number assigned to any vertex of G . The radio number of G , denoted by $rn(G)$, is the minimum value of $rn(\Upsilon)$ taken over all labelings Υ of G . Computing such a problem for graphs with diameter two itself is NP-hard [3]. For the past two decades, several authors have studied the radio number problem for general graphs and interconnection networks. In the recent years, researchers have introduced the variations of radio labelling and called them as a different labelling either by changing $|\Upsilon(x) - \Upsilon(y)|$ as $(\Upsilon(x) + \Upsilon(y))/2$, $\sqrt{\Upsilon(x)\Upsilon(y)}$, $|\Upsilon(x)\Upsilon(y)|/2$, and $(2\Upsilon(x)\Upsilon(y))/(\Upsilon(x) + \Upsilon(y))$ or $\text{diam}(G)$ as $\text{diam}(G) - 1$, $\text{rad}(G)$, etc.

Motivated by the radio labelling problem, in order to increase the number of channels by splitting the given geographical area into two subregions, Selvam et al. [4] brought in the radial radio labelling concept in 2017. A mapping $\Upsilon: V(G) \longrightarrow N \cup \{0\}$ for a connected graph $G =$

(V, E) is called a radial radio labelling if this satisfies the inequality $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq \text{rad}(G) + 1 \quad \forall x, y \in V(G)$, where $\text{rad}(G)$ is the radius of the graph G . The radial radio number of Υ denoted by $rr(\Upsilon)$ is the maximum number mapped under Υ . The radial radio number of G denoted by $rr(G)$ is equal to $\min \{rr(\Upsilon) / \Upsilon \text{ is a radial radio labelling of } G\}$. It is obvious from the definitions that, for any connected graph G , $rn(G) \geq rr(G)$. However, the radial radio number is reduced to the radio number for any self-centered graphs, which is for the graphs that satisfy $\text{rad}(G) = \text{diam}(G)$. For example, the radio number and radial radio number for the complete graphs K_n and complete bipartite graphs $K_{m,n}$ are n and $m + n + 1$, respectively. Hence, $rn(G) > rr(G)$ for any graph G which is not self-centered. Selvam et al. [4] proved that $rr(G) \geq n$ for any self-centered graph G . In addition, Selvam et al. [5] proved few results connecting the clique number ω and $rr(G)$ as follows: (i) $\omega(G) \leq rr(G)$, (ii) for $m \geq 1$, $\exists$ a graph G which satisfies $rr(G) = m + \omega$ and $\omega = 3$, and (iii) $\exists$ a graph G with $rr(G) = \omega + 1$, whenever $\omega \geq 4$. Yenoke [6] determined the upper bounds for the radial radio number of certain uniform cyclic and split graphs. Moreover, Arputha Jose and Giridharan [7] proved that $rr(\text{MT}(n)) \leq 2n + 1$ and $rr(D(n)) \leq 2n + 2$, where $\text{MT}(n)$ is the Mongolian tent and $D(n)$ is the diamond graph. This research article highlights the newly defined enhanced hexagonal difference hexagonal network. Also, the radial radio number for HX_η , $\text{HDN}(\eta)$, and $\text{EDH}(\eta)$ was resolved.

2. Hexagonal and Its Derived Networks

In 2D geometry, the hexagonal network is the triangular tessellation of the Euclidean plane, and this was broadly analysed in [8–10]. In the graph theoretical approach, a hexagonal network of dimension η is denoted by HX_η which contains $6(\eta - 2)$ vertices of degree 4, 6 corner vertices of degree 3, and $3\eta^2 - 9\eta + 7$ vertices of degree 6. It was identified that each side of a network is equal to the dimension η . Also, $\exists$ a unique centre vertex at a distance $\eta - 1$ from the corner vertices. Therefore, $\text{diam}(\text{HX}_\eta)$ and $\text{rad}(\text{HX}_\eta)$ are $2\eta - 2$ and $\eta - 1$, respectively. In addition, it has $9\eta^2 - 15\eta + 6$ edges and $3\eta^2 - 3\eta + 1$ vertices. See Figure 1(a).

Manuel et al. [11] placed a vertex in the face of each triangle and then joined it by the corresponding three vertices of the triangle and derived an architecture from the hexagonal network called an enhanced hexagonal network,

and it is denoted by $\text{HDN}(\eta)$. In addition, $|E(\text{HDN}(\eta))| = 9\eta^2 - 15\eta + 7$ and $|V(\text{HDN}(\eta))| = 27\eta^2 - 51\eta + 24$. See Figure 1(b). Also, the diameter and radius are the same as in HX_η . This paper studies a new network named enhanced hexagonal difference hexagonal network which is obtained from the enhanced hexagonal network by removing all the edges of HX_η from $\text{HDN}(\eta)$. That is, $\text{EDH}(\eta) = \text{HDN}(\eta) - E(\text{HX}_\eta)$. It is denoted by $\text{EDH}(\eta)$. The number of vertices and edges in $\text{EDH}(\eta)$ is $9\eta^2 - 15\eta + 7$ and $(27\eta^2 - 51\eta + 24) - (9\eta^2 - 15\eta + 6) = 18\eta^2 - 36\eta + 18$, respectively. See Figure 1(c). Furthermore, the diameter and the radius for $\text{EDH}(\eta)$ are $4(\eta - 1)$ and $2(\eta - 1)$, respectively.

Even though the vertices of the three axes α, β , and γ of HX_η are already defined in the literature, for the requirement of the proof, we have renamed the vertices of the vertical lines (β - lines) from the left most top to the right most bottom as $y_1^1, y_2^1, \dots, y_{\kappa}^1, y_1^2, y_2^2, \dots, y_{\eta+1}^2, \dots, y_1^{\eta+1}, y_2^{\eta+1}, \dots, y_{2\kappa-1}^{\eta+1}, \dots, y_1^{2\eta-1}, y_2^{2\eta-1}, \dots, y_{\eta}^{2\eta-1}$. Furthermore, the face vertices of $\text{HDN}(\eta)$ are named in the same manner from the left most top to the right most bottom as $x_1^1, x_2^1, \dots, x_{\eta-1}^1, x_1^2, x_2^2, \dots, x_{\eta}^2, x_1^3, x_2^3, \dots, x_{\eta}^3, x_1^4, x_2^4, \dots, x_{\eta+1}^4, \dots, x_1^{2\eta-2}, x_2^{2\eta-2}, \dots, x_{2\eta-2}^{2\eta-2}, x_1^{2\eta-1}, x_2^{2\eta-1}, \dots, x_{2\eta-1}^{2\eta-1}, \dots, x_1^{4(\eta-1)}, x_2^{4(\eta-1)}, \dots, x_{\eta-1}^{4(\eta-1)}$. See Figure 2(a).

3. Radial Radio Number of HX_η , $\text{HDN}(\eta)$, and $\text{EDH}(\eta)$

In this section, we have determined the upper bounds for the radial radio number of the hexagonal network, the enhanced hexagonal network, and the newly defined network $\text{EDH}(\eta)$.

Theorem 1. Let HX_η be a hexagonal network of dimension η ; then, the radial radio number of HX_η satisfies $rr(\text{HX}_\eta) \leq (\eta - 1)(\eta^2 - \eta - 1) + 1$.

Proof. First, we partition the vertex set $V(\text{HX}_\eta)$ into five disjoint sets $W_1 = \{y_\sigma^\lambda / \sigma = 1, 2, \dots, \eta, \lambda = 1, 2, \dots, \eta - 1\}$, $W_2 = \{y_\sigma^{(\eta-1)+\lambda} / \sigma = 1, 2, \dots, \eta, \lambda = 1, 2, \dots, \eta - 1\}$, $W_3 = \{y_{\eta+\sigma}^{\lambda+1} / \sigma = 1, 2, \dots, \lambda, \lambda = 1, 2, \dots, \eta - 2\}$, $W_4 = \{y_{\eta+\sigma}^{(\eta-1)+\lambda} / \sigma = 1, 2, \dots, \eta - \lambda, \lambda = 1, 2, \dots, \eta - 1\}$, and $W_5 = \{y_\sigma^{2\eta-1} / \sigma = 1, 2, \dots, \eta\}$.

Define a mapping $\Upsilon: V(\text{HX}_\eta) \longrightarrow N \cup \{0\}$ as follows:

$$\begin{aligned} \Upsilon(y_\sigma^\lambda) &= (\eta - 1)(\sigma - 1) + (\lambda - 1)(\eta - 1)\eta, \quad \sigma = 1, 2, \dots, \eta, \lambda = 1, 2, \dots, \eta - 1, \\ \Upsilon(y_\sigma^{(\eta-1)+\lambda}) &= (\eta - 1)(\sigma - 1) + (\eta - 1)\eta(\lambda - 1) + 1, \quad \sigma = 1, 2, \dots, \eta, \lambda = 1, 2, \dots, \eta - 1, \\ \Upsilon(y_{\eta+\sigma}^{\lambda+1}) &= (\eta - 1)(\sigma + \lambda\eta - 1), \quad \sigma = 1, 2, \dots, \lambda, \lambda = 1, 2, \dots, \eta - 2, \\ \Upsilon(y_{\eta+\sigma}^{(\eta-1)+\lambda}) &= 1 + (\eta - 1)(-1 + \sigma + \eta\lambda), \quad \sigma = 1, 2, \dots, \eta - \lambda, \lambda = 1, 2, \dots, \eta - 1, \\ \Upsilon(y_\sigma^{2\eta-1}) &= (\eta - 1)(\sigma - 1), \quad \sigma = 1, 2, \dots, \eta. \end{aligned} \quad (1)$$

 FIGURE 1: HX_η , $HDN(\eta)$, and $EDH(\eta)$ for $\eta = 6$.

 FIGURE 2: Naming of vertices in $HDN(3)$ and a radial radio labelling of $HDN(4)$ which attains the bound.

See Figure 2(b).

Next, we claim that ∇ satisfies the radial radio labelling condition. Since $\text{rad}(HX_\eta) = \eta - 1$, we must verify that $|\nabla(x) - \nabla(y)| + d(x, y) \geq \eta - 1 + 1 = \eta \forall x, y \in V(HX_\eta)$.

Let $x, y \in HX_\eta$.

Case 1: if $x, y \in W_1$, then $x = y_r^s$ and $y = y_p^q$, where $1 \leq r, p \leq \eta$, $1 \leq s$, and $q \leq \eta - 1$. Therefore, $\nabla(x) = (\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1)$, $\nabla(y) = (\eta - 1)(p - 1) + (\eta - 1)\eta(q - 1)$, and $d(x, y) \geq 1$. If $s = q$ and $r \neq p$, then

$$\begin{aligned} |\nabla(x) - \nabla(y)| + d(x, y) &\geq |((\eta - 1)(r - 1) + (\eta - 1)\eta, \\ &\quad \cdot (s - 1)) - ((\eta - 1)(p - 1) \\ &\quad + (\eta - 1)\eta(q - 1))| + 1 \\ &\geq |(\eta - 1)(r - p)| + 1 \\ &\geq \eta. \end{aligned}$$

(2)

If $r = p$ and $s \neq q$, then $|\neg(x)\neg(y)| + d(x, y) \geq |(\eta - 1)\eta(s - q)| + 1 > \eta$. Otherwise, the condition is obviously true.

Case 2: if $x, y \in W_2$, then $x = y_r^{\eta+s}$ and $y = y_p^{(\eta-1)+q}$, where $1 \leq r, p \leq \eta$, $1 \leq s$, and $q \leq \eta - 1$. Here, $d(x, y) \geq 1$. Furthermore, $|\neg(x)\neg(y)| = |((\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1) + 1) - ((\eta - 1)(p - 1) + (\eta - 1)\eta(q - 1) + 1)|$. If both $r \neq p$ and $s \neq q$, then there is nothing to verify. If $s = q$ and $r \neq p$ or $s \neq q$ and $r = p$, then as in the previous case, $|\neg(x) - \neg(y)| + d(x, y) \geq |(\eta - 1)(r - p)| + 1 \geq \eta$ and $|\neg(x) - \neg(y)| + d(x, y) \geq |(\eta - 1)\eta(s - q)| + 1 > \eta$.

Case 3: suppose $x, y \in W_3$; then, $d(x, y) \geq 1$, $\neg(x) = (\eta - 1)(r + \eta s - 1)$, and $\neg(y) = (\eta - 1)(p + \eta q - 1)$, where $1 \leq r, p, s, q \leq \eta - 2$. If either $r \neq p$ or $s \neq q$, then $|\neg(x)\neg(y)| + d(x, y) \geq |(\eta - 1)(r - p)| + 1 \geq \eta$ or $|\neg(x)\neg(y)| + d(x, y) \geq |(\eta - 1)(\eta s - \eta q)| + 1 > \eta$.

Case 4: if $x, y \in W_4$, then $\neg(x) = \neg(y_{\eta+r}^{(\eta-1)+s}) = (\eta - 1)(r + \eta s - 1) + 1$ and $\neg(y) = \neg(y_{\eta+p}^{(\eta-1)+q}) = (\eta - 1)(p + \eta q - 1) + 1$, where $1 \leq r, p, s, q \leq \eta - 1$. Moreover, $d(x, y) \geq 1$. Hence, as in Case 3, we can verify that $|\neg(x) - \neg(y)| + d(x, y) \geq \eta$.

Case 5: suppose x and y lie in the set W_5 ; then, $x = y_s^{2\eta-1}$ and $y = y_t^{2\eta-1}$, where $1 \leq s \neq t \leq \eta$. Hence, the values of x and y under $\neg$ are $(\eta - 1)(s - 1)$ and $(\eta - 1)(t - 1)$, respectively. In addition, $d(x, y) \geq 1$. Hence, $|\neg(x) - \neg(y)| + d(x, y) \geq |(\eta - 1)(s - t)| + 1 \geq \eta$ since $s \neq t$.

Case 6: if $x \in W_1$ and $y \in W_2$, then $x = y_r^s$ and $y = y_p^{(\eta-1)+q}$, where $1 \leq r, p \leq \eta$, $1 \leq s$, and $q \leq \eta - 1$. Therefore, $\neg(x)$ and $\neg(y)$ values are $(\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1)$ and $(\eta - 1)(p - 1) + (\eta - 1)\eta(q - 1) + 1$, respectively. Since x and y are at a minimum distance, $\eta - 1$ and $|\neg(x) - \neg(y)| \geq 1$, we get $|\neg(x) - \neg(y)| + d(x, y) \geq 1 + \eta - 1 \geq \eta$.

Case 7: suppose $x \in W_1$ and $y \in W_3$; then, $\neg(y_r^s) = (\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1)$ and $\neg(y_{\eta+p}^{q+1}) = (\eta - 1)(p + \eta q + -1)$, where $1 \leq r \leq \eta$, $1 \leq s \leq \eta - 1$, and $1 \leq p, q \leq \eta - 2$. If $r = p$ and $s = q$, then $d(x, y) = \eta$. Again, if either $r \neq p$ or $s \neq q$, then $|\neg(x)\neg(y)| \geq \eta - 1$ and $d(x, y) \geq 1$. Hence, for the above possibilities, $|\neg(x) - \neg(y)| + d(x, y) \geq \eta$.

Case 8: if $x \in W_1$ and $y \in W_4$, then the verification is the same as Case 6 since $|\neg(x) - \neg(y)| \geq 1$ and $d(x, y) \geq \eta - 1$.

Case 9: if $x \in W_1$ and $y \in W_5$, then $x = y_r^s$ and $y = y_p^{2\eta-1}$, where $1 \leq r, p \leq \eta$, and $1 \leq s \leq \eta - 1$. Since $d(x, y) \geq \eta + 1$, the condition is trivially satisfied.

Case 10: if $x \in W_2$ and $y \in W_3$, then $x = y_r^{(\eta-1)+s}$ and $y = y_{\eta+p}^{q+1}$, where $1 \leq r \leq \eta$, $1 \leq p, q \leq \eta - 2$, and $1 \leq s \leq \eta - 1$. Therefore, $\neg(x) = ((\eta - 1)(r - 1) + (\eta - 1)(s - 1)\eta + 1)$, $\neg(y) = (\eta - 1)(p + \eta q + -1)$, and

$d(x, y) \geq \eta - 1$. Hence, $|\neg(x) - \neg(y)| + d(x, y) \geq |((\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1) + 1) - ((\eta - 1)(p + \eta q + -1))| + \eta - 1 \geq \eta$.

Case 11: if $x \in W_2$ and $y \in W_4$, then the verification is similar to Case 7.

Case 12: if $x \in W_2$ and $y \in W_5$, then $|\neg(x) - \neg(y)| = |((\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1) + 1) - ((\eta - 1)(t - 1))|$, $|\neg(x)\neg(y)| = |((\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1) + 1) - ((\eta - 1)(t - 1))|$, where $x = y_t^{(\eta-1)+s}$ and $y = y_t^{2\eta-1}$, $1 \leq r, t \leq \eta$, $1 \leq s \leq \eta - 1$.

If $s = 1$, then $d(x, y) = \eta - 1$. Therefore, $|\neg(x) - \neg(y)| + d(x, y) \geq |((\eta - 1)(r - 1) + 1) - (\eta - 1)(t - 1)| + \eta - 1 \geq \eta$.

If $s > 1$, then $d(x, y) \geq 1$, and hence, $|\neg(x) - \neg(y)| + d(x, y) \geq |((\eta - 1)(r - 1) + (\eta - 1)\eta + 1) - (\eta - 1)(t - 1)| \geq \eta(\eta - 1) + 1 > \eta$.

Case 13: assume $x \in W_3$ and $y \in W_4$; then, $\neg(x = y_{\eta+r}^{s+1}) = (\eta - 1)(r + \eta s - 1)$ and $\neg(y = y_{\eta+p}^{(\eta-1)+q}) = (\eta - 1)(p + \eta - 1) + 1$, where $1 \leq p, q \leq \eta - 1$, $1 \leq r, s \leq \eta - 2$. If $p = r + 1$, then $d(x, y) \geq \eta - 1$ and $|\neg(x) - \neg(y)| \geq 1$; otherwise, $d(x, y) \geq 1$ and $|\neg(x) - \neg(y)| \geq \eta(\eta - 1)$. Hence, we get $|\neg(x) - \neg(y)| + d(x, y) \geq \eta$.

Case 14: let $x \in W_3$ and $y \in W_5$; then, $\neg(x)\neg(y) = ((\eta - 1)(r + \eta s - 1)) - (\eta - 1)(t - 1)$, where $1 \leq p, q \leq \eta - 2$, $1 \leq t \leq \eta$. If $p = r + 1$, then $d(x, y) \geq \eta - 1$ and $|\neg(x)\neg(y)| \geq 1$; otherwise, $d(x, y) \geq 1$ and $|\neg(x)\neg(y)| \geq \eta(\eta - 1)$. Hence, we get $|\neg(x)\neg(y)| + d(x, y) \geq \eta$.

Case 15: let $x \in W_4$ and $y \in W_5$; then, the verification is the same as in Case 12.

Thus, $|\neg(x) - \neg(y)| + d(x, y) \geq \eta \quad \forall x, y \in V(HX_\eta)$.

Hence, $\neg$ is a valid radial radio labelling, and the vertex $y_\eta^{(2\eta-2)}$ attains the maximum value, $(\eta - 1)(\eta - 1) + (\eta - 1)\eta(\eta - 1 - 1) + 1 = (\eta - 1)((\eta - 1) + \eta(\eta - 2)) + 1 = ((\eta - 1)\eta^2 - \eta - 1) + 1$. Therefore, $rr(HX_\eta) \leq (\eta - 1)(\eta^2 - \eta - 1) + 1$. $\square$

Theorem 2. Let $HDN(\eta)$ be an enhanced hexagonal network of dimension κ ; then, the radial radio number of $HDN(\eta)$ satisfies $rr(HDN(\eta)) \leq (\eta - 1)(\eta^2 - \eta - 1) + (2\eta - 3)(\eta - 2) + (\eta - 2)^2 + 2$.

Proof. First, let us partition the face vertices of $HDN(\eta)$ into four disjoint sets $V_1 = \{x_\sigma^1/\sigma = 1, 2, \dots, \eta - 1, 1 = 1, 2, \dots, 2(\eta - 1)\}$, $V_2 = \{x_\sigma^{2(\eta-1)+1}/\sigma = 1, 2, \dots, \eta - 1, 1 = 1, 2, \dots, 2(\eta - 1)\}$, $V_3 = \{x_{\eta+\sigma-1}^{1+1}/\sigma = 1, 2, \dots, (1/2), 1 = 1, 2, \dots, 2\eta - 3\}$, and $V_4 = \{x_{\eta+\sigma-1}^{1+2(\eta-1)}/\sigma = 1, 2, \dots, \eta - (1 + 1)/2, 1 = 1, 2, \dots, 2\eta - 3\}$. The remaining vertices in $HDN(\eta)$ are partitioned into 5 disjoint sets as in Theorem 1.

Now, we define a mapping $\neg: V(HDN(\eta)) \rightarrow N \cup \{0\}$ as follows:

$$\begin{aligned}
\Upsilon(x_{\sigma}^{\lambda}) &= (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((\sigma - 1) + (\eta - 1)(\lambda - 1)) + 2, \quad \sigma = 1, 2, \dots, \eta - 1, \lambda = 1, 2, \dots, 2(\eta - 1), \\
\Upsilon(x_{\sigma}^{2(\eta-1)+\lambda}) &= (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((\sigma - 1) + (\eta - 1)(\lambda - 1)) + 2, \quad \sigma = 1, 2, \dots, \eta - 1, \lambda = 1, 2, \dots, 2(\eta - 1), \\
\Upsilon(x_{\eta+\sigma-1}^{\lambda+1}) &= (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((\sigma - 1) + (\eta - 1)(\lambda - 1)) + 2, \quad \sigma = 1, 2, \dots, \frac{\lambda}{2}, \lambda = 1, 2, \dots, 2\eta - 3, \\
\Upsilon(x_{\sigma}^{2(\eta-1)+\lambda}) &= (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((\sigma - 1) + (\eta - 1)(\lambda - 1)) + 2, \quad \sigma = 1, 2, \dots, \eta - \frac{\lambda + 1}{2}, \lambda = 1, 2, \dots, 2\eta - 3.
\end{aligned} \tag{3}$$

The remaining vertices are labelled as in Theorem 1. See Figure 3(b).

Next, we claim that $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq \eta$ $\forall x, y \in V(\text{HDN}(\eta))$.

Let $x, y \in V(\text{HDN}(\eta))$.

Case 1: if $x, y \in V_1$, then $x = x_r^s$ and $y = x_p^q$, where $1 \leq r, p \leq \eta - 1$, $1 \leq s, q \leq 2(\eta - 1)$. Therefore, $\Upsilon(x) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2$, $\Upsilon(y) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2$, and $d(x, y) \geq 2$. If $r \neq p$ and $s = q$, then

$$\begin{aligned}
&|\Upsilon(x) - \Upsilon(y)| + d(x, y), \\
&\geq |(\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2 - ((\eta - 1)(\eta^2 - \eta - 1) \\
&\quad + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2)| + 2, \\
&\geq |(-2 + \eta)(r - p)| + 2, \\
&\geq \eta.
\end{aligned} \tag{4}$$

If $s \neq q$ and $r = p$, then $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq |(\eta - 2)(\eta - 1)(s - q)| + 2 > \eta$. If both $r \neq p$ and $s \neq q$, the condition is obviously true.

Case 2: if $x, y \in V_2$, then $x = x_r^{2(\eta-1)+s}$ and $y = x_p^{2(\eta-1)+q}$, where $1 \leq r, p \leq \eta - 1$, $1 \leq s, q \leq 2(\eta - 1)$. Here, $d(x, y) \geq 2$. Furthermore, $|\Upsilon(x) - \Upsilon(y)| = |(\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2 - ((\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2)|$. The verification for different possibilities is the same as in Case 1.

Case 3: suppose $x, y \in V_3$; then, $d(x, y) \geq 2$, $\Upsilon(x) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2$, and $\Upsilon(y) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2$, where $1 \leq r, p \leq (2\eta - 3)/2$, $1 \leq s, q \leq 2(\eta - 1)$. If either $r \neq p$ or $s \neq q$, then $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq |(\eta - 2)(r - p)| + 2 \geq \eta$ or $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq |(\eta - 2)\eta(s - q)| + 2 > \eta$.

Case 4: if $x, y \in V_4$, then $\Upsilon(x) = \Upsilon(x_{\eta+r-1}^{2(\eta-1)+s}) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2$ and $\Upsilon(y) = \Upsilon(x_{\eta+p-1}^{2(\eta-1)+q}) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2$, where $1 \leq r, p \leq \eta - ((2\eta - 3) + 1)/2$, $1 \leq s, q \leq 2\eta - 3$. Moreover, $d(x, y) \geq 2$. Hence, as in Case 3, we can verify that $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq \eta$.

Case 5: if $x \in V_1$ and $y \in V_2$, then $x = x_r^s$ and $y = x_p^{q+2(\eta-1)}$, where $1 \leq r, p \leq \eta - 1$, $1 \leq s, q \leq 2(\eta - 1)$. Therefore, $\Upsilon(x) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2$ and $\Upsilon(y) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2$, respectively. Since $d(x, y) \geq \eta$ and $|\Upsilon(x) - \Upsilon(y)| \geq 0$, we get $|\Upsilon(x) - \Upsilon(y)| + d(x, y) \geq 0 + \eta \geq \eta$.

Case 6: suppose $x \in V_1$ and $y \in V_3$; then,

$$\begin{aligned}
\Upsilon(x_r^s) &= (1 + \eta)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2, \\
\Upsilon(x_{\eta+p-1}^{q+1}) &= (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2,
\end{aligned} \tag{5}$$

FIGURE 3: A radial radio labelling of HDN(4) and EDH(4) which attains the bound.

where $1 \leq r \leq -1$, $1 \leq s \leq 2(\eta - 1)$, $1 \leq p \leq (2\eta - 3)/2$, and $1 \leq q \leq 2\eta - 3$. If $r = p$ and $s = q$, then $d(x, y) = \eta$. Again, if either $r \neq p$ or $s \neq q$, then $|\nabla(x) \nabla(y)| \geq \eta - 2$ and $d(x, y) \geq 2$. From the above possibilities, $|\nabla(x) - \nabla(y)| + d(x, y) \geq \eta$.

Case 7: if $x \in V_1$ and $y \in V_4$, then $\nabla(x_r^s) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((r - 1) + (\eta - 1)(s - 1)) + 2$ and $\nabla(x_{\eta+p-1}^{2(\eta-1)+q}) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2$, where $1 \leq r \leq \eta + 1$, $1 \leq s \leq 2(\eta - 1)$, $1 \leq p \leq \eta - ((2\eta - 3) + 1)/2$, and $1 \leq q \leq 2\eta - 3$. Since $d(x, y) \geq \eta$, there is nothing to verify.

Case 8: if $x \in V_2$ and $y \in V_3$, then $x = x_r^{2(\eta-1)+s}$ and $y = x_{\eta+p-1}^{q+1}$, where $1 \leq r \leq \eta - 1$, $1 \leq s \leq 2(\eta - 1)$, $1 \leq p \leq (2\eta - 3)/2$, and $1 \leq q \leq 2\eta - 3$. Therefore, $\nabla(x) = ((\eta - 1) + (\eta - 1)\eta(s - 1) + 1)$, $\nabla(y) = (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (\eta - 1)(q - 1)) + 2$, and $d(x, y) \geq \eta$. Hence, $|\nabla(x) - \nabla(y)| + d(x, y) \geq \eta$.

Case 9: if $x \in V_2$ and $y \in V_4$, then the verification is similar to Case 6.

Case 10: if $x \in V_3$ and $y \in V_4$, then $|\nabla(x) - \nabla(y)| = |((\eta - 1)(r - 1) + (\eta - 1)\eta(s - 1) + 1) - (\eta - 1)(\eta^2 - \eta - 1) + (\eta - 2)((p - 1) + (-1 + \eta)(q - 1)) + 2|$, where $x = x_{\eta+r-1}^{2(\eta-1)+s}$ and $y = x_{\eta+p-1}^{2(\eta-1)+q}$,

$1 \leq r \leq (2\eta - 3)/2$, $1 \leq s \leq 2\eta - 3$, $1 \leq p \leq \eta - ((2\eta - 3) + 1)/2$, and $1 \leq q \leq 2\eta - 3$. If $s = p$, then $d(x, y) = \eta$. Therefore, $|\nabla(x) \nabla(y)| + d(x, y) \geq |((\eta - 1)(r - 1) + 1) - (\eta - 1)(q - 1)| + \eta - 1 \geq \eta$. If $s \neq p$, then $d(x, y) \geq 2$ and $|\nabla(x) \nabla(y)| \geq (\eta - 1)(\eta - 2)$ which implies $|\nabla(x) - \nabla(y)| + d(x, y) \geq \eta$.

Case 11: let $x \in V_s$, $1 \leq s \leq 4$, and $y \in W_\lambda$, $1 \leq \lambda \leq 5$; then, $|\nabla(x) \nabla(y)| \geq (\eta - 1)(\eta^2 - \eta - 1) + 1 > \eta$.

Case 12: if $x, y \in W_\lambda$, $1 \leq \lambda \leq 5$, then the proof is the same as in Theorem 1.

Thus, $|\nabla(x) - \nabla(y)| + d(x, y) \geq \eta \forall x, y \in V(\text{HDN}(\eta))$. Hence, ∇ is a valid radial radio labelling, and all the three vertices $x_{\eta-1}^{2(\eta-1)}$, $x_{2(\eta-1)}^{2(\eta-1)}$, and $x_{\eta-1}^{4(\eta-1)}$ yield the maximum value, $(\eta - 1)(\eta^2 - \eta - 1) + (\eta + 2)((\eta - 1) - 1) + (\eta - 1)((2\eta - 1) - 1) + 2 = (-1 + \eta)(\eta^2 - \eta - 1 + (2\eta - 3)(-2 + \eta) + (\eta - 2)^2) + 2$. Therefore, $rr(\text{HDN}(\eta)) \leq (\eta - 1)(\eta^2 - \eta - 1) + (2\eta - 3)(\eta - 2) + (\eta - 2)^2 + 2$. $\square$

Theorem 3. The radial radio number of $\text{EDH}(\eta)$ satisfies $rr(\text{EDH}(\eta)) \leq (2\eta - 3)\{\eta(\eta - 2) + 2(\eta - 1)^2\} + 1, \eta > 1$.

Proof. Define a mapping $\nabla: V(\text{EDH}(\eta)) \longrightarrow N \cup \{0\}$ as follows:

$$\begin{aligned}
\Upsilon(y_{\sigma}^{\lceil}) &= (2\eta - 3)(\sigma - 1) + (2\eta - 3)(\eta - 1)(\lceil - 1), \quad \sigma = 1, 2, \dots, \eta - 1, \lceil = 1, 2, \dots, \eta - 1, \\
\Upsilon(y_{\sigma}^{\eta+\lceil}) &= (2\eta - 3)(\sigma - 1) + (2\eta - 3)(\eta - 1)\lceil + 2, \quad \sigma = 1, 2, \dots, \eta - 1, \lceil = 1, 2, \dots, \eta - 2, \\
\Upsilon(y_{\eta+\sigma-1}^{\lceil}) &= (2\eta - 3)(\sigma - 1) + (2\eta - 3)(\lceil - 1)(\eta - 1) + 1, \quad \sigma = 1, 2, \dots, \lceil, \lceil = 1, 2, \dots, \eta - 2, \\
\Upsilon(y_{\eta+\sigma-1}^{\eta+\lceil}) &= (2\eta - 3)(\sigma - 1) + (2\eta - 3)(\lceil - 1)(\eta - 1) + 3, \quad \sigma = 1, 2, \dots, \eta - \lceil, \lceil = 1, 2, \dots, \eta - 2, \\
\Upsilon(y_{(\lceil-1)(\eta-1)+\sigma}^{\eta}) &= (2\eta - 3)(\sigma - 1) + 2, \quad \sigma = 1, 2, \dots, \eta - 1, \lceil = 1, 2, \\
\Upsilon(y_{\sigma}^{2\eta-1}) &= (2\eta - 3)(\sigma - 1), \quad \sigma = 1, 2, \dots, \eta - 1, \\
\Upsilon(y_{2\eta-1}^{\eta}) &= 2, \\
(y_{\eta}^{2\eta-1}) &= 1, \\
\Upsilon(x_{\sigma}^{\lceil}) &= \eta(\eta - 2)(2\eta - 3) + (2\eta - 3)\sigma + (2\eta - 3)(\eta - 1)(\lceil - 1), \quad \sigma = 1, 2, \dots, \eta - 1, \lceil = 1, 2, \dots, 2(\eta - 1), \\
\Upsilon(x_{\sigma}^{2(\eta-1)+\lceil}) &= \eta(\eta - 2)(2\eta - 3) + (2\eta - 3)\sigma + (2\eta - 3)(\eta - 1)(\lceil - 1) + 1, \quad \sigma = 1, 2, \dots, \eta - 1, \lceil = 1, 2, \dots, 2(\eta - 1), \\
\Upsilon(x_{\eta+\sigma-1}^{\lceil+1}) &= \eta(\eta - 2)(2\eta - 3) + (2\eta - 3)\sigma + (2\eta - 3)(\eta - 1)\lceil + 1, \quad \sigma = 1, 2, \dots, \frac{\lceil}{2}, \lceil = 1, 2, \dots, 2\eta - 3, \\
\Upsilon(x_{\sigma}^{2(\eta-1)+\lceil}) &= \eta(\eta - 2)(2\eta - 3) + (2\eta - 3)\sigma + (2\eta - 3)(\eta - 1)(\lceil - 1) + 2, \quad \sigma = 1, 2, \dots, \eta - \frac{\lceil+1}{2}, \lceil = 1, 2, \dots, 2\eta - 3.
\end{aligned}$$

(6)

See Figure 2(b).

As the remaining part of the proof is similar to Theorem 2, we have left the proof to the reader. $\square$

4. Conclusion

In this research study, we have introduced the enhanced hexagonal difference hexagonal network from the existing enhanced hexagonal network. Furthermore, we have investigated

$$rr(EDH(\eta)) \leq (2\eta - 3)\{\eta(\eta - 2) + 2(\eta - 1)^2\} + 1, \eta > 1.$$

Also, the radial radio number of HDN(η) and EDH(η) was presented. These bounds will motivate other researchers to conduct further research studies on the applications of the enhanced hexagonal network and its derived networks. Furthermore, these studies can be extended to the radio k -chromatic number and its variations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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