Research Article

Solving Poisson Equation by Distributional HK-Integral: Prospects and Limitations

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In this paper, we present some properties of integrable distributions which are continuous linear functional on the space of test function $\mathscr{D}(\mathbb{R}^2)$. Here, it uses two-dimensional Henstock–Kurzweil integral. We discuss integrable distributional solution for Poisson's equation in the upper half space $\mathbb{R}^3_+$ with Dirichlet boundary condition.

1. Introduction

In this paper, we find an integrable distributional solution for Poisson's equation in the upper half space. Poisson's equation is a second-order elliptic partial differential equation which has many applications in physics and engineering. There are three types of boundary conditions for Poisson's equation: Dirichlet condition, Neumann conditions, and Robin conditions. This article is mainly concerning on Poisson's equation with Dirichlet boundary condition (1):

$$\begin{cases} \triangle u = f, & \text{in } \Omega \subset \mathbb{R}^n, \\ u = g, & \text{on } \partial\Omega. \end{cases} \quad (1)$$

Partial differential equations (PDEs) are more difficult to solve than ordinary differential equations (ODEs). Therefore, numerical approximations are widely used in application. There are standard numerical methods such as finite difference and finite element etc. to solve (1). In numerical approaches to solving PDEs, particularly in finite difference methods, we may face two ways of accuracy loss because of discretization of domain and approximating partial derivatives by difference formulas. Finite element methods will improve the accuracy of problems with irregular boundaries, but still abide by approximations as one has to do minimizing functionals with smaller class of functions [1]. Also, artificial neural network methods can be used for approximate nonsmooth solutions. See [2] for application of artificial neural network methods for Poisson's equation with boundary value problems in domain $\mathbb{R}^2$. This method trains data to optimize the algorithm which gives numerical solutions to partial differential equations. Convergent properties and accuracy of solution are not discussed in the given method of [2]. Finite element method can be used to solve elliptic problems with Henstock–Kurzweil integrable functions. Such an approximation is used in [3] for ODEs, where existence and uniqueness of the solution are not discussed. The proposed analytical approach in our paper will direct to a solution, where if its explicit form is not available, one has to only do a numerical integration.

There are several methods to find analytical solution for Dirichlet problem such as Green's function, Dirichlet's principle, layer potentials, energy methods, and Perron's method. In 1850 [4], George Green gives theoretical approach for Dirichlet problem which is now called as Green's function. This assumes that Green's function exists for any domain which is not true. However, this idea influences

modern techniques and distributional theory. Using Green's function, one can obtain a general representation solution for Poisson's equation.

General representation solution depends on the domain Ω and nature of the functions f and g. For Lebesgue integration, it is required that the conditions $g \in C(\partial\Omega)$ and $f \in C(\Omega)$ get a unique analytical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for (1), where Ω needs to be open and bounded in $\mathbb{R}^n$ ([5], pg. 28). In classical solution, it is worth to use Henstock–Kurzweil integral (HK-integral) instead of the Lebesgue integral.

HK-integral is more advanced, and it includes the Lebesgue and Riemann integral. This integral was first introduced by Henstock and Kurzweil in 1957. It has many advantages such as convergence theorems, integration on unbounded intervals/functions, Fubini's theorem, and fundamental theorem of calculus with full generality, for details see [6–9]. If $\mathscr{H}\mathscr{K}$ is the space of HK-integrable functions, then it has proper inclusion $L^1 \subsetneq \mathscr{H}\mathscr{K}$. For an example, a highly oscillating function $f(x) = (1/x)$ $\sin(1/x^3)$ is neither Lebesgue nor Riemann integrable. But f is HK-integrable. There are only few articles which use HK-integration to find analytical solution for Dirichlet problem [10, 11]. Talvila uses HK-integration to find solution for Dirichlet problem in the upper half plane, see example in [10]. Also, the same author obtains useful results of Poisson kernel in the unit disk via HK-integral with application into Dirichlet problem [11]. However, both articles [10, 11] prove its results in $\mathbb{R}^2$ and do not extend into $\mathbb{R}^3$. It is an obvious fact that properties of HK-integration are not easily extended for higher dimension. Even the HK-integration is superior to Lebesgue integration, there are a few drawbacks. Most significantly, there is no known natural topology for $\mathscr{H}\mathscr{K}$, so that it is not a Banach space. Therefore, there are not much functional analysis tools to work with the problem. However, there is a natural seminorm on $\mathscr{H}\mathscr{K}$ defined by Alexiewicz [12].

Looking at an alternative, we moved into distribution theory; specially the space of integrable distribution $\mathscr{A}_C$ which is isomorphic to completion of the space $\mathscr{H}\mathscr{K}$, which is presented in Section 2. It seems the concept of integrable distribution is first introduced by Mikusiński and Ostaszewski [13] and further developed by several authors [14, 15]. In recent years, integrable distributional solution for differential equations has been used by many authors [16–19]. We define derivative (in distributional sense) of $\mathscr{A}_C$ and the integral on it. In Section 3, the space of Hardy–Krause variation functions is identified as a multiplier for $\mathscr{H}\mathscr{K}$. Here it uses Henstock–Stieltjes integral and integration by parts defined on $\mathscr{A}_C$. In [15], example 14.4, Talvila discussed the Dirichlet boundary condition of Laplace equation in the half space. There it uses Poisson Kernel, and proof of integrable distributional solution for the Laplace

equation is not given. We extend the proof of this problem in Section 4 and obtain integrable distributional solution for (1) when $f = 0$.

2. Integrable Distributions

Distribution theory was frequently used with the advent of Laurent Schwartz [20] in 1950. Distributions or generalized functions are an important functional analysis tool in modern analysis, especially in the field of PDE's. When functions are nonsmooth, then distribution theory allows to perform operations: translation, differentiation, convolution etc. Here, we use integrable distributions which are a special class of distributions that will be important for applications of distribution theory to partial differential equations, for details see [13, 14, 21]. There are several articles on solving ordinary and partial differential equations using integrable distributions [16, 17, 19]. Integrable distribution has used to find a distributional solution for Poisson's equation with Dirichlet boundary condition for the upper half plane. This article extends the distributional solution for Poisson's equation for the 3-dimensional upper half space.

Let the space of test functions is $\mathscr{D}(\mathbb{R}^2) = \{\phi : \mathbb{R}^2 \longrightarrow \mathbb{R} | \phi \in C^\infty$ and ϕ has compact support$\}$. Then, $\mathscr{D}(\mathbb{R}^2)$ forms a normed space under usual pointwise operations. The dual space, $\mathscr{D}'(\mathbb{R})$, is called as the space of distributions. That is, the distributions are the continuous linear functionals on $\mathscr{D}(\mathbb{R}^2)$. For a given distribution T, denotes its action on test function $\phi \in \mathscr{D}(\mathbb{R}^2)$ by $\langle T, \phi \rangle$. For an example, if f is a locally HK-integrable function, then its corresponding distribution is defined $\langle T_f, \phi \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f\phi$. Hereafter, all integral will be HK-integral unless stated otherwise. Next, consider derivative of a distribution, sometimes called as distributional derivative or weak derivative. If F is continuously differentiable, then $\int_{-\infty}^{\infty} F'\phi = -\int_{-\infty}^{\infty} F\phi'$ for any test function ϕ. The compact support of ϕ was used in this integration by parts. With this suggestion, we define the distributional derivative of a distribution $F \in \mathscr{D}'(\mathbb{R})$ by $\langle \partial^\alpha F, \phi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \phi \rangle$ for any $\phi \in \mathscr{D}(\mathbb{R}^2)$. Therefore, in particular, $\langle \partial_{12} F, \phi \rangle = \langle F, \partial_{12} \phi \rangle$, where $\partial_{12} = \partial^2/\partial y \, \partial x$. Now, consider a specific type of distributions called as integrable distributions. This definition seems to have been first introduced by Mikusiński and Ostaszewski [13]. After that, it was developed in detail in the plane by Ang et al. [14] and on $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}^2$ by Talvila [15, 21].

For the extended real numbers $\overline{\mathbb{R}}$, consider $C^0(\overline{\mathbb{R}}^2)$ be the space of continuous functions so that both $\lim_{x \longrightarrow -\infty} F(x, y)$ and $\lim_{y \longrightarrow -\infty} F(x, y)$ exit for any $F \in C(\overline{\mathbb{R}}^2)$. Then, define $\mathscr{B}_C(\overline{\mathbb{R}}^2) = \{F \in C^0(\overline{\mathbb{R}}^2) | F(-\infty, y) = F(x, -\infty) = 0, \forall x, y \in \overline{\mathbb{R}}\}$. Then, $\mathscr{B}_C(\overline{\mathbb{R}}^2)$ forms a Banach space under the uniform norm $\|\cdot\|_\infty$. Next, define the

integrable distribution as the derivative of function in $\mathscr{B}_C(\overline{\mathbb{R}}^2)$.

Definition 1. Let $f \in \mathscr{D}'(\mathbb{R}^2)$ be distribution. The space of integrable distributions is defined and denoted by

$$\mathscr{A}_C\left(\overline{\mathbb{R}}^2\right) = \left\{f \in \mathscr{D}'\left(\mathbb{R}^2\right) | f = \partial_{12}F, F \in \mathscr{B}_C\left(\overline{\mathbb{R}}^2\right)\right\}. \quad (2)$$

Here, $f = \partial_{12}F$ be the sense of distributional derivative, i.e., $\langle f, \phi \rangle = \langle \partial_{12}F, \phi \rangle = \langle F, \partial_{12}\phi \rangle$ for $\phi \in \mathscr{D}(\mathbb{R}^2)$. The function F is called as a primitive of f. Generally, primitive is not unique and differed by a constant. However, with our choice $F(-\infty, y) = F(x, -\infty) = 0$, primitive is unique for any given $f \in \mathscr{D}'(\mathbb{R}^2)$.

Theorem 1. *Any given $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ has a unique primitive in $\mathscr{B}_C(\overline{\mathbb{R}}^2)$.*

Proof. Let F_1 and F_2 are primitives of $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$. Then, $f = \partial_{12}F_1 = \partial_{12}F_2$ so that $\partial_{12}(F_1 - F_2) = 0$. This implies $F_1 - F_2 = X(x) + Y(y)$ for some $X, Y \in C(\overline{\mathbb{R}}^2)$. Since $(F_1 - F_2) \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$, for a fixed y_0

$$0 = \lim_{x \to -\infty}(F_1 - F_2) = \lim_{x \to -\infty}X(x) + \lim_{x \to -\infty}Y(y_0). \quad (3)$$

Therefore, $\lim_{x \to -\infty}X(x) = -Y(y_0)$ for any $x \in \mathbb{R}$. This implies that $X(x)$ is a constant function. For fixed x_0, letting $y \to -\infty$ gives $Y(y)$ is a constant function for the same argument. Those constant functions sums into 0, and hence $F_1 = F_2$. □

The uppercase letter is denoted the primitive of any integrable distribution f. If $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ with the primitive $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$, then for any test function $\phi \in \mathscr{D}(\mathbb{R}^2)$,

$$\langle f, \phi \rangle = \langle \partial_{12}F, \phi \rangle = \langle F, \partial_{12}\phi \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F\partial_{12}\phi, \quad (4)$$

where it uses HK-integral, and here onward the same, unless otherwise stated. Since the linearity of derivatives, $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ forms a vector space with usual operations of functionals. To get Banach space structure of $\mathscr{A}_C(\overline{\mathbb{R}}^2)$, it requires to define the suitable norm. Using the unique primitive of $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$, we define the Alexiewicz norm as

$$\|f\| = \sup\left\{\left|\int_{-\infty}^{x}\int_{-\infty}^{y}f\right|: (x, y) \in \mathbb{R}^2\right\} = \sup_{\mathbb{R}^2}|F| = \|F\|_\infty. \quad (5)$$

Note that the space $\mathscr{H}\mathscr{K}$ with the Alexiewicz norm is a normed space, but not a Banach space [12, 22]. Absence of Banach space structure of $\mathscr{H}\mathscr{K}$ is main disadvantage on it. It is one of the reasons to move into $\mathscr{A}_C(\overline{\mathbb{R}}^2)$.

Theorem 2. *$\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is a Banach space with the Alexiewicz norm $\|\cdot\|$.*

Proof. First, prove $\|\cdot\|$ is a norm.

(i) If $f \equiv 0$, then clearly $\|f\| = 0$. If $\|f\| = 0$ and then $\|F\|_\infty = 0$. This implies $F \equiv 0$ and hence $f = \partial_{12}F \equiv 0$.

(ii) For any $\lambda \in \mathbb{R}$ and $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$,

$$\|\lambda f\| = \sup\left\{\left|\int_{-\infty}^{x}\int_{-\infty}^{y}\lambda f\right|: (x, y) \in \mathbb{R}^2\right\}$$
$$= |\lambda|\sup\left\{\left|\int_{-\infty}^{x}\int_{-\infty}^{y}f\right|: (x, y) \in \mathbb{R}^2\right\} = |\lambda|\|f\|. \quad (6)$$

(iii) If $f, g \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$, the primitive of $(f + g)$ is $(F + G)$. Then,

$$\|f + g\| = \|F + G\|_\infty \leq \|F\|_\infty + \|G\|_\infty = \|f\| + \|g\|. \quad (7)$$

Therefore, $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is a normed space. To show completeness, let any $\{f_n\} \subseteq \mathscr{A}_C(\overline{\mathbb{R}}^2)$ be Cauchy sequence in $\|\cdot\|$. Then, $\{F_n\} \subseteq \mathscr{B}_C(\overline{\mathbb{R}}^2)$ is a Cauchy sequence in $\|\cdot\|_\infty$ because $\|f_n - f_m\| = \|F_n - F_m\|_\infty$. But $\mathscr{B}_C(\overline{\mathbb{R}}^2)$ is a Banach space, so that there exists a $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$ and $F_n \to F$ in $\|\cdot\|_\infty$. Let $f = \partial_{12}F$. Then, $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ and $\|f_n - f\| = \|F_n - F\|_\infty \to 0$. This shows $f_n \to f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ in $\|\cdot\|$ and hence the lemma. □

Alexiewicz norm dose not induce an inner product in $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ because the norm does not satisfy the parallelogram law. Therefore, $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is not a Hilbert space under the norm $\|\cdot\|$. However, the space $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ with $\|\cdot\|$ is isomorphic to $\mathscr{B}_C(\overline{\mathbb{R}}^2)$ with $\|\cdot\|_\infty$. The mapping $f \to F$ gives a linear isomorphism between the spaces. Uniqueness of primitive gives the mapping is injective, and by the definition it is surjective mapping. Furthermore, the mapping preserves norm since $\|f\| = \|F\|_\infty$. Thus, the space $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is identified with the space of continuous functions vanish at infinity.

Proposition 1. *$(\mathscr{A}_C(\overline{\mathbb{R}}^2), \|\cdot\|)$ is isometrically isomorphic to $(\mathscr{B}_C(\overline{\mathbb{R}}^2), \|\cdot\|_\infty)$.*

Next, we define the integral of an integrable distribution, as it uses the primitive $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$ of $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$.

Definition 2 (see [15], Definition 4.3) Let $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ with the primitive $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$. We define integral of f on $I = [a, b] \times [c, d] \subseteq \overline{\mathbb{R}}^2$ by

$$\int_I f = \int_a^b \int_c^d f = F(a, c) + F(b, d) - F(a, d) - F(b, c). \quad (8)$$

This integral is uniquely defined by uniqueness of the primitive, Theorem 1. Also, it is very general and includes the Riemann integral, Lebesgue integral, and HK-integral. Norm preservation of Proposition 1 gives that the corresponding integral values agree on each integral as presented in Theorem 3.

Theorem 3. *For any test function $\phi \in \mathscr{D}(\mathbb{R}^2)$, if $f \in L^1(\mathbb{R}^2)$ is identity with the corresponding distribution $f: \phi \longmapsto (L)\int_{\mathbb{R}^2} f\phi$ (or with HK-integral), then*

    (i) $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$

    (ii) $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is the completion of $L^1(\mathbb{R}^2)$ (or $\mathscr{HK}(\mathbb{R}^2)$) with respect to the norm:

$$\|f\| = \sup\left\{ \left| (L)\int_{-\infty}^{x}\int_{-\infty}^{y} f \right| : (x,y) \in \mathbb{R}^2 \right\}. \quad (9)$$

    (or in the sense of HK-integral)

    (iii) $\int_{\mathbb{R}^2} f = (L)\int_{\mathbb{R}^2} f$ (or $= (HK)\int_{\mathbb{R}^2} f$)

Proof

    (i) Let $f \in L^1(\mathbb{R}^2)$. For any $(x,y) \in \overline{\mathbb{R}}^2$, we define $F^L(x,y) = (L)\int_{-\infty}^{x}\int_{-\infty}^{y} f$. Then, F^L is a primitive of f in the Lebesgue integral (or HK-integral). Then, $F^L \in C^0(\mathbb{R}^2)$ since the primitive of Lebesgue integral (or HK-integral) is continuous. Clearly, $F^L(-\infty, y) = F^L(x, -\infty) = 0$ for any $(x,y) \in \overline{\mathbb{R}}^2$. Therefore, $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$.

    (ii) From (i), $L^1(\mathbb{R}^2) \subset \mathscr{A}_C(\overline{\mathbb{R}}^2)$. Let any $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$. Then, it needs to show $f \in \overline{L^1(\mathbb{R}^2)}$. We prove this by showing f is a limit point of $\overline{L^1(\mathbb{R}^2)}$. Since $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ there is $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$. Now choose a sequence $\{F_n\} \subset C^2(\overline{\mathbb{R}}^2)$ so that $\|F_n - F\|_\infty \longrightarrow 0$. Since F_n converges to F uniformly and $F(-\infty, y) = F(x, -\infty) = 0$, we can assume that $F_n(-\infty, y) = F_n(x, -\infty) = 0$ for each n. Now, take $f_n = \partial F_n$ in the distributional sense. Then, $f_n = \partial F_n \in C(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$. Moreover, $\|f_n - f\| = \|F_n - F\|_\infty \longrightarrow 0$. Hence, f is a limit point of $\overline{L^1(\mathbb{R}^2)}$.

    (iii) Any Lebesgue integrable function is HK-integrable with the same value. Thus, the result. $\qquad\square$

Note that if $f \in L^1(\mathbb{R}^2)$, then its Alexiewicz norm and L^1 norm are not equivalent, for an example see [15] (pg. 9). The norms are equivalent if $f \geq 0$ for almost everywhere. From the definition of integral of $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ and its unique primitive, we obtain following fundamental theorem of calculus.

Theorem 4 (*fundamental theorem of calculus*).

    (i) If $F \in C(\mathbb{R}^2)$ then $\int_{-\infty}^{x}\int_{-\infty}^{y} \partial F = F(x,y) + F(\infty, \infty) - F(-\infty, y) - F(x, -\infty)$

    (ii) Let $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ and $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f$. Then $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$ and $f = \partial F$

Proof. See [15]. $\qquad\square$

3. Multiplier for Integrable Distributions

We will turn now to our discussion on multipliers for $\mathscr{A}_C(\overline{\mathbb{R}}^2)$. A multiplier for a class of functions $\mathscr{F}$ is a function g such that $fg \in \mathscr{F}$ for each $f \in \mathscr{F}$. For an example, in Lebesgue integral, $g \in L^\infty(\mathbb{R}^n)$ if and only if $fg \in L^1(\mathbb{R}^n)$ for each $f \in L^1(\mathbb{R}^n)$. i.e., the space $L^\infty(\mathbb{R}^n)$ is multipliers for $L^1(\mathbb{R}^n)$. For HK-integral in one-dimensional case 2, space of bounded variation functions, $\mathscr{BV}$, is multipliers for $\mathscr{HK}$ [9], (Theorem 6.1.5). In multidimension case, there is no unique way to extend the notion of variation to function. Adams and Clarkson [23] give six such extensions but mostly recognized by Vitali variation and Hardy–Krause variation. To establish integration by parts formula and multipliers for $\mathscr{A}_C(\overline{\mathbb{R}}^2)$, we begin with Hardy–Krause variation.

Two intervals in $\overline{\mathbb{R}}^2$ are nonoverlapping if their intersection is of Lebesgue measure zero. A division of $\overline{\mathbb{R}}^2$ is a finite collection of nonoverlapping intervals whose union is $\overline{\mathbb{R}}^2$. If $\varphi: \overline{\mathbb{R}}^2 \longrightarrow \mathbb{R}$, then its total variation (in the sense of Vitali) is given by

$$V_{12}\varphi = \sup_D \sum_i |\varphi(a_i, c_i) + \varphi(b_i, d_i) - \varphi(a_i, d_i) - \varphi(b_i, c_i)|, \quad (10)$$

where the supremum is taken over all divisions D of $\overline{\mathbb{R}}^2$ and interval $I_i = [a_i, b_i] \times [c_i, d_i] \in D$.

Definition 3 (Hardy–Krause bounded variation). A function $\varphi: \overline{\mathbb{R}}^2 \longrightarrow \mathbb{R}$ is said to be Hardy–Krause bounded variation if the following conditions are satisfied.

    (i) $V_{12}\varphi$ is finite

    (ii) The function $x \longmapsto \varphi(x, y_0)$ is bounded variation on $\overline{\mathbb{R}}$ for some $y_0 \in \overline{\mathbb{R}}$

    (iii) The function $y \longmapsto \varphi(x_0, y)$ is bounded variation on $\overline{\mathbb{R}}$ for some $x_0 \in \overline{\mathbb{R}}$

The set of functions in Hardy–Krause variation is denoted by $\mathscr{BV}_{HK}(\overline{\mathbb{R}}^2)$. As in [15], $\mathscr{BV}_{HK}(\overline{\mathbb{R}}^2)$ forms a Banach space with norm defined by $\|\varphi\|_{\mathscr{BV}} = \|\varphi\|_\infty + \|V_1\varphi\|_\infty + \|V_2\varphi\|_\infty + V_{12}\varphi$. Here, $V_1\varphi$ and $V_2\varphi$ are bounded variation to the functions of (ii) and (iii) in Definition 3. For any multidimensional case, any function of

Hardy–Krause variation is a multiplier for HK-integrable functions, see [9]. Therefore, define following integration by parts, as a two-dimensional case which is presented in ([9], Theorem 6.5.9).

Definition 4 (integration by parts). Let $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ with its primitive $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$. If $g \in \mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$, then define integration by parts on $[a,b] \times [c,d] \subseteq \overline{\mathbb{R}}^2$.

$$\int_a^b \int_c^d fg = F(a,c)g(a,c) + F(b,d)g(b,d) - F(a,d)g(a,d) - F(b,c)g(b,c)$$

$$- \int_a^b F(x,d)d_1g(x,d) + \int_a^b F(x,c)d_1g(x,c)$$

$$- \int_c^d F(b,y)d_2g(b,y) + \int_c^d F(a,y)d_2g(a,y)$$

$$+ \int_a^b \int_c^d F(x,y)d_{12}g(x,y). \tag{11}$$

Here, $dg(x,y)$ is Henstock–Stieltjes integral of relevant variables, see in detail [24]. This integration by parts formula induces the space of Hardy–Krause bounded variation functions, $\mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$, as a multiplier for $\mathscr{A}_C(\overline{\mathbb{R}}^2)$. This multiplier is important to define convolution and obtain convergence theorem.

It needs to obtain continuity of the Alexiewicz norm since the Dirichlet boundary condition is taken on it. For any $(s,t) \in \mathbb{R}^2$, the translation of $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2) \subseteq \mathscr{D}\prime(\mathbb{R}^2)$ is defined by $\langle \tau_{(s,t)}f, \phi \rangle = \langle f, \tau_{(-s,-t)}\phi \rangle$ for $\phi \in \mathscr{D}(\mathbb{R}^2)$, where $\tau_{(s,t)}\phi(x,y) = \phi(x-s, y-t)$. Translation is invariant and continuous under the Alexiewicz norm.

Theorem 5. *Let $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ and $(s,t) \in \mathbb{R}^2$. Then, (i) $\tau_{(s,t)}f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$, (ii) $\|f\| = \|\tau_{(s,t)}f\|$, and (iii) $\|f - \tau_{(s,t)}f\| \longrightarrow 0$ as $(s,t) \longrightarrow (0,0)$. i.e., continuity of f in the Alexiewicz norm.*

Proof. Let $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$ be the primitive of $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$. Then, $\tau_{(s,t)}F$ is a continuous function.

(i) For any $\phi \in \mathscr{D}(\mathbb{R}^2)$, applying change of variable in integral,

$$\langle \tau_{(s,t)}f, \phi \rangle = \langle f, \tau_{(-s,-t)}\phi \rangle$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\phi(x+s, y+t)dxdy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-s, y-t)\phi(x,y)dxdy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_{12}F(x-s, y-t)\phi(x,y)dxdy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_{12}\big(\tau_{(s,t)}F(x,y)\big)\phi(x,y)dxdy. \tag{12}$$

Therefore, $\tau_{(s,t)}F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$ is the primitive of $\tau_{(s,t)}f$.

(ii) Clearly, $\|F\|_\infty = \|\tau_{(s,t)}F\|_\infty$ for any $(s,t) \in \mathbb{R}^2$. Therefore, $\|f\| = \|\tau_{(s,t)}f\|$.

(iii) If $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$, then $(f - \tau_{(s,t)}f) \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ so that

$$\|f - \tau_{(s,t)}f\| = \|F - \tau_{(s,t)}F\|_\infty = \sup_{(x,y)\in\mathbb{R}^2} |F(x,y) - F(x-s, y-t)| \longrightarrow 0, \tag{13}$$

when $(s,t) \longrightarrow (0,0)$ because of uniform continuity of F. $\square$

4. Integrable Distributional Solution for Poisson Equation

Poisson's kernel and its convolution are used to find a classical solution to the Dirichlet problem. In this section, we use convolution and convergence theorem on $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ to find integrable distributional solution. If $g \in \mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$,

then $\tau_{(-s,-t)}g \in \mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$. From integration by parts, the convolution

$$f * g(x,y) = \int_{\mathbb{R}} \int_{\mathbb{R}} f\tau_{(-s,-t)}g\,dsdt$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-s, y-t)g(x,y)dsdt \tag{14}$$

exists for $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ and $g \in \mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$. Proposition 2 gives some basic properties of convolution which help with our problem.

Proposition 2. *If $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ and $g \in \mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$, then (i) $f * g$ exists on $\mathbb{R}^2$ and (ii) $f * g = g * f$.*

Proof. See [15], Theorem 14.1.   □

Convergence theorems on Banach space are important when solving partial differential equations. There are several convergence theorems in $\mathscr{A}_C$ such as strong convergence in $\mathscr{A}_C$, weak/strong convergence in $\mathscr{BV}$, and weak/strong convergence in $\mathscr{D}$. Details are discussed in the plane by Ang, Schmitt and Vy [14] and on $\overline{\mathbb{R}}$ by Talvila [2]. Now, we move into the convergence theorem on $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ which will help to interchange limit of integral.

Proposition 3. *Let φ_n be a sequence in $\mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$ so that $\|\varphi_n\|_{\mathscr{BV}} < \infty$ and $\lim_{n \to \infty} \varphi_n = \varphi$ pointwise on $\overline{\mathbb{R}}^2$ for a function φ. If $g \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$, then $g \in \mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$ and*

$$\lim_{n \to \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} g \varphi_n = \int_{\mathbb{R}} \int_{\mathbb{R}} g \varphi. \tag{15}$$

Proof. See [15], Proposition 9.1.   □

Let us move into the main problem. As in [25] (pg. 84), Poisson's equation with Dirichlet boundary condition is easily reduced to the case where either $f = 0$ or $g = 0$ in (1). Therefore, consider integrable distributional solution for the following problem:

$$\begin{cases} \triangle u = 0, & \text{in } \mathbb{R}^3_+, \\ u = g, & \text{on } \partial \mathbb{R}^3_+, \end{cases} \tag{16}$$

where $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$. For $z > 0$, consider Poisson's kernel for the upper half space in 3-dimension

$$K(x, y, z) = \frac{z(x^2 + y^2 + z^2)^{-(3/2)}}{2\pi}. \tag{17}$$

Note that $\int_{\mathbb{R}} \int_{\mathbb{R}} K \mathrm{d}x \mathrm{d}y = 1$. If $g \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then classical solution for the problem is given by the following convolution:

$$\begin{aligned} u(x, y, z) &= K(x, y, .) * g(x, y) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - s, y - t, .) g(s, t) \mathrm{d}t \mathrm{d}s. \end{aligned} \tag{18}$$

This convolution gives integrable distributional solution for (16).

Theorem 6. *Let $g \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ and define $u(x, y, z)$ by (18). Then,*

(i) For each $x, y \in \mathbb{R}$ and $z > 0$, the convolution $K(x, y, .) * g(x, y) = u(x, y, z)$ exists and $u \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$

(ii) $\triangle u(x, y, z) = 0$ in $\mathbb{R}^3_+$

(iii) $\lim_{z \to 0^+} \|u(x, y, z) - g(x, y)\| = 0$, for $x, y \in \mathbb{R}$

Proof

(i) Fix $x, y \in \mathbb{R}$ and $z > 0$. Then, the function $(s, t) \longmapsto K(x - s, y - t, z)$ is of Hardy–Krause variation on $\mathbb{R}^2$. From Proposition 2, $K * g$ exists on the upper half space. Also, $u = K * g \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ because any function of Hardy–Krause variation is a multiplier for HK-integrable functions.

(ii) Fix $x, y \in \mathbb{R}$ and $z > 0$. To get u_x, let $x_n \longrightarrow x$ be a sequence so that $x_n \neq x$. For any $(s, t) \in \mathbb{R}^2$, define

$$\varphi_n(s, t) = \frac{K(x_n - s, y - t, z) - K(x - s, y - t, z)}{x_n - x}. \tag{19}$$

Then, $u_x = \lim_n \int_{\mathbb{R}} \int_{\mathbb{R}} g(s, t) \varphi_n(s, t) \mathrm{d}s \mathrm{d}t$. Since $K(x - s, y - t, z)$ of Hardy–Krause bounded variation, so does $\varphi_n(s, t)$ for each n. Since $g \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$, from Proposition 3

$$\begin{aligned} u_x &= \int_{\mathbb{R}} \int_{\mathbb{R}} \lim_{n \to \infty} g(s, t) \varphi_n(s, t) \mathrm{d}s \mathrm{d}t \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} g(s, t) K_x(x - s, y - t, z) \mathrm{d}s \mathrm{d}t. \end{aligned} \tag{20}$$

Repeat the same argument to obtain u_{xx}, u_y, u_{yy}, u_z, and u_{zz}. From the linearity of integral,

$$\triangle u(x, y, z) = \int_{\mathbb{R}} \int_{\mathbb{R}} g(s, t) \triangle K(x - s, y - t, z) \mathrm{d}s \mathrm{d}t. \tag{21}$$

But Poisson kernel is harmonic, so that $\triangle u(x, y, z) = 0$ in $\mathbb{R}^3_+$.

(iii) Let any $\alpha, \beta \in \mathbb{R}$. For Poisson kernel $\int_{\mathbb{R}} \int_{\mathbb{R}} K \mathrm{d}x \mathrm{d}y = 1$. Using Fubini–Tonelli theorem and interchanging iterated integral,

$$
\begin{aligned}
\int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} (u - g) \mathrm{d}x \mathrm{d}y &= \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} [u(x, y, z) - g(x, y)] \mathrm{d}x \mathrm{d}y \\
&= \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} [g(x, y) * K(x, y, z) - g(x, y)] \mathrm{d}x \mathrm{d}y \\
&= \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} \int_{\mathbb{R}} \int_{\mathbb{R}} K(s, t, z) [g(x - s, y - t) - g(x, y)] \mathrm{d}s \mathrm{d}t \mathrm{d}x \mathrm{d}y \\
&= \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{z}{2\pi \left(s^2 + t^2 + z^2 \right)^{3/2}} [g(x - s, y - t) - g(x, y)] \mathrm{d}s \mathrm{d}t \mathrm{d}x \mathrm{d}y \\
&= \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi \left(s^2 + t^2 + 1 \right)^{3/2}} [g(x - zs, y - zt) - g(x, y)] \mathrm{d}s \mathrm{d}t \mathrm{d}x \mathrm{d}y \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{2\pi \left(s^2 + t^2 + 1 \right)^{3/2}} \int_{-\infty}^{\alpha} \int_{-\infty}^{\beta} [g(x - zs, y - zt) - g(x, y)] \mathrm{d}x \mathrm{d}y \mathrm{d}s \mathrm{d}t \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} K(s, t, 1) \|g(x - zs, y - zt) - g(x, y)\| \mathrm{d}s \mathrm{d}t \\
&\leq 2\|g\|.
\end{aligned}
\tag{22}
$$

Since α and β are arbitrary,

$$
\|u(x, y, z) - g(x, y)\| \leq \int_{\mathbb{R}} \int_{\mathbb{R}} K(s, t, 1) \|g(x - zs, y - zt) - g(x, y)\| \mathrm{d}s \mathrm{d}t \leq 2\|g\|.
\tag{23}
$$

To let $z \longrightarrow 0^+$, apply dominated convergence theorem (HK-integral) for the sequence of functions:

$$
g_n(s, t) = K(s, t, 1) \left\| g\left(x - \frac{1}{n}s, y - \frac{1}{n}t \right) - g(x, y) \right\|.
\tag{24}
$$

Then, $|g_n(s, t)| \leq 2\|g\| K(s, t, 1) \in \mathscr{H}\mathscr{K}(\mathbb{R}^2)$ since $\int_{\mathbb{R}} \int_{\mathbb{R}} K(s, t, 1) \mathrm{d}s \mathrm{d}t = 1$. It gives

$$
\begin{aligned}
\lim_{n \to \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} g_n(s, t) \mathrm{d}s \mathrm{d}t &= \lim_{n \to \infty} \int_{\mathbb{R}} \int_{\mathbb{R}} K(s, t, 1) \|g(x - zs, y - zt) - g(x, y)\| \mathrm{d}s \mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \lim_{n \to \infty} g_n(s, t) \mathrm{d}s \mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} K(s, t, 1) \lim_{n \to \infty} \left\| g\left(x - \frac{1}{n}s, y - \frac{1}{n}t \right) - g(x, y) \right\| \mathrm{d}s \mathrm{d}t.
\end{aligned}
\tag{25}
$$

From part (iii) of Theorem 5, we get $\lim_{n \to \infty} \|g(x - (1/n)s, y - (1/n)t) - g(x, y)\| = 0$ so that limit of (23), $\lim_{z \to 0^+} \|u(x, y, z) - g(x, y)\| = 0$, and this completes the theorem. $\quad\square$

Next, we give an example which will apply Theorem 6.

Example 1. Consider the problem (16) in which

$$g(x,y) = \begin{cases} \dfrac{\sin x \sin y}{xy}, & x \neq 0 \text{ and } y \neq 0, \\ \\ 0, & x = 0 \text{ or } y = 0. \end{cases} \quad (26)$$

Then, g is HK-integrable but not absolutely integrable. Hence, $g \in \mathscr{A}_C(\overline{\mathbb{R}}^2) \smallsetminus L^1(\mathbb{R}^2)$. From Theorem 6,

$$\begin{aligned} u(x,y,z) &= \int_{\mathbb{R}} \int_{\mathbb{R}} K(x-s, y-t, .) g(s,t) dt ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{z}{2\pi} \left[(x-s)^2 + (y-t)^2 + z^2 \right]^{-(3/2)} \left(\frac{\sin s \sin t}{st} \right) dt ds, \end{aligned} \quad (27)$$

exists and gives solution for the example. Note that $Kg \in \mathscr{HK}(\mathbb{R}^2)$ since $K \in \mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$ is a multiplier for $\mathscr{HK}(\mathbb{R}^2)$. However, the solution does not exist with the Lebesgue integral. For contraposition, suppose that $Kg \in L^1(\mathbb{R}^2)$. Then, it follows from Fubini's theorem [9] (Theorem 2.5.5) that the function

$$h: s \longmapsto K(x-s, y-t, z) g(s,t), \quad (28)$$

belongs to $L^1(\mathbb{R}^1)$ for almost all $t \in \mathbb{R}$. That is,

$$\begin{aligned} h(s) &= \left(\frac{z \sin t}{2\pi t} \right) \frac{\sin s}{s \left[(x-s)^2 + (y-t)^2 + z^2 \right]^{3/2}} \in L^1(\mathbb{R}^1) \\ &= h_0(s) \left(\frac{\sin s}{s} \right), \end{aligned} \quad (29)$$

where $h_0(s) = (z \sin t / 2\pi t)(1/[(x-s)^2 + (y-t)^2 + z^2]^{3/2})$. This contradicts that $h(s) \notin L^1(\mathbb{R}^1)$. To see that, suppose $h(s) \in L^1[0,1] \subset L^1(\mathbb{R}^1)$. Since $h_0(s) \in L^\infty[0,1]$ is multiplier for $L^1[0,1]$, it follows that $\sin s/s \in L^1[0,1]$. This implies that $\sin s/s$ absolutely integrable on $[0,1]$, which is a contradiction. Hence, $h(s) \notin L^1(\mathbb{R}^1)$.

Even HK-integral exists in (27), it may not be able to evaluate and get explicit function. In that case, numerical integration can be applied. Numerical integration should be done with respect to s and t while fixing x, y, and z, details in [26]. This reduces solving the partial differential equation into applying numerical integration for the double integral (27).

5. Discussion

In this work, integrable distributional solution is obtained for Poisson equation with the Dirichlet boundary condition in the upper half space. Boundary conditions are taken in the Alexiewicz norm. Lebesgue integral uses for the classical solution to Poisson equation. Because of the proper inclusion $L^1 \subsetneq \mathscr{HR}$, we may use HK-integral to solve Poisson equation. However, there are some limitations due to the absence of any natural topology. That is, $\mathscr{HK}$ is not a Banach space; nevertheless, there is a natural seminorm called Alexiewicz. This is one of the main reasons to move into the space of integrable distribution $\mathscr{A}_C(\overline{\mathbb{R}}^2)$, which contains $\mathscr{HK}$ and leads to a Banach space with Alexiewicz norm.

Integrable distribution is defined as derivative (distributional sense) of function in $\mathscr{B}_C(\overline{\mathbb{R}}^2)$. In this definition, any $f \in \mathscr{A}_C(\overline{\mathbb{R}}^2)$ identifies with the unique primitive $F \in \mathscr{B}_C(\overline{\mathbb{R}}^2)$. This unique primitive allows to define a Alexiewicz norm $\|\cdot\|$ on $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ and leads to a Banach space. Alexiewicz norm does not induce an inner product in $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ because it does not satisfy the parallelogram law. Therefore, $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is not a Hilbert space under the norm $\|\cdot\|$. However, the space $(\mathscr{A}_C(\overline{\mathbb{R}}^2), \|\cdot\|)$ is isometrically isomorphic to the space $(\mathscr{B}_C(\overline{\mathbb{R}}^2), \|\cdot\|_\infty)$. Integral on $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is uniquely defined by uniqueness of the primitive. $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ is the completion of $L^1(\mathbb{R}^2)$ (or $\mathscr{HK}(\mathbb{R}^2)$) with respect to the corresponding Alexiewicz norm.

In one-dimensional HK-integral, space of bounded variation functions, $\mathscr{BV}$, is multipliers for $\mathscr{HK}$. In multidimension case, there are several extensions to variation of function. To establish integration by parts formula and multipliers for $\mathscr{A}_C(\overline{\mathbb{R}}^2)$, we begin with Hardy–Krause variation. The space of Hardy–Krause bounded variation functions, $\mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$, as a multiplier for $\mathscr{A}_C(\overline{\mathbb{R}}^2)$. This multiplier is important to define convolution and obtain convergence theorem. $\mathscr{BV}_{\mathscr{HK}}(\overline{\mathbb{R}}^2)$ is used to define convolution and obtain convergence theorem on $\mathscr{A}_C(\overline{\mathbb{R}}^2)$. There are several convergence theorems in $\mathscr{A}_C$ such as strong convergence in $\mathscr{A}_C$, weak/strong convergence in $\mathscr{BV}$, and weak/strong convergence in $\mathscr{D}$. Convergence theorem on $\mathscr{A}_C(\overline{\mathbb{R}}^2)$ and dominated convergence theorem in $\mathscr{HK}(\mathbb{R}^2)$ are used to obtain integrable distributional solution. Finding of this article gives solution for Poisson equation for broader initial condition in the upper half space. Dirichlet problem in $\mathbb{R}^3$ is considered, and the solution has obtained by HK-integration where it is not possible from Lebesgue integral. Because of function's complexity, it may not be able to have solution in explicit

form. It would help keep an analytical outlook in the solution, which would be better than solving the partial differential equation by numerical methods such as finite difference methods and finite element methods.

There is potential to find integrable distributional solution for Poisson equation in other boundary conditions, Neumann and Robin. But it needs to develop suitable convergence theorems on $\mathscr{A}_C$ and its multipliers $\mathscr{BV}_{\mathscr{HK}}$. Also, it is possible to consider Poisson equation in unit ball with the relevant Poisson kernel. Talvila, in [15], gives some construction to develop integrable distribution on $\mathbb{R}^n$, integration by parts, and mean value theorem and defines $\mathscr{A}_C(\overline{\mathbb{R}}^n)$ and $\mathscr{B}_C(\overline{\mathbb{R}}^n)$ for multidimensional case. Hence, integrable distributional solution is possible to obtain for Poisson equation with Dirichlet boundary condition in $\mathbb{R}^n_+$ with suitable convergence theorem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] R. L. Burden and J. D. Faires, *Numerical Analysis*, Thomson Brooks/Cole, USA, 8th edition, 2005.

[2] C. Anitescu, E. Atroshchenko, N. Alajlan, and T. Rabczuk, "Artificial neural network methods for the solution of second order boundary value problems," *Computers, Materials & Continua*, vol. 59, no. 1, pp. 345–359, 2019.

[3] D. A. León-Velasco, M. M. Morín-Castillo, J. J. Oliveros-Oliveros, T. Pérez-Becerra, and J. A. Escamilla-Reyna, "Numerical solution of some differential equations with Henstock–Kurzweil functions," *Journal of Function Spaces*, vol. 2019, Article ID 8948570, 9 pages, 2019.

[4] G. Green, "An essay on the application of mathematical analysis to the theories of electricity and magnetism," *Journal für die reine und angewandte Mathematik (Crelles Journal)*, vol. 1850, no. 39, pp. 73–89, 1850.

[5] L. C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, USA, 1998.

[6] R. Bartle, *A Modern Theory of Integration*, American Mathematical Society, R.I., Providence, RI, USA, 2001.

[7] R. Henstock, *The General Theory of Integration*, Oxford University Press, Oxford, UK, 1991.

[8] C. Swartz, *Introduction to Gauge Integrals*, World Scientific, Singapore, 2001.

[9] T.-Y. Lee, *Henstock–Kurzweil Integration on Euclidean Spaces*, World Scientific, Singapore, 2011.

[10] E. Talvila, "Continuity in the Alexiewicz norm," *Mathematica Bohemica*, vol. 131, no. 2, pp. 189–196, 2006.

[11] E. Talvila, "Estimates of henstock-kurzweil Poisson integrals," *Canadian Mathematical Bulletin*, vol. 48, no. 1, pp. 133–146, 2005.

[12] A. Alexiewicz, "Linear functionals on Denjoy-integrable functions," *Colloquium Mathematicum*, vol. 1, no. 4, pp. 289–293, 1948.

[13] P. Mikusiński and K. Ostaszewski, "Embedding Henstock integrable functions into the space of Schwartz distributions," *Real Analysis Exchange*, vol. 14, no. -89, pp. 24–29, 1988.

[14] D. D. Ang, K. Schmitt, and L. K. Vy, "A multidimensional analogue of the denjoy–perron–henstock–kurzweil integral," *The Bulletin of the Belgian Mathematical Society — Simon Stevin*, vol. 4, pp. 355–371, 1997.

[15] E. Talvila, "The continuous primitive integral in the plane," *Real Analysis Exchange*, vol. 45, no. 2, pp. 283–326, 2020.

[16] S. Heikkilä, "On nonlinear distributional and impulsive Cauchy problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 2, pp. 852–870, 2012.

[17] B. Liang, G. Ye, W. Liu, and H. Wang, "On second order nonlinear boundary value problems and the distributional Henstock-Kurzweil integral," *Boundary Value Problem*, vol. 2015, 2015.

[18] A. Sikorska-Nowak, "Existence theory for integrodifferential equations and henstock-kurzweil integral in Banach spaces," *Journal of Applied Mathematics*, vol. 2007, Article ID 31572, 12 pages, 2007.

[19] M. Tvrdý, "Linear distributional differential equations of the second order," *Math. Bohem.* vol. 119, no. 4, pp. 415–436, 1994.

[20] L. Schwartz, *Théorie des Distributions*, Hermann, Paris, France, 1950.

[21] E. Talvila, "The distributional denjoy integral," *Real Analysis Exchange*, vol. 33, no. 1, pp. 51–82, 2008.

[22] K. Ostaszewski, "The space of Henstock integrable functions of two variables," *International Journal of Mathematics and Mathematical Sciences*, vol. 11, no. 1, pp. 15–21, 1988.

[23] C. R. Adams and J. A. Clarkson, "On definitions of bounded variation for functions of two variables," *Transactions of the American Mathematical Society*, vol. 35, no. 4, pp. 824–854, 1933.

[24] B. Bongiorno, "The henstock–kurzweil integral," in *Handbook of Measure Theory*, E. Pap, Ed., pp. 589–615, North Holland, Dordrecht, Netherlands, 2002.

[25] G. B. Folland, *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NY, USA, 1995.

[26] S. M. Aljassas, "Numerical methods for evaluation of double integrals with continuous integrands," *International Journal of Pure and Applied Mathematics*, vol. 119, no. 10, 2018.