Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2021, Article ID 5524430, 7 pages
https://doi.org/10.1155/2021/5524430

Research Article

Fourier Transformation and Stability of a Differential Equation on $L^1(\mathbb{R})$

Hamid Rezaei,[1] **Zahra Zafarasa,**[1] **and Lotfollah Karimi**[2]

[1]*Department of Mathematics, College of Sciences, Yasouj University, Yasouj-75914-74831, Iran*
[2]*Department of Mathematics, College of Sciences, Hamedan University of Technology, Hamedan-65169-13733, Iran*

Correspondence should be addressed to Hamid Rezaei; rezaei@yu.ac.ir

Received 26 February 2021; Revised 19 May 2021; Accepted 5 July 2021; Published 15 July 2021

Academic Editor: Theodore E. Simos

In the present paper, by the Fourier transform, we show that every linear differential equation with constant coefficients of n-th order has a solution in $L^1(\mathbb{R})$ which is infinitely differentiable in $\mathbb{R}\setminus\{0\}$. Moreover the Hyers–Ulam stability of such equations on $L^1(\mathbb{R})$ is investigated.

1. Introduction

One of the main mathematical problems in the study of functional equations is the Hyers–Ulam stability of equations. S. M. Ulam was the first one who investigated this problem by the question under what conditions does there exist an additive mapping near an approximately additive mapping? Hyers in [1] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. Let X_1, X_2 be two real Banach spaces, and $\varepsilon > 0$ is given. Then, for every mapping, $f: X_1 \longrightarrow X_2$ satisfying

$$\|f(x+y) - f(x) - f(y)\| < \varepsilon \quad \text{for all } x; y \in X_1. \tag{1}$$

There exists a unique additive mapping $g: X_1 \longrightarrow X_2$ with the property

$$\|f(x) - g(x)\| < \varepsilon, \quad \text{for all } x \in X_1. \tag{2}$$

After Hyers result, these problems have been extended to other functional equations [2]. This may be the most important extension in the Hyers–Ulam stability of the differential equations.

The differential equation $\varphi(y, y', \dots, y^{(n)}) = f$ has Hyers–Ulam stability on normed space X if, for given $\varepsilon > 0$ and a function y such that $\|\varphi(y, y', \dots, y^{(n)}) - f\| < \varepsilon$, there is a solution $y_a \in X$ of the differential equation such that $\|y - y_a\| < K(\varepsilon)$ and $\lim_{\varepsilon \longrightarrow 0} K(\varepsilon) = 0$.

In the theory of differential equations, we search for a classical symmetric solution or a weak or generalized solution. Usually, these solutions at least satisfy the equation almost everywhere. It seems that the ϵ-solution provides us a wider notion of a solution, and for some physical applications, it models the underlying physics more appropriate. We hope this paper is a beginning for further research about the ϵ-solution of a differential equation. In the theory of differential equations, it is particularly important for problems arising from physical applications that we would prefer our solution changes only a little when the data of the problem change a little. It is called the continuity of the equation with respect to data. See [3] and section 3.2 in [4]. It is straightforward to check that the Hyers–Ulam stability of our problems is equivalent to the continuity of the equations $\varphi(y, y\prime, \dots, y^{(n)}) = f$ with respect to the right-hand side f.

Alsina and Ger [5] were the first authors who investigated the Hyers–Ulam stability of a differential equation. The result of the work of Alsina and Ger was extended to the Hyers–Ulam stability of a higher-order differential equation with constant coefficients in [6]. We mention here only the recent contributions in this subject of some mathematicians [6–15].

Rus has proved some results on the stability of ordinary differential and integral equations using Gronwall lemma and the technique of weakly Picard operators [13, 14]. Using the method of integral factors, the Hyers–Ulam stability of some ordinary differential equations of first and second order with constant coefficients has been proved in [9, 15].

We recall that the Laplace transform is a powerful integral transform used to switch a function from the time domain to the s-domain. It maps a function to a new function on the complex plane. The Laplace transform can be used in some cases to solve linear differential equations with given initial conditions. Applying the Laplace transform method, Rezaei et al. [16] investigated the Hyers–Ulam stability of the linear differential equations of functions defined on $(0, +\infty)$.

Fourier transforms can also be applied to the solution of differential equations of functions with domain $(-\infty, +\infty)$. They can convert a function to a new function on the real line. Since the Laplace transform cannot be used for the functions defined on $(-\infty, +\infty)$, in the present paper, we apply the Fourier transforms to show that every n-order linear differential equation with constant coefficients has a solution in $L^1(\mathbb{R})$ which is infinitely differentiable in $\mathbb{R}\setminus\{0\}$. Moreover, it proves the Hyers–Ulam stability of the equation on $L^1(\mathbb{R})$.

It seems that authors of [17], Theorem 3.1, by using the Fourier transform, proved the Hyers–Ulam stability of the linear differential equation of functions on $(0, +\infty)$. But, this is not a new result because the restriction of the Fourier transform on the space of functions with domain in $(0, +\infty)$ is the Laplace transform, and it has already been proven that the Laplace transform established the Hyers–Ulam stability of the linear differential equation of functions on $(0, +\infty)$ (see [16]).

2. Fourier Transform and Inversion Formula

Throughout this paper, $\mathbb{F}$ will denote either the real field, $\mathbb{R}$, or the complex field, $\mathbb{C}$. Assume that $f\colon (-\infty, +\infty) \longrightarrow \mathbb{F}$ is absolutely integrable. Then, the Fourier transform associated to f is a mapping $\mathcal{F}(f)\colon \mathbb{R} \longrightarrow \mathbb{C}$ defined by

$$\mathcal{F}(f)(w) = \int_{-\infty}^{\infty} e^{-iwt} f(t)\mathrm{d}t. \tag{3}$$

Also, the inverse Fourier transform associated to a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}^{-1}(f)(t) = \int_{-\infty}^{\infty} e^{iwt} f(w)\mathrm{d}w. \tag{4}$$

By the Fourier Inversion Theorem (see Theorem A.14 in [18]) if $f \in L^1(\mathbb{R})$ and f and $f\prime$ are piecewise continuous on $\mathbb{R}$, that is, both are continuous in any finite interval except possibly for a finite number of jump discontinuities, then at each point t where f is continuous,

$$\mathcal{F}^{-1}(\mathcal{F}(f))(t) = f(t). \tag{5}$$

In particular, if $f, \mathcal{F}(f) \in L^1(\mathbb{R})$, then the above-mentioned relation holds almost everywhere on $\mathbb{R}$ (see Theorem 9.11 in [19]).

In the following, some required properties of the Fourier transform are presented.

Proposition 1. *Let $f \in L^1(\mathbb{R})$, $F = \mathcal{F}(f)$, and u be the step function defined by $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$. Then,*

(i) $\mathcal{F}(e^{-zt}u(t))(w) = (1/iw + z)$ *provided that* $\mathrm{Re}(z) > 0$
(ii) $\mathcal{F}(f(t - t_0/k)(w)) = ke^{-iwt_0}F(kw)$
(iii) $\mathcal{F}(f(t)e^{iw_0t})(w) = F(w - w_0)$
(iv) $\mathcal{F}((-it)^n f(t))(w) = F^{(n)}(w)$
(v) $\mathcal{F}(y^{(n)}(t))(w) = (iw)^n \mathcal{F}(y)(w)$.

Proof. Parts (i)–(iii) are obtained by the definition of Fourier transform. For part (iv), see Theorem 1.6, page 136, in [20]. For part (v), see Theorem 3.3.1, part (f), in [21]. □

We recall that the convolution of two functions $f, g \in L^1(\mathbb{R})$ is defined by

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t - x)g(x)\mathrm{d}x. \tag{6}$$

Then, $f * g \in L^1(\mathbb{R})$, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, and

$$\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g). \tag{7}$$

Moreover, we have the following theorem.

Theorem 1. *Let $f, g \in L^1(\mathbb{R})$; then,*

(i) $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.
(ii) $\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}(f) * \mathcal{F}^{-1}(g)$.
(iii) If either f or g is differentiable, then $f * g$ is differentiable. If f' exists and is continuous, then $(f * g)' = f' * g$.

Proof. For part (i), see Theorems 1.3 and 1.5, page 135, in [20]. Part (ii) follows from the definition of $\mathcal{F}^{-1}$ and the Fourier Inversion Theorem. For part (iii), see Proposition 2.1, page 68, in [22]. □

Note that if u is the step function, then

$$(f * u)(t) = (u * f)(t) = \int_{-\infty}^{+\infty} u(t - x)f(x)\mathrm{d}x = \int_{-\infty}^{t} f(x)\mathrm{d}x. \tag{8}$$

The following corollary is deduced from Theorem 1.5, page 135, in [20].

Corollary 1. *Let $f \in L^1(\mathbb{R})$; then,*

$$\mathcal{F}(f * u)(w) = \frac{\mathcal{F}(f)(w)}{iw}, \quad (w \neq 0). \tag{9}$$

3. Hyers–Ulam Stability of the Linear Differential Equation

Before stating the main theorem, we need the following important proposition.

Proposition 2. *Let $f \in L^1(\mathbb{R})$ and p be a polynomial with the complex roots $w_0, w_1, \ldots, w_{k-1}$, $k \geq 1$. Then, there is a function $y_0 \in L^1(\mathbb{R})$ which is infinitely differentiable in $\mathbb{R} \setminus \{0\}$, $(k-1)$-times differentiable at zero, and satisfying*

$$\mathcal{F}(y_0)(w) = \frac{\mathcal{F}(f)(w)}{p(w)}, \quad (w \neq w_0, w_1, \ldots, w_{k-1}). \tag{10}$$

Proof. First assume that $p(w) = (w - w_0)^k$ and $\operatorname{Im}(w_0) \neq 0$. Put $z_1 := -iw_0$ when $\operatorname{Im}(w_0) > 0$ and $z_2 := iw_0$ when $\operatorname{Im}(w_0) < 0$. Since $\operatorname{Re}(z_i) > 0, i = 1, 2$, Proposition 1, part (i), implies that

$$\mathcal{F}\left(e^{-z_1 t} u(t)\right)(w) = \frac{1}{iw + z_1} = \frac{1}{iw - iw_0} = \frac{1}{i(w - w_0)},$$
$$\mathcal{F}\left(e^{-z_2 t} u(t)\right)(w) = \frac{1}{iw + z_2} = \frac{1}{iw + iw_0} = \frac{1}{i(w + w_0)}. \tag{11}$$

In the second case, using part (ii) of Proposition 1 for $t_0 = 0$ and $k = -1$, we get

$$\mathcal{F}\left(e^{z_2 t} u(-t)\right)(w) = \frac{1}{-iw + iw_0} = \frac{-1}{i(w - w_0)}. \tag{12}$$

According to the abovementioned computations, the function $f_0: (-\infty, +\infty) \longrightarrow \mathbb{F}$ is defined by

$$f_0(t) = \begin{cases} ie^{iw_0 t} u(t), & \text{if} \quad \operatorname{Im}(w_0) > 0, \\ -ie^{iw_0 t} u(-t), & \text{if} \quad \operatorname{Im}(w_0) < 0. \end{cases} \tag{13}$$

It satisfies the equation

$$\mathcal{F}(f_0)(w) = \frac{1}{w - w_0}, \tag{14}$$

and by part (iv) of Proposition 1, we see that

$$\mathcal{F}\left((-it)^{k-1} f_0\right)(w) = \left[\frac{1}{(w - w_0)}\right]^{(k-1)} = \frac{(-1)^{k-1}(k-1)!}{(w - w_0)^k}. \tag{15}$$

Now, we put

$$y_0 := \left(\frac{(-it)^{k-1}}{(k-1)!} f_0\right) * f. \tag{16}$$

Then, $y_0 \in L^1(\mathbb{R})$, by Theorem 1 part (ii), y_0 is infinitely differentiable in $\mathbb{R} \setminus \{0\}$, $y_0^{(k-1)}(0)$ exists, and

$$\mathcal{F}(y_0)(w) = \mathcal{F}\left(\frac{(-it)^{k-1}}{(k-1)!} f\right)(w) \mathcal{F}(y)(w) = \frac{\mathcal{F}(y)(w)}{(w - w_0)^m}, \tag{17}$$

for $w \neq w_0$. In the next case, we suppose that $\operatorname{Im}(w_0) = 0$, namely, w_0 is a real number. In this case, the argument is different. Indeed, let $f_0 = f$ and

$$f_{i+1}(t) = e^{iw_0 t}\left[\left(f_i e^{-iw_0 t}\right) * u(t)\right], \tag{18}$$

for $i = 0, 1, 2, \ldots, k$. Then, by Corollary 1, we have

$$\mathcal{F}(f * u)(w) = \frac{\mathcal{F}(f)(w)}{iw}, \quad (w \neq 0), \tag{19}$$

and so, by (iii) of Proposition 1,

$$\mathcal{F}\left(f(t)e^{-iw_0 t} * u\right)(w) = \frac{\mathcal{F}\left(f(t)e^{-iw_0 t}\right)(w)}{iw} = \frac{\mathcal{F}(y)(w + w_0)}{iw}. \tag{20}$$

Hence,

$$\mathcal{F}(f_1)(w) = \mathcal{F}\left(e^{iw_0 t}\left[\left(f(t)e^{-iw_0 t}\right) * u(t)\right]\right)(w) = \frac{\mathcal{F}(f)(w)}{i(w - w_0)}. \tag{21}$$

Continuing in this way, we get

$$\mathcal{F}(f_{i+1})(w) = \frac{\mathcal{F}(f_i)(w)}{i(w - w_0)}. \tag{22}$$

Therefore,

$$\mathcal{F}(f_k)(w) = \frac{\mathcal{F}(w)(y)}{i^k (w - w_0)^k}. \tag{23}$$

Finally, we put $y_0 := i^k f_k$. Then, y_0 has the requested properties and satisfies in (10) for $p(w) = (w - w_0)^k$, and in this case, the proof is completed.

Now, we suppose that $p(w)$ expresses as a product of linear factors

$$p(w) = (w - w_1)^{k_1}, (w - w_2)^{k_2}, \ldots, (w - w_k)^{n_k}, \tag{24}$$

for some complex number $w_i, i = 1, \ldots, k$ and some integer $k_i, i = 1, 2, \ldots, k$. Applying the partial fraction decomposition of $1/P(w)$, we obtain

$$\frac{1}{p(w)} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_{ij}}{(w - w_i)^j}, \tag{25}$$

where λ_{ij} is a complex number for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$. Considering the first part of proof, there exists a $y_{ij} \in L^1(\mathbb{R})$ which is k-times differentiable on $\mathbb{R} \setminus \{0\}$ and

$$\mathcal{F}\left(y_{ij}\right)(w) = \frac{\mathcal{F}(y)}{(w - w_i)^j}, \quad (w \neq w_i), \tag{26}$$

for every integer $1 \leq i \leq k$ and $1 \leq j \leq n_i$. Then, we put

$$y_0(w) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{ij} y_{ij}(w), \quad (w \neq w_1, w_2, \ldots, w_k) \tag{27}$$

and use the linearity of Fourier transform and (26) to obtain

$$\mathcal{F}(y_0)(w) = \mathcal{F}\left(\sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{ij} y_{ij}(w)\right) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \lambda_{ij} \mathcal{F}\left(y_{ij}\right)(w),$$
$$= \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\lambda_{ij} \mathcal{F}(y)}{(w - w_i)^j} = \frac{\mathcal{F}(y)(w)}{p(w)}. \tag{28}$$

□

Theorem 2. *consider the differential equation*

$$y'(t) + a_0 y(t) = f(t), \quad (t \in \mathbb{R}), \tag{29}$$

where a_0 is a nonzero scalar such that $\mathrm{Re}(a_0) \neq 0$ and $f \in L^1(\mathbb{R})$. Then, there exists a constant M with the following property: for every $y \in L^1(\mathbb{R})$ and for given $\varepsilon > 0$ satisfying

$$\left\| y' + a_0 y - f \right\|_1 \leq \varepsilon, \tag{30}$$

there exists a differentiable solution $y_a \in L^1(\mathbb{R})$ of the equation such that $\|y_a - y\|_1 \leq M\varepsilon$.

Proof. Let

$$h(t) = y'(t) + a_0 y(t) - f(t). \tag{31}$$

Then,

$$\begin{aligned}\mathcal{F}(h)(w) &= \mathcal{F}(y')(w) + a_0 \mathcal{F}(y)(w) - \mathcal{F}(f)(w),\\ &= (iw)\mathcal{F}(y)(w) + a_0\mathcal{F}(y)(w) - \mathcal{F}(f)(w),\\ &= (iw + a_0)\mathcal{F}(y)(w) - \mathcal{F}(f)(w).\end{aligned} \tag{32}$$

Hence,

$$\mathcal{F}(y)(w) - \frac{\mathcal{F}(f)(w)}{iw + a_0} = \frac{\mathcal{F}(h)(w)}{iw + a_0}. \tag{33}$$

By the preceding proposition, there exists a function $y_a \in L^1(\mathbb{R})$ such that

$$\mathcal{F}(y_a)(w) = \frac{\mathcal{F}(f)(w)}{iw + a_0}. \tag{34}$$

According to (33), for $h = 0$, we find that y_a, in fact, is a solution of the equation. Without loss of generality, we suppose that $\mathrm{Re}(a_0) > 0$. Then, by considering (33) and part (i) of Proposition 1, we obtain

$$\begin{aligned}\mathcal{F}(y - y_a)(w) &= \mathcal{F}(y)(w) - \mathcal{F}(y_a)(w),\\ &= \mathcal{F}(y)(w) - \frac{\mathcal{F}(f)(w)}{iw + a_0},\\ &= \frac{\mathcal{F}(h(t))(w)}{iw + a_0} = \mathcal{F}\left(e^{-a_0 t}u(t) * h(t)\right)(w).\end{aligned} \tag{35}$$

Consequently, $y(t) - y_a(t) = e^{-a_0 t}u(t) * h(t)$ and

$$\begin{aligned}\|y - y_a\|_1 &= \int_{-\infty}^{+\infty} \left| y(t) - y_a(t) \right| \mathrm{d}t,\\ &= \int_{-\infty}^{+\infty} \left| e^{-a_0 t}u(t) * h(t) \right| \mathrm{d}t\\ &\leq \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} e^{-xa_0} u(x) h(t - x) \mathrm{d}x \right| \mathrm{d}t\\ &\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left| e^{-xa_0} \right| |u(x)| |h(t - x)| \mathrm{d}x \mathrm{d}t\\ &\leq \int_{\infty}^{+\infty} \int_{0}^{+\infty} e^{-x\mathrm{Re}(a_0)} |h(t - x)| \mathrm{d}x \mathrm{d}t\\ &\leq \int_{0}^{+\infty} \int_{-\infty}^{+\infty} e^{-x\mathrm{Re}(a_0)} |h(t - x)| \mathrm{d}t \mathrm{d}x \leq \frac{1}{\mathrm{Re}(a_0)} \|h\|_1 < \frac{\varepsilon}{\mathrm{Re}(a_0)}.\end{aligned} \tag{36}$$

This completes the proof. □

We denote by $C^n(\mathbb{R})$ the space of all n-times differentiable continuous functions on $\mathbb{R}$.

Theorem 3. *consider the differential equation*

$$y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) = f(t), \tag{37}$$

where $f \in L^1(\mathbb{R})$, $n > 1$, and $a_0, a_1, \ldots, a_{n-1}$ are given scalars such that $\mathrm{Re}(a_{n-1}) \neq 0$. Then, there exists a constant M with the following property: for every $y \in L^1(\mathbb{R})$ and for given $\varepsilon > 0$ satisfying

$$\left\| y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} - f \right\|_1 \leq \varepsilon, \tag{38}$$

there exists a solution y_a of equation (37) such that $y_a \in C^n(\mathbb{R}) \cap L^1(\mathbb{R})$ and

$$\left\| y_a - y \right\|_1 \leq M\varepsilon, \quad (t \in \mathbb{R}). \tag{39}$$

Proof. Let

$$h(t) = y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) - f(t). \tag{40}$$

Applying Proposition 1, part (v), we may derive

$$\begin{aligned}\mathcal{F}(h)(w) &= \mathcal{F}\left(y^{(n)}\right)(w) + \sum_{k=0}^{n-1} a_k \mathcal{F}\left(y^{(k)}\right)(w) - \mathcal{F}(f)(w),\\ &= (iw)^n \mathcal{F}(y) + \sum_{k=0}^{n-1} a_k (iw)^k \mathcal{F}(y) - \mathcal{F}(f)(w),\\ &= p(iw)\mathcal{F}(y)(w) - \mathcal{F}(f)(w),\end{aligned} \tag{41}$$

where p is a complex polynomial determined by

$$p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k. \tag{42}$$

Applying relations (40) and (41) for $h = 0$, we find that y is a solution of (37) if and only if

$$\mathcal{F}(y)(w) = \frac{\mathcal{F}(f)(w)}{p(iw)}. \tag{43}$$

Now, from (41), we deduce that

$$\frac{\mathcal{F}(h)(w)}{p(iw)} = \mathcal{F}(y)(w) - \frac{\mathcal{F}(f)(w)}{p(iw)}, \quad (iw \neq w_0, w_1, \ldots, w_{n-1}), \tag{44}$$

where $w_0, w_1, \ldots, w_{n-1}$ are the roots of the polynomial $p(w)$. By Proposition 2, there exists a function $y_a \in L^1(\mathbb{R})$ such that

$$\mathcal{F}(y_a)(w) = \frac{\mathcal{F}(f)(w)}{p(iw)}. \tag{45}$$

Since every function satisfying (43) is a solution of (37), we find that y_a is a solution of equation (37), and from (44), we obtain

$$\mathcal{F}(y)(w) - \mathcal{F}(y_a)(w) = \frac{\mathcal{F}(h)(w)}{p(iw)}, \tag{46}$$

and consequently,

$$\left|y(t) - y_a(t)\right| = \left|\mathcal{F}^{-1}\left(\frac{\mathcal{F}(h)(w)}{p(iw)}\right)(t)\right|. \tag{47}$$

By the definition of h and the inequality (38), $\|h\|_1 \leq \varepsilon$, so

$$|\mathcal{F}(h)(w)| \leq \int_{-\infty}^{+\infty} \left|e^{-iwt}\right| |h(t)| \mathrm{d}t \leq \varepsilon. \tag{48}$$

Now, by considering part (i) of Theorem 1, we have

$$\begin{aligned}\left\|y - y_a\right\|_1 &= \int_{-\infty}^{+\infty} \left|y(t) - y_a(t)\right| \mathrm{d}t,\\ &= \int_{-\infty}^{+\infty} \left|\mathcal{F}^{-1}\left(\frac{\mathcal{F}(h)(w)}{p(iw)}\right)\right| \mathrm{d}t,\\ &= \int_{-\infty}^{+\infty} \left|\mathcal{F}^{-1}(\mathcal{F}(h)(w))(t) * \mathcal{F}^{-1}\left(\frac{1}{p(iw)}\right)(t)\right| \mathrm{d}t,\\ &= \int_{-\infty}^{+\infty} \left|\left(h * \mathcal{F}^{-1}\left(\frac{1}{p(iw)}\right)\right)(t)\right| \mathrm{d}t,\\ &= \left\|h * \mathcal{F}^{-1}\left(\frac{1}{p(iw)}\right)\right\|_1,\\ &\leq \|h\|_1 \left\|\mathcal{F}^{-1}\left(\frac{1}{p(iw)}\right)\right\|_1 = M\varepsilon,\end{aligned} \tag{49}$$

where

$$M = \left\|\mathcal{F}^{-1}\left(\frac{1}{p(iw)}\right)\right\|_1 < +\infty, \quad (\text{because } \deg(p) > 1). \tag{50}$$

To see the last relation, looking at (42) and the fact that $w_0, w_1, \ldots, w_{n-1}$ are the roots of polynomial p, we get

$$\begin{gathered} w_0 + w_1 + \cdots w_{n-1} = -a_{n-1},\\ p(iw) = (iw - w_0)(iw - w_1)\ldots(iw - w_{n-1}). \end{gathered} \tag{51}$$

Since $\operatorname{Re}(a_{n-1}) \neq 0$, there exist some $0 \leq j \leq n-1$ such that $\operatorname{Re}(w_j) \neq 0$. Then, by Proposition 1, part (i), there exist some $y_j \in L^1(\mathbb{R})$ such that

$$\mathcal{F}\left(y_j\right)(w) = \frac{1}{iw - w_j}. \tag{52}$$

Let $q(w) = \prod_{k \neq j}(iw - w_k)$. Then,

$$\frac{1}{p(iw)} = \frac{\left(1/\left(iw - w_j\right)\right)}{q(w)} = \frac{\mathcal{F}\left(y_j\right)(w)}{q(w)}. \tag{53}$$

Since $y_j \in L^1(\mathbb{R})$, by Proposition 2, there exist some $y_j \in L^1(\mathbb{R})$ such that

$$\mathcal{F}\left(y_j\right)(w) = \frac{\mathcal{F}\left(y_j\right)(w)}{q(w)}. \tag{54}$$

Comparing the abovementioned relations, we see that

$$\mathscr{F}\left(y_j\right)(w)=\frac{1}{p(iw)}, \tag{55}$$

and hence,

$$\frac{1}{p(iw)}=\mathscr{F}^{-1}\left(y_j(w)\right)\in L^1(\mathbb{R}). \tag{56}$$

Therefore, (37) has Hyers–Ulam stability, and the proof is completed. □

Remark 1. Note that Proposition 2, in fact, states that equation (37) has a solution in $L^1(\mathbb{R})$ which is infinitely differentiable in $\mathbb{R}\setminus\{0\}$, $(k-1)$-times differentiable at zero.

Remark 2. In Theorem 2, it is easy to see that $M=1/|\mathrm{Re}(a_0)|$ is the minimal Hyers–Ulam constant. To see this, let the assumptions of Theorem 2 hold with the Hyers–Ulam constant M. Suppose that $\mathrm{Re}(a_0)>0$ and $\delta=\mathrm{Re}(a_0)\varepsilon$. Putting $f_\delta(t):=f(t)+\delta e^{-a_0 t}u(t)$, we see that $f_\delta\in L^1(\mathbb{R})$, and by Remark 1, the equation $y'(t)+a_0y(t)-f_\delta(t)=0$ has a solution $y_\delta\in L^1(\mathbb{R})$ satisfying

$$\begin{aligned}\left\|y_\delta'+a_0y_\delta-f\right\|_1&=\left\|y_\delta'+a_0y_\delta-f_\delta+f_\delta-f\right\|_1,\\&=\left\|f_\delta-f\right\|_1=\left\|\delta e^{-2a_0t}u(t)\right\|_1=\frac{\delta}{\mathrm{Re}(a_0)}\le\varepsilon.\end{aligned} \tag{57}$$

In this case, the function h in the proof of Theorem 2 is obtained by the following equation:

$$\begin{aligned}h(t)&=y_\delta'(t)+a_0y_\delta(t)-f(t)=y_\delta'(t)+a_0y_\delta(t)-f_\delta(t)\\&\quad+\delta e^{-a_0t}u(t)=\delta e^{-a_0t}u(t).\end{aligned} \tag{58}$$

Applying Theorem 2, there is a solution y_a of the equation $y'+a_0y-f=0$ such that $\|y_\delta-y_a\|_1\le M\varepsilon$. Considering the proof of Theorem 2 for $h=\delta e^{-a_0t}u(t)$ and y_δ instead of y, we obtain

$$\begin{aligned}M\varepsilon\ge\left\|y_\delta-y_a\right\|_1&=\int_{-\infty}^{+\infty}\left|y_\delta(t)-y_a(t)\right|\mathrm{d}t\\&\ge\int_0^{+\infty}\left|y_\delta(t)-y_a(t)\right|\mathrm{d}t,\\&=\int_0^{+\infty}\left|e^{-a_0t}u(t)*h(t)\right|\mathrm{d}t,\\&=\int_0^{+\infty}\left|\int_{-\infty}^{+\infty}e^{-xa_0}u(x)\delta e^{-(t-x)\mathrm{Re}(a_0)}u(t-x)\mathrm{d}x\right|\mathrm{d}t,\\&=\int_0^{+\infty}\left|\int_0^t\delta e^{-ta_0}\mathrm{d}x\right|\mathrm{d}t,\\&=\int_0^{+\infty}\left|\delta te^{-ta_0}\right|\mathrm{d}t,\\&=\delta\int_0^{+\infty}te^{-t\mathrm{Re}(a_0)}\mathrm{d}t=\frac{\delta}{\mathrm{Re}(a_0)^2}.\end{aligned} \tag{59}$$

Hence, $(\delta/\mathrm{Re}(a_0)^2)\le M\varepsilon$, and considering $\delta=\mathrm{Re}(a_0)\varepsilon$, we obtain $(1/\mathrm{Re}(a_0))\le M$.

The following example shows that the condition $\mathrm{Re}(a_{n-1})\ne 0$ in the abovementioned theorem is necessary and cannot be removed.

Example 1. Every nonzero solution of equation $y\prime(t)-iy(t)=0$ has the form $y=\lambda e^{it}$ which is not in $L^1(\mathbb{R})$. Hence, $y=0$ is the only solution of equation in $L^1(\mathbb{R})$. Now, for $\epsilon>0$, consider the function

$$y_\varepsilon(t)=e^{(i-1)t}u(t)+\frac{\varepsilon}{\sqrt{2}}e^{-t}u(t), \tag{60}$$

which is in $L^1(\mathbb{R})$. To see this, note that

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|y_\varepsilon(t)\right|\mathrm{d}t&=\int_{-\infty}^{+\infty}\left|e^{(i-1)t}u(t)+\frac{\varepsilon}{\sqrt{2}}e^{-t}u(t)\right|\mathrm{d}t\\&\le\int_{-\infty}^{+\infty}\left|e^{(i-1)t}u(t)\right|\mathrm{d}t+\frac{\varepsilon}{\sqrt{2}}\int_{-\infty}^{+\infty}\left|e^{-t}u(t)\right|\mathrm{d}t,\\&=\int_0^{+\infty}\left|e^{it}\right|\left|e^{-t}\right|\mathrm{d}t+\frac{\varepsilon}{\sqrt{2}}\int_0^{+\infty}\left|e^{-t}\right|\mathrm{d}t,\\&=\int_0^{+\infty}e^{-t}\mathrm{d}t+\frac{\varepsilon}{\sqrt{2}}\int_0^{+\infty}e^{-t}\mathrm{d}t,\quad<+\infty.\end{aligned} \tag{61}$$

On the other hand, for $t<0$, $y_\varepsilon'(t)=y_\varepsilon(t)=0$. Thus,

$$\begin{aligned}\left\|y_\varepsilon'-iy_\varepsilon\right\|_1&=\int_{-\infty}^{+\infty}\left|y_\varepsilon'(t)-iy_\varepsilon(t)\right|\mathrm{d}t,\\&=\int_{-\infty}^{+\infty}\left|ie^{it}-\frac{\varepsilon}{\sqrt{2}}e^{-t}u(t)-iMe^{it}-\frac{\varepsilon}{\sqrt{2}}ie^{-t}u(t)\right|,\\&=\frac{\varepsilon}{\sqrt{2}}\int_0^{+\infty}|1+i|e^{-t}\mathrm{d}t=\varepsilon.\end{aligned} \tag{62}$$

If the equation, $y\prime(t)-iy(t)=0$ has Hyers–Ulam stability, and there must be a solution $y_a\in L^1(\mathbb{R})$ of the differential equation such that $\|y_\varepsilon-y_a\|_1<K(\varepsilon)$ and $\lim_{\varepsilon\longrightarrow0}K(\varepsilon)=0$. Since the only solution of equation in $L^1(\mathbb{R})$ is zero function, $y_a=0$, and on the other hand,

$$\begin{aligned}\left\|y_\varepsilon-y_a\right\|_1=\left\|y_\varepsilon\right\|_1&=\int_0^{+\infty}\left|e^{(i-1)t}+\frac{\varepsilon}{\sqrt{2}}e^{-t}\right|\\&\ge\int_0^{+\infty}\left|e^{(i-1)t}\right|dt-\frac{\varepsilon}{\sqrt{2}}\int_0^{+\infty}e^{-t}\mathrm{d}t=1-\frac{\varepsilon}{\sqrt{2}}.\end{aligned} \tag{63}$$

Therefore,

$$1-\frac{\varepsilon}{\sqrt{2}}\le K(\varepsilon), \tag{64}$$

which is a contradiction with the fact that $\lim_{\varepsilon\longrightarrow0}K(\varepsilon)=0$.

4. Conclusions

The main conclusions of this paper are as follows:

(1) Every differential equation $y^{(n)}(t) + \sum_{k=0}^{n-1} a_k y^{(k)}(t) = f(t)$, where $f \in L^1(\mathbb{R})$, $n > 1$, and $\mathrm{Re}(a_{n-1}) \neq 0$ has a n-times differentiable solution in $C^n(\mathbb{R}) \cap L^1(\mathbb{R})$

(2) Equation (37) is stable on $L^1(\mathbb{R})$ in the sense of Hyers–Ulam

Data Availability

No data were used to support this study.

Disclosure

This manuscript has been submitted as a preprint in the following link: http://export.arxiv.org/pdf/2005.03296.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences*, vol. 27, no. 4, pp. 222–224, 1941.

[2] S. M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, New York, NY, USA, 2011.

[3] T. Mynt-U and L. Debenath, *Linear Partial Differential Equations for Scientists and Engineers*, Birkhauser, Boston, MA, USA, fourth edition, 2007.

[4] C. P. Lia and F. R. Zhang, "A survey on the stability of fractional differential equations," *The European Physical Journal Special Topics*, vol. 193, no. 1, pp. 27–47, 2011.

[5] C. Alsina and R. Ger, "On some inequalities and stability results related to the exponential function," *Journal of Inequalities and Applications*, vol. 2, pp. 373–380, 1998.

[6] S.-E. Takahasi, H. Takagi, T. Miura, and S. Miyajima, "The Hyers-Ulam stability constants of first order linear differential operators," *Journal of Mathematical Analysis and Applications*, vol. 296, no. 2, pp. 403–409, 2004.

[7] D. S. Cîmpean and D. Popa, "On the stability of the linear differential equation of higher order with constant coefficients," *Applied Mathematics and Computation*, vol. 217, no. 8, pp. 4141–4146, 2010.

[8] Y. Guo, M. Chen, X.-B. Shu, and F. Xu, "The existence and Hyers-Ulam stability of solution for almost periodical fractional stochastic differential equation with fBm," *Stochastic Analysis and Applications*, vol. 39, no. 4, pp. 643–666, 2020.

[9] Y. Li and Y. Shen, "Hyers-Ulam stability of linear differential equations of second order," *Applied Mathematics Letters*, vol. 23, no. 3, pp. 306–309, 2010.

[10] S. Li, L. Shu, X.-B. Shu, and F. Xu, "Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays," *Stochastics*, vol. 91, no. 6, pp. 857–872, 2018.

[11] S. Muthaiah, D. B. N. G. Thangaraj, D. Baleanu, and N. Gopal Thangaraj, "Existence and Hyers-Ulam type stability results for nonlinear coupled system of Caputo-Hadamard type fractional differential equations," *Aims Mathematics*, vol. 6, no. 1, pp. 168–194, 2021.

[12] T. Miura, S. Miyajima, and S.-E. Takahasi, "A characterization of Hyers-Ulam stability of first order linear differential operators," *Journal of Mathematical Analysis and Applications*, vol. 286, no. 1, pp. 136–146, 2003.

[13] I. A. Rus, "Remarks on Ulam stability of the operatorial equations," *Fixed Point Theory*, vol. 10, pp. 305–320, 2009.

[14] I. A. Rus, "Ulam stability of ordinary differential equations," *Stud. Univ. Babes-Bolyai Math*, vol. 54, pp. 125–134, 2009.

[15] G. Wang, M. Zhou, and L. Sun, "Hyers-Ulam stability of linear differential equations of first order," *Applied Mathematics Letters*, vol. 21, no. 10, pp. 1024–1028, 2008.

[16] H. Rezaei, S.-M. Jung, and T. M. Rassias, "Laplace transform and Hyers-Ulam stability of linear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 403, no. 1, pp. 244–251, 2013.

[17] M. Ramdoss, P. Selvan-Arumugam, and C. Park, "Ulam stability of linear differential equations using Fourier transform," *AIMS Mathematics*, vol. 5, no. 2, pp. 766–780, 2020.

[18] J. L. Schif, *The Laplace Transform, Theory and its Application*, Springer, New York, NY, USA, 1999.

[19] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, NY, USA, 1987.

[20] Y. Katzenelson, *An Introduction to Harmonic Analysis*, Cambrige University Press, Cambridge, UK, 2002.

[21] A. Deitmar, *A First Course in Harmonic Analysis*, Springer, Berlin, Germany, 2005.

[22] J. B. Conway, *Functions of One Complex Variable*, Springer, New York, NY, USA, 1978.