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Research Article

Representation Theory of Groups and *D*-Modules

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In this paper, we study a decomposition D-module structure of the polynomial ring. Then, we illustrate a geometric interpretation of the Specht polynomials. Using Brauer's characterization, we give a partial generalization of the fact that factors of the discriminant of a finite map π: spec$B \longrightarrow$ spec$A$ generate the irreducible factors of the direct image of B under the map π.

1. Introduction

The main purpose of this paper is to generalize results on modules over the Weyl algebra which appeared in [1]. Results in [1] have been obtained in a geometric context. This paper is partially expository in nature. Section 2.2 has been presented at the 9th International Conference on Mathematical Modeling in Physical Sciences to describe the action of the rational quantum Calogero–Moser system on polynomials. For the sake of clarity, we reformulate it here in a more algebraic context.

A prevailing idea in representation theory is that larger structures can be understood by breaking them up into their smallest pieces. Also the natural framework of algebraic geometry is one of the polynomials and, the development of modern algebra has given a particular status to polynomials. In this vein, we study polynomial rings as modules over a ring of invariant differential operators by elaborating its irreducible submodules. We know that the direct image of a simple module under a proper map π is semisimple by the decomposition theorem [2]. The simplest case is when the map π: $X = \mathrm{spec}B \longrightarrow Y = \mathrm{spec}A$ is finite; in such case, it is easy to give an elementary and wholly algebraic proof, using essentially the generic correspondence with the differential Galois group, which equals the ordinary group G. The irreducible submodules of the direct image are in one-to-one correspondence with the irreducible representations of G (see [1]). In the case of the invariants of the symmetric group, $B = \mathbb{C}[x_1, \ldots, x_n]A = \mathbb{C}[x_1, \ldots, x_n]\mathcal{S}^n = \mathbb{C}[y_1, \ldots, y_n]$, an explicit basis of the A-module structure of B is given by $\mathrm{H}_n = \{x^\alpha | \alpha_i \leq n - i,\ 1 \leq i \leq n\}$. In what follows, we endowed B with a differential structure by using directly the action of the Weyl algebra associated to A after a localization. We use the representation theory of symmetric groups to exhibit the generators of its simple components.

The approach in this paper is different from the one in [1].

Secondly, we give a geometric interpretation of the ordinary Specht polynomials which are defined as combinatorial objects [3–5].

Finally, using Brauer's characterization of characters, we give a partial generalization to arbitrary finite maps of the fact that factors of the discriminant of the finite map generate the irreducible factors of the direct image $\pi_+\mathcal{O}_X$.

1.1. Preliminaries: Specht Polynomials and Specht Modules. In this section, we recall some general facts about the actions of symmetric group on polynomial ring. The symmetric group $\mathcal{S}_n$ is the group of permutations of the set of variables $\{x_1, \ldots, x_n\}$. Let $g \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial, and $\sigma \in \mathcal{S}_n$; we define

$$(\sigma g)(x_1, \ldots, x_n) = g(\sigma x_1, \ldots, \sigma x_n). \tag{1}$$

Peel gave the construction of irreducible submodules of $\mathbb{C}[x_1, \ldots, x_n]$ in the following way [4].

By a partition of n, we mean a sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that

$$\lambda_1 \geq \lambda_2 \geq \dots \lambda_r > 0 \text{ and } \lambda_1 + \lambda_2 + \cdots + \lambda_r = n. \tag{2}$$

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition; we arrange the variables $x_1, \dots, x_n$ in an array, with r rows and λ_1 columns, containing a variable in the first λ_i positions of the ith row; each variable occurs exactly once in the array. For example, one such array for the partition $(4, 2, 1)$ of 7 is

$$\begin{array}{|c|c|c|c|} \hline x_1 & x_4 & x_6 & x_7 \\ \hline x_2 & x_5 & \multicolumn{2}{c}{} \\ \cline{1-2} x_3 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \tag{3}$$

and such an array is called a λ-tableau. There are $n!$ λ-tableaux for each partition λ of n. We shall denote such tableaux by t. Suppose that the variables $a_1, \dots, a_l$ occur in jth column of λ-tableau t, with a_i in the ith row. We form the difference product $\Delta(a_1, \dots, a_l) = \prod_{i<k}(a_i - a_k)$, if $l > 1$, and if $l = 1$, $\Delta(a_1) = 1$. Multiplying these difference products for all the columns of t, we obtain a polynomial which we denote by $f(t)$. For $\sigma \in \mathcal{S}_n$, let σt be the tableau obtained from t by replacing x_i in t by σx_i. Then, $\sigma f(t) = f(\sigma t)$. It follows that the set of all linear combinations of the $n!$ polynomials $f(t)$, obtained from the λ-tableaux t, is a cyclic $\mathbb{C}[\mathcal{S}_n]$-module generated by any $f(t)$. We denote this module by S^λ. A λ-tableau is said to be standard if the variables occur in increasing order ($x_i > x_j$ if $i > j$) along each row from left to right and down each column. Peel proved the following in [4].

Theorem 1. *$B^\lambda = \{f(t)$: t is a standard λ-tableau$\}$ is a basis of S^λ.*

We call $f(t)$ the Specht polynomial corresponding to the λ-tableau t, we call S^λ the Specht module corresponding to the partition λ, and $f(t)$ is a standard Specht polynomial if t is a standard tableau.

Theorem 2. *S^λ for $\lambda \vdash n$ forms a complete list of irreducible $\mathcal{S}_n$-module over the complex field.*

2. Geometric Interpretation of the Specht Polynomials

In this section, we establish a decomposition theorem and give a geometric interpretation of the Specht polynomials.

2.1. Notation. Let $D_X = \mathbb{C}\langle x_1, \dots, x_n, \partial/\partial x_1, \dots, \partial/\partial x_n \rangle$ be the ring of differential operators associated to $\mathcal{O}_X = \mathbb{C}[x_1, \dots, x_n]$, and let $\mathcal{O}_Y = \mathbb{C}[x_1, \dots, x_n]^{\mathcal{S}_n} = \mathbb{C}[y_1, \dots, y_n]$ be the ring of invariant under the symmetric group $\mathcal{S}_n$ where

$$y_j = \sum_{i=1}^{n} \frac{x_i^j}{j}, \quad \text{for } j = 1, \dots, n. \tag{4}$$

We denote by $D_Y = \mathbb{C}\langle y_1, \dots, y_n, \partial/\partial y_1, \dots, \partial/\partial y_n \rangle$ the ring of differential operators associated to $\mathcal{O}_Y = \mathbb{C}[y_1, \dots, y_n]$. Since $\mathcal{O}_X$ is a simple D_X-module [6], the direct image $\pi_+(\mathcal{O}_X)$ of $\mathcal{O}_X$ under the map $\pi\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n/\mathcal{S}_n$ is semisimple [2]. We would like to study $\mathcal{O}_X$ as a D_Y-module without the machinery of the direct image structure but by the direct actions of D_Y on $\mathcal{O}_X$. By localization, $\mathcal{O}_X$ can be turned into a D_Y-module, as the following lemma states.

Lemma 1. *Let $A = (x_i^{j-1})_{1 \leq i,j \leq n}$, $\Delta = det(A)$, and $\widetilde{\mathcal{O}_X} = \mathcal{O}_X[\Delta^{-1}]$ and let $\widetilde{D_Y} = D_Y[\Delta^{-2}]$ be the localization of D_Y at Δ^2. Then, $\widetilde{\mathcal{O}_X}$ is a $\widetilde{D_Y}$-module.*

Proof. Let us make clear the actions of $\widetilde{D_Y}$ on $\widetilde{\mathcal{O}_X}$.

We have $y_j = \sum_{i=1}^n x_i^j/j$, $j = 1, \dots, n$, and hence $\partial/\partial x_i = \sum_{j=1}^n x_i^{j-1} \partial/\partial y_j$, $i = 1, \dots, n$. We get the following equation:

$$\begin{pmatrix} \dfrac{\partial}{\partial x_n} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{pmatrix} = A \begin{pmatrix} \dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial y_n} \end{pmatrix}. \tag{5}$$

Since $\Delta \neq 0$, it follows that

$$\begin{pmatrix} \dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial y_n} \end{pmatrix} = A^{-1} \begin{pmatrix} \dfrac{\partial}{\partial x_n} \\ \vdots \\ \dfrac{\partial}{\partial x_n} \end{pmatrix}, \tag{6}$$

and it is clear that $\widetilde{\mathcal{O}_X}$ is a $\widetilde{D_Y}$-module.

What are the simple components $\widetilde{\mathcal{O}_X}$ as $\widetilde{D_Y}$-module and their multiplicities? ☐

Example 1. For $n = 2$, $\widetilde{D_Y} = \mathbb{C}\langle x_1, x_2, \partial/\partial y_1, \partial/\partial y_2, \Delta^{-2} \rangle$ and $\widetilde{\mathcal{O}_X} = \mathbb{C}[x_1, x_2, \Delta^{-1}]$ where $\Delta = x_1 - x_2$ and

$$\begin{cases} \dfrac{\partial}{\partial y_1} & = \dfrac{x_2}{x_2 - x_1} \dfrac{\partial}{\partial x_1} - \dfrac{x_1}{x_2 - x_1} \dfrac{\partial}{\partial x_2}, \\[2ex] \dfrac{\partial}{\partial y_2} & = \dfrac{-1}{x_2 - x_1} \dfrac{\partial}{\partial x_1} + \dfrac{1}{x_2 - x_1} \dfrac{\partial}{\partial x_2}. \end{cases} \tag{7}$$

We have that

$$\widetilde{\mathcal{O}_X} = M_1 \oplus M_2, \tag{8}$$

where $M_1 = \mathcal{O}_Y[\Delta^{-2}]$ and $M_2 = \widetilde{D}_Y(x_1 - x_2)$ are $\widetilde{D}_Y$-simple modules.

2.2. Simple Components and Their Multiplicities. In this section, we state our first main result. We use the representation theory of symmetric groups to yield results on modules over the ring of differential operators. It is well known that

$$\mathcal{O}_X = \mathbb{C}[\mathcal{S}_n] \otimes \mathcal{O}_Y \text{ as } \mathcal{O}_Y - \text{modules}. \tag{9}$$

Let us consider the multiplicative closed set $S = \left\{\Delta^k\right\}_{k\in\mathbb{N}} \subset \mathcal{O}_X$. It follows that

$$S^{-1}\mathcal{O}_X = \mathbb{C}[\mathcal{S}_n] \otimes S^{-1}\mathcal{O}_Y \text{ as } S^{-1}\mathcal{O}_Y - \text{modules}, \tag{10}$$

where $S^{-1}\mathcal{O}_X$ and $S^{-1}\mathcal{O}_Y$ are the localization of $\mathcal{O}_X$ and $\mathcal{O}_Y$ at S, respectively. But $S^{-1}\mathcal{O}_X = \widetilde{\mathcal{O}}_X$ and $S^{-1}\mathcal{O}_Y = \widetilde{\mathcal{O}_Y}$, whereby we get

$$\widetilde{\mathcal{O}}_X = \mathbb{C}[\mathcal{S}_n] \otimes \widetilde{\mathcal{O}_Y} \text{ as } \mathcal{S}_n - \text{modules}. \tag{11}$$

Lemma 2. *There exists an injective map*

$$\mathbb{C}[\mathcal{S}_n] \longrightarrow \mathrm{Hom}_{\mathbb{C}}\left(\widetilde{\mathcal{O}}_X, \widetilde{\mathcal{O}}_X\right). \tag{12}$$

Proof. The $\mathcal{S}_n$-module $\mathbb{C}[\mathcal{S}_n]$ acts on itself by multiplication, and this multiplication yields an injective map $\mathbb{C}[\mathcal{S}_n] \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[\mathcal{S}_n], \mathbb{C}[\mathcal{S}_n])$. Since $\widetilde{\mathcal{O}_Y}$ is invariant under this action of $\mathbb{C}[\mathcal{S}_n]$, we get the expected injective map. ☐

Proposition 1. *There exists an injective map*

$$\mathbb{C}[\mathcal{S}_n] \longrightarrow \mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}\left(\widetilde{\mathcal{O}}_X, \widetilde{\mathcal{O}}_X\right). \tag{13}$$

Proof. Since $\widetilde{\mathrm{D}_Y} = \mathbb{C}\langle y_1, \ldots, y_n, \partial y_1, \ldots, \partial y_n, \Delta^{-2}\rangle$, we only need to show that every element of $\mathbb{C}[\mathcal{S}_n]$ commute with $y_1, \ldots, y_n, \partial y_1, \ldots, \partial y_n$.

(i) It is clear that every element of $\mathbb{C}[\mathcal{S}_n]$ commutes with y_i, $i = 1, \ldots, n$.

(ii) Let us show that every element of $\mathbb{C}[\mathcal{S}_n]$ commutes with ∂y_i, $i = 1, \ldots, n$. Let $\mathbf{D}$ be a derivation on the field $\overline{\mathcal{O}}_Y = \mathbb{C}(y_1, \ldots, y_n)$ (the field of fractions of $\mathcal{O}_Y$); then, $(\overline{\mathcal{O}}_Y, \mathbf{D})$ is a differential field. Since $\overline{\mathcal{O}}_X = \mathbb{C}(x_1, \ldots, x_n)$ (the field of fractions of $\widetilde{\mathcal{O}}_X$) is a Galois extension of $\overline{\mathcal{O}}_Y$, by [7, Theorem 6.2.6] there exists a unique derivation on $\overline{\mathcal{O}}_X$ which extends $\mathbf{D}$; then, $(\overline{\mathcal{O}}_X, \mathbf{D})$ is also a differential ring. In the same way, $\sigma^{-1}\mathbf{D}\sigma = \mathbf{D}$ for every $\sigma \in \mathcal{S}_n$. Therefore, $\sigma\mathbf{D} = \mathbf{D}\sigma$ and σ commute with $\mathbf{D}$. ☐

Corollary 1

$$\mathbb{C}[\mathcal{S}_n] \cong \mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}\left(\widetilde{\mathcal{O}}_X, \widetilde{\mathcal{O}}_X\right). \tag{14}$$

Proof see [[1], Corollary 2.6]. Before we state our first main result, let us recall some facts.

By Maschke's theorem [[9], Chap XVIII], we know that $\mathbb{C}[\mathcal{S}_n]$ is a semisimple ring, and

$$\mathbb{C}[\mathcal{S}_n] = \underset{\lambda\vdash n}{\oplus} R_\lambda, \tag{15}$$

where the sum is taken over all the partitions of n and R_λ are simple rings. We have the following corresponding decomposition of the identity element of $\mathbb{C}[\mathcal{S}_n]$:

$$1 = \sum_{\lambda\vdash n} r_\lambda, \tag{16}$$

where r_λ is the identity element of R_λ, $r_\lambda^2 = 1$ and $r_\lambda r_\mu = 0$ if $\lambda \neq \mu$, and the set $\{r_\lambda\}_{\lambda\vdash n}$ is the set of central idempotents of $\mathbb{C}[\mathcal{S}_n]$.

Let n be a positive integer, λ be a partition of n, $\mathrm{Tab}\,(\lambda)$ be the set of standard tableau of shape λ, and $\mathrm{Tab}\,(n) = \cup_{\lambda\vdash n}\mathrm{Tab}\,(\lambda)$. We have $r_\lambda = \sum_{t\in\mathrm{Tab}\,(\lambda)} e_t$ where e_t is the primitive idempotent associated to the standard tableau $t \in \mathrm{Tab}\,(\lambda)$ (see [10]). ☐

Theorem 3. *For every primitive idempotent $e_i \in \mathbb{C}[\mathcal{S}_n]$,*

(1) $e_i\widetilde{\mathcal{O}}_X$ is a nontrivial $\widetilde{\mathrm{D}}_Y$-submodule of $\widetilde{\mathcal{O}}_X$

(2) The $\widetilde{\mathrm{D}}_Y$-module $e_i\widetilde{\mathcal{O}}_X$ is simple

(3) There exist a partition $\lambda\vdash n$ and a Specht polynomial p_λ associated to a standard tableau of shape λ such that $e_i\widetilde{\mathcal{O}}_X = \widetilde{\mathrm{D}}_Y p_\lambda$

Proof

(1) Let $e_i \in \mathbb{C}[\mathcal{S}_n]$ be a primitive idempotent; by [10, Theorem 4.3], there exists a Specht polynomial p_λ such that $e_i p_\lambda$ is a scalar multiple of p_λ; then, $0 \neq p_\lambda \in e_i\widetilde{\mathcal{O}}_X$ and $e_i\widetilde{\mathcal{O}}_X \neq \{0\}$. Since e_i commutes with every element of $\widetilde{\mathrm{D}}_Y$ and $\widetilde{\mathcal{O}}_X$ is a $\widetilde{\mathrm{D}}_Y$-module, it follows that $e_i\widetilde{\mathcal{O}}_X$ is a $\widetilde{\mathrm{D}}_Y$-module.

(2) Assume that $1 = \sum_{i=1}^s e_i$ where $\{e_i\}_{1\leq i\leq s}$ is the set of primitive idempotents of $\mathbb{C}[\mathcal{S}_n]$; then, $\widetilde{\mathcal{O}}_X = \sum_{i=1}^s e_i\widetilde{\mathcal{O}}_X$. Let $m \in e_i\widetilde{\mathcal{O}}_X \cap e_j\widetilde{\mathcal{O}}_X$ with $i \neq j$; then, $m = e_i m_i$ and $m = e_j m_j$, but $e_i e_j = 0$ then $e_i m = e_i e_j m_j = 0$, and hence $m = 0$. Therefore, $\widetilde{\mathcal{O}}_X = \oplus_{i=1}^s e_i\widetilde{\mathcal{O}}_X$, and we get

$$\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}\left(\widetilde{\mathcal{O}}_X, \widetilde{\mathcal{O}}_X\right) \cong \oplus_{i,j=1}^s \mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}\left(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X\right), \tag{17}$$

and by Corollary 1, we get

$$\mathbb{C}[\mathcal{S}_n] \cong \underset{i,j=1}{\overset{s}{\oplus}} \mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}\left(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X\right). \tag{18}$$

We also have, by [10, Proposition 3.29], that $\mathbb{C}[\mathcal{S}_n] \cong \oplus_{\lambda\vdash n}\mathrm{End}_{\mathbb{C}}\,(S^\lambda)$ where S^λ is the Specht module associated with the partition $\lambda\vdash n$. But

$$\mathbb{C}[\mathcal{S}_n] = \oplus_{\lambda\vdash n} r_\lambda \mathbb{C}[\mathcal{S}_n] \text{ and } r_\lambda \mathbb{C}[\mathcal{S}_n] \cong \mathrm{End}_{\mathbb{C}}\left(\mathbb{C}^{f^\lambda}\right), \tag{19}$$

where $f^\lambda = \dim S^\lambda$. We recall that each standard tableau t_i is associated with an idempotent e_i.

Let us show that $\mathbb{C}[\mathcal{S}_n] \cong \oplus_{\lambda\vdash n} (\oplus_{t_i,t_j\in \mathrm{Tab}(\lambda)} \mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X))$. Let x be an element of $\mathbb{C}[\mathcal{S}_n]$ and r_λ be a central idempotent with $\lambda\vdash n$. Then, x induces an $\widetilde{\mathrm{D}}_Y$-homomorphism $\widetilde{\mathcal{O}}_X \longrightarrow \widetilde{\mathcal{O}}_X, m\mapsto x\cdot m$. Since r_λ is in the center of $\mathbb{C}[\mathcal{S}_n]$, $x\cdot r_\lambda\widetilde{\mathcal{O}}_X = (x\cdot r_\lambda)\widetilde{\mathcal{O}}_X \subset r_\lambda\widetilde{\mathcal{O}}_X$, which means $x\in\oplus_{\lambda\vdash n}\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(r_\lambda\widetilde{\mathcal{O}}_X, r_\lambda\widetilde{\mathcal{O}}_X)$. Then, $\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X) = \{0\}$ if $t_i\in\mathrm{Tab}(\lambda_i)$, $t_j\in\mathrm{Tab}(\lambda_j)$ and $\lambda_i\neq\lambda_j$. We get

$$\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}\left(\widetilde{\mathcal{O}}_X, \widetilde{\mathcal{O}}_X\right) \cong \oplus_{\lambda\vdash n}\left(\oplus_{t_i,t_j\in\mathrm{Tab}(\lambda)}\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}\left(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X\right)\right). \tag{20}$$

The number of direct factors in the sum $\oplus_{t_i,t_j\in\mathrm{Tab}(\lambda)}\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X)$ is $(f^\lambda)^2$.

Let us show that $\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X)\cong\mathbb{C}$ if $t_i, t_j\in\mathrm{Tab}(\lambda)$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[\mathcal{S}_n] \arrow[r, "\phi"'] \arrow[d, "\alpha_\lambda"'] & \mathrm{Hom}_{\tilde{\mathcal{D}}_Y}(\tilde{\mathcal{O}}_X, \tilde{\mathcal{O}}_X) \arrow[d, "\beta_\lambda"] \\
r_\lambda\mathbb{C}[\mathcal{S}_n] \arrow[r, "\psi"'] & \mathrm{Hom}_{\tilde{\mathcal{D}}_Y}(r_\lambda\tilde{\mathcal{O}}_X, r_\lambda\tilde{\mathcal{O}}_X)
\end{tikzcd}$$

where β_λ: $\oplus_{\mu\vdash n}\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(r_\mu\widetilde{\mathcal{O}}_X, r_\mu\widetilde{\mathcal{O}}_X)\longrightarrow\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(r_\lambda\widetilde{\mathcal{O}}_X, r_\lambda\widetilde{\mathcal{O}}_X)$ and α_λ: $\oplus_{\mu\vdash n}r_\mu\mathbb{C}[\mathcal{S}_n]\longrightarrow r_\lambda\mathbb{C}[\mathcal{S}_n]$ are canonical projections and ϕ is the isomorphism of Corollary 1. It follows that ψ is an isomorphism, and hence $r_\lambda\mathbb{C}[\mathcal{S}_n]\cong\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(r_\lambda\widetilde{\mathcal{O}}_X, r_\lambda\widetilde{\mathcal{O}}_X)$. Now we identify $r_\lambda\mathbb{C}[\mathcal{S}_n]$ with the set $\mathrm{Mat}_{f^\lambda}(\mathbb{C})$ of square matrices of order f^λ with coefficients in $\mathbb{C}$ or with $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{f^\lambda})$.

Let E_{ij} be the square matrix of order f^λ with 1 at the position (i,j) and 0 elsewhere and $E_i = E_{i,i}$, and then we identify the idempotent $e_i\in r_\lambda\mathbb{C}[\mathcal{S}_n]$ with E_i in $\mathrm{Mat}_{f^\lambda}(\mathbb{C})$. Let $B=(a_{ij})\in\mathrm{Mat}_{f^\lambda}(\mathbb{C})$; we get $B=\sum_{i,j}a_{i,j}E_{i,j}=\sum_{i,j}E_iBE_j$; in fact $E_i\mathrm{B}E_j$ is the matrix with $a_{i,j}$ in the position (i,j) and 0 elsewhere; if $R=\mathrm{Mat}_{f^\lambda}$, we get that $E_iRE_j\cong\mathbb{C}$. This isomorphism ψ implies that $\oplus_{t_i,t_j\in\mathrm{Tab}(\lambda)}E_iRE_j\cong\oplus_{t_i,t_j\in\mathrm{Tab}(\lambda)}\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X)$; the restriction of ψ to E_iRE_j yields a map $E_iRE_j\longrightarrow\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X)$ and this map is surjective; moreover, we have $E_iRE_j\cong\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X)$. Therefore, $\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_i\widetilde{\mathcal{O}}_X)\cong\mathbb{C}$. Let us assume that $e_i\widetilde{\mathcal{O}}_X$ is not a simple $\widetilde{\mathrm{D}}_Y$-module; then, $e_i\widetilde{\mathcal{O}}_X$ may be written as $e_i\widetilde{\mathcal{O}}_X=\oplus_{j\in J}N_j$ where N_j are simple $\widetilde{\mathrm{D}}_Y$-modules and $|J|>1$. It follows that $\dim_{\mathbb{C}}(\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_i\widetilde{\mathcal{O}}_X))\geq|J|$ but $\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_i\widetilde{\mathcal{O}}_X)\cong\mathbb{C}$, so we obtain that $J=1$, which necessarily implies that $e_i\widetilde{\mathcal{O}}_X$ is a simple $\widetilde{\mathrm{D}}_Y$-module.

(3) By proof (i), there exists a Specht polynomial $p_\lambda\in e_i\widetilde{\mathcal{O}}_X, \lambda\vdash n$ such that $e_i\widetilde{\mathcal{O}}_X=\widetilde{\mathrm{D}}_Y p_\lambda$. ☐

Corollary 2. *With the above notations, $e_i\widetilde{\mathcal{O}}_X\cong_{\widetilde{\mathrm{D}}_Y}e_j\widetilde{\mathcal{O}}_X$ if t_i and t_j have the same size (if there is a partition $\lambda\vdash n$ such that $t_i, t_j\in Tab(\lambda)$).*

Proof. The $\widetilde{\mathrm{D}}_Y$-modules $e_i\widetilde{\mathcal{O}}_X$ are simple and $\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X)\cong\mathbb{C}$ whenever there exists a partition $\lambda\vdash n$ such that $t_i, t_j\in\mathrm{Tab}(\lambda)$. Since $\mathrm{Hom}_{\widetilde{\mathrm{D}}_Y}(e_i\widetilde{\mathcal{O}}_X, e_j\widetilde{\mathcal{O}}_X)\neq\{0\}$, we conclude by using the Schur lemma. ☐

Proposition 2. *For every young tableau t, let $f(t)$ be the associated Specht polynomial; then, we have*

(1) $\widetilde{\mathcal{O}}_X=\oplus_{t\in Tab(n)}\widetilde{\mathrm{D}}_Y f(t)$.

(2) $\widetilde{\mathcal{O}}_X\cong_{\widetilde{\mathrm{D}}_Y}\oplus_{\lambda\vdash n}f^\lambda\widetilde{\mathrm{D}}_Y f(t_\lambda)$, where $t_\lambda\in Tab(\lambda)$.

Proof. We have by the proof of Theorem 3 that

$$\widetilde{\mathcal{O}}_X=\oplus e_i\widetilde{\mathcal{O}}_X, \tag{21}$$

and $e_i\widetilde{\mathcal{O}}_X$ are simple $\widetilde{\mathrm{D}}_Y$-modules. Since to each e_i corresponds a partition $\lambda\vdash n$ and a λ-tableau t_i such that $e_i\widetilde{\mathcal{O}}_X=\widetilde{\mathrm{D}}_Y f(t)$, then $\widetilde{\mathcal{O}}_X=\oplus_{t\in\mathrm{Tab}(n)}\widetilde{\mathrm{D}}_Y f(t)$. If t and t' are two λ-tableaux, by Corollary 2, $\widetilde{\mathrm{D}}_Y f(t)\cong_{\widetilde{\mathrm{D}}_Y}\widetilde{\mathrm{D}}_Y f(t')$. Therefore, $\widetilde{\mathcal{O}}_X\cong\oplus_{\lambda\vdash n}f^\lambda\widetilde{\mathrm{D}}_Y f(t_\lambda)$ with $t_\lambda\in\mathrm{Tab}(\lambda)$. ☐

2.3. Example.

2.3. Example. We consider the case $n=3,4$

For $n=3$, the Specht polynomials corresponding to standard tableaux are

$$\begin{aligned} &f(t_1)=1, f(t_2)=(x_1-x_2), f(t_3)=(x_1-x_3),\\ &f(t_4)=(x_1-x_2)(x_1-x_3)(x_2-x_3). \end{aligned} \tag{22}$$

Correspondingly, we have that

$$\begin{aligned} \widetilde{\mathcal{O}}_X &= \widetilde{\mathcal{O}}_Y\oplus\widetilde{\mathrm{D}}_Y(x_1-x_2)\oplus\widetilde{\mathrm{D}}_Y(x_1-x_3)\oplus\widetilde{\mathrm{D}}_Y\\ &\quad\cdot(x_1-x_2)(x_1-x_3)(x_2-x_3). \end{aligned} \tag{23}$$

For $n=4$, the Specht polynomials corresponding to standard tableaux are

$$\begin{aligned} &f(t_1)=1, f(t_2)=(x_1-x_2), f(t_3)=(x_1-x_3),\\ &f(t_4)=(x_1-x_4),\\ &f(t_5)=(x_1-x_2)(x_3-x_4), f(t_6)=(x_1-x_3)(x_2-x_4),\\ &f(t_7)=(x_1-x_2)(x_1-x_3)(x_2-x_3),\\ &f(t_8)=(x_1-x_2)(x_1-x_4)(x_2-x_4),\\ &f(t_9)=(x_1-x_3)(x_3-x_4)(x_3-x_4),\\ &f(t_{10})=(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4). \end{aligned} \tag{24}$$

Correspondingly, we have that

$$\begin{aligned}\widetilde{\mathcal{O}}_X = {} & \widetilde{\mathcal{O}}_Y \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_2) \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_3) \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_4) \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_2)(x_3 - x_4) \\ & \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_3)(x_2 - x_4) \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \\ & \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_2)(x_1 - x_4)(x_2 - x_4) \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_3)(x_3 - x_4)(x_3 - x_4) \\ & \oplus \widetilde{\mathrm{D}}_Y (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4).\end{aligned} \tag{25}$$

2.4. Geometric Interpretation of Specht Polynomials. Specht polynomials were introduced as combinatoric objects [3, 4]. In fact, the Specht polynomials were first used by Wilhem Specht to generate rational representations of the symmetric group $\mathcal{S}_n$ [5].

We give a geometric interpretation of those polynomials as follows.

Let $\mathcal{O}_X = \mathbb{Z}[x_1, \ldots, x_n]$ and $\mathcal{O}_Y = \mathcal{O}_X^{\mathcal{S}_n}$. Let $I \subset \{1, \ldots, n\}$, and we define

$$\Delta(I) = \prod_{i,j \in I, i<j} (x_i - x_j). \tag{26}$$

Let $J = P_1 \cup \ldots \cup P_k$ be a partition of $\{1, 2, \ldots, n\}$ as a set. To such partition, we associated the subgroup $\mathcal{S}_J = \mathcal{S}_{P_1} \times \cdots \times \mathcal{S}_{P_k}$ of $\mathcal{S}_n$ and the ring $\mathcal{O}_{Y_J} = \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_J}$. Let $Y_J = \mathrm{spec}\mathcal{O}_{Y_J}$, $X = \mathrm{spec}\mathcal{O}_X$ and $Y = \mathrm{spec}\mathcal{O}_Y$. The maps $\pi' \colon X \longrightarrow Y_J$ and $\pi'' \colon Y_J \longrightarrow Y$ are the obvious ones, and we get the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "\pi"] \arrow[d, "\pi'"'] & Y \\
Y_J \arrow[ur, "\pi''"'] &
\end{tikzcd}$$

We clearly have the map $\pi_{IJ} \colon Y_I \longrightarrow Y_J$, whenever I is a refinement of J. The Jacobian of π' is

$$p_J = \prod_{i=1}^{k} \Delta(P_i), \tag{27}$$

a product of Vandermonde determinants on each of the set of variables with subscript in P_i. This is a Specht polynomial. Now J defines a numerical partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ where each $\lambda_i = |P_{i_s}|$ for some i_s. This partition induces a Ferrers diagram where the first column has λ_1 boxes, the second columns has λ_2 boxes, and so on. Moreover, one gets an induced tableau by filling (in increasing order) the numbers in P_1 in the first column, the numbers in P_2 in the second column, and so on, where P_1 is the subset of integers in the first columns, P_2 is the subset of integers in the second, and so on. Conversely to every tableau corresponds a partition of $\{1, \cdots, n\}$ given by letting the integers in the different columns form the partition Hence, to every tableau T, we can associate a Specht polynomial p_T. These are geometrical objects, since they are Jacobians of certain polynomials maps.

Proposition 3. *From the previous section, we get the following fact:*

(a) $p_J^2 \in \mathcal{O}_{Y_J}$.

(b) The submodules $(\mathcal{O}_{Y_J})_{p_J^2}$ *and* $(\mathcal{O}_{Y_J})_{p_J^2} \cdot p_J$ *are factors of rank one in the decomposition of* $(\mathcal{O}_X)_{p_J}$ *as* $(\mathcal{O}_{Y_J})_{p_J^2}$*-module, i.e.,*

$$(\mathcal{O}_X)_{p_J} = \left(\mathcal{O}_{Y_J}\right)_{p_J^2} \oplus \left(\mathcal{O}_{Y_J}\right)_{p_J^2} \cdot p_J \oplus \cdots. \tag{28}$$

3. A Generalization

Consider the map $\pi \colon X = \mathrm{spec}\mathcal{O}_X \longrightarrow Y = \mathrm{spec}\mathcal{O}_Y$. We proved in [1, Theorem 2.10] that the irreducible D-module factors of the direct image $\pi_+(\mathcal{O}_X)$ are generated by the Specht polynomials which are divisors of the Jacobian of π.

We will now consider a general finite map $\pi \colon X \longrightarrow Y$. A consequence of that situation is that the simple submodules of $\pi_+\mathcal{O}_X$ are generated by divisors of the Jacobian of π. A natural question is in what generality this is true. We will prove a similar though weaker result in general. To describe this generalization, let us recall some facts established in [1].

Let k be an algebraically closed field of characteristic 0. Denote for a k-algebra B, by $T_{B/K}$ the k-linear derivations of B.

There is a general correspondence between representations of the differential Galois group of a D_X-module M, defined using a Picard–Vessiot extension and the category of modules generated by M (for the case of one variable see [8]).

Let L and K be two fields, say that a $T_{K/k}$-module M is *L-trivial* if $L \otimes_K M \cong L^n$ as $T_{L/k}$-modules. Denote by $\mathrm{Mod}^L(T_{K/k})$ the full subcategory of finitely generated $T_{K/k}$-modules that are L-trivial.

If G is a finite group, let $\mathrm{Mod}(k[G])$ be the category of finite-dimensional representations of $k[G]$. Let now $k \longrightarrow K \longrightarrow L$ be a tower of fields such that $K = L^G$.

Proposition 4 (see [1]). *The functor*

$$\nabla \colon \mathrm{Mod}(k[G]) \longrightarrow \mathrm{Mod}(T_{K/k}), \quad V \mapsto (L \otimes_k V)^G \tag{29}$$

is fully faithful and defines an equivalence of categories

$$\mathrm{Mod}(k[G]) \longrightarrow \mathrm{Mod}^L(T_{K/k}). \tag{30}$$

The quasi-inverse of ∇ is the functor

$$\mathrm{Loc} \colon \mathrm{Mod}^L(T_{K/k}) \longrightarrow \mathrm{Mod}(k[G]), \mathrm{Loc}(M) = (L \otimes_K M)^{\phi(T_{K/k})}. \tag{31}$$

The above equivalence

$$\mathrm{Mod}(T_K) \longrightarrow {}^{\mathrm{Loc}}\mathrm{Mod}(G) \xrightarrow{\nabla} \mathrm{Mod}(T_K), \tag{32}$$

can be extended to a Galois correspondence. Fix K and L and consider intermediate fields L^{E^K}. Given two such fields ${E_1}^{E_2}$, we have the categories $\mathrm{Mod}^L(T_{E_1/k})$ and $\mathrm{Mod}^L(T_{E_2/k})$. The map π: $\mathrm{spec}E_1 \longrightarrow \mathrm{spec}E_2$ induces an isomorphism $E_2 \otimes_{E_1} T_{E_2/k} \cong T_{E_1/k}$, in particular a canonical lifting $\mathrm{D}_{E_2} \longrightarrow \mathrm{D}_{E_1}$. Corresponding to this ring homomorphism, we have the usual pair of adjoint functors. First the inverse image:

$$\pi^+\colon \mathrm{Mod}^L\left(T_{E_2/k}\right) \longrightarrow \mathrm{Mod}^L\left(T_{E_1/k}\right), \tag{33}$$

is given by

$$M \longrightarrow E_1 \otimes_{E_2} M. \tag{34}$$

It is immediate by $L \otimes_{E_1} (E_1 \otimes_{E_2} M) \cong L \otimes_{E_2} M$ that the image of the inverse image lies in $\mathrm{Mod}^L(T_{E_1/k})$. Secondly, we have the direct image functor π_+, between the same categories, given by restricting the action on M to $T_{E_2/k}$ using the canonical lifting $T_{E_2/k} \longrightarrow T_{E_1/k}$. The direct image is right adjoint to the inverse image.

Thus, the direct image landing in $\mathrm{Mod}^L(T_{E_2/k})$ is clear, e.g., in the following way. By the proposition in the preceding section, it suffices to prove this for E_1, since it then follows for any direct factor. Now the category $\mathrm{Mod}^L(T_{L/k})$ is closed under submodules and quotients, and $L \otimes_{E_2} E_1$ is a submodule of $L \otimes_{E_2} L$, which is a quotient of $L \otimes_K L \in \mathrm{Mod}^L(T_{L/k})$. So $L \otimes_{E_2} E_1 \in \mathrm{Mod}^L(T_{E_2/k})$. By the proposition, there are equivalences of categories $\mathrm{Mod}(k[H_i]) \longrightarrow \mathrm{Mod}(T_{E_i/k})$, where $E_i = L^{H_i}$, and we now want to express the direct and inverse images of D-modules in terms of the corresponding group representation categories and functors.

Proposition 5 (see [1])

(i) *Define* π^+: *Mod*$(k[H_2]) \longrightarrow$ *Mod*$(k[H_1])$ *as the restriction associated to the injection* $k[H_1] \subset k[H_2]$. *Then,*

$$\pi^+(\nabla(V)) = \nabla\left(\pi^+(V)\right). \tag{35}$$

(ii) *Define* π_+: *Mod*$(k[H_1]) \longrightarrow$ *Mod*$(k[H_2])$ *as the coinduction associated to the injection* $k[H_1] \subset k[H_2]$, *defined in the following way:*

$$\pi_+(V) = \mathrm{Hom}_{k[H_1]}\left(k[H_2], V\right). \tag{36}$$

Then,

$$\pi_+(\nabla(V)) = \nabla\left(\pi_+(V)\right). \tag{37}$$

The study of the decomposition factors of $\pi_+\mathcal{O}_X$ can be reduced to the behavior of the direct image over the complement to the branch locus or even over the generic point. Let j: $U \subset X$ and i: $V \subset Y$ be the inclusions.

Proposition 6 (see [1]). *Let* π: $X \longrightarrow Y$ *be a finite map. Then,*

(i) $\pi_+\mathcal{O}_X$ *is semisimple as a* D_Y*-module.*

(ii) *If* $\pi_+\mathcal{O}_X = \oplus M_k$, $k \in I$ *is a decomposition into simple (non-zero)* D_Y*-modules, then* $\pi_+\mathcal{O}_U = \oplus i^* M_k$, $k \in I$, *is a decomposition of* $\pi_+\mathcal{O}_U$ *into simple (nonzero)* D_V*-modules.*

For simplicity, we assume that we are in a generic situation, working with fields. Consider a factorization

$$\pi = \pi_2 \circ \pi_1\colon \mathrm{X} \longrightarrow \mathrm{Z} \longrightarrow \mathrm{Y}. \tag{38}$$

Generically, this corresponds to extensions of fields $K \subset E \subset L$, and we have isomorphisms $T_{L/k} \cong L \otimes_K T_{K/k}$. Assume that $K \subset L$ is Galois with group G. The field E corresponds to a subgroup H, and $(\pi_1)_+\mathcal{O}_X$ to $1\uparrow_1^H \cong k[H]$. Maps λ: $H \longrightarrow k^*$ may be factored through the quotient H/H' with the commutator subgroup $H' = [H, H]$ and give rise to one-dimensional representations. Corresponding functions $x_\lambda \in L$ exist using the functor ∇, characterized by the property that $hx_\lambda = \lambda(h)x_\lambda$. Hence, $(\pi_1)_+\mathcal{O}_X$ contains as direct factor the D_E-submodule $\sum_\lambda E x_\lambda$, where the sum runs over all homomorphisms λ: $H \longrightarrow k^*$. This is actually the sum of all one-dimensional D_E-submodules of $\pi_1\mathcal{O}_X$. Each gives rise to submodules $(\pi_2)_+ E x_\lambda$. As is seen in the example below, these functions may sometimes be thought of as powers of discriminants (corresponding to the extension $E \subset L^{H'}$). In the case of the symmetric group, the Specht polynomials were such functions for the sign representation $S_\lambda \longrightarrow k^*$.

An immediate consequence of Brauer's characterization of characters [11, Theorem 20], saying that any character of a finite group is a linear sum of monomial characters with integer coefficients, is then the generic case of the following theorem.

Theorem 4. *Let* π: $X \longrightarrow Y$ *be an affine finite map. The Grothendieck group generated by the* D_Y *submodules of* $\pi_+\mathcal{O}_X$, *is also generated by* $(\pi_2)_+R$, *where* R *is a rank 1* D_Z *submodule of* $(\pi_1)_+\mathcal{O}_X$ *and* Z *is an intermediate factorization (38).*

Proof. The corresponding generic result follows from Brauer's theorem and the correspondence of irreducible modules with group representations. Then, the result follows by Proposition 6. ☐

3.1. Example: Cyclic Affine Covering. Consider the affine finite map f: $\mathbf{A}^1 \longrightarrow \mathbf{A}^1$ given by the map on rings $\mathbb{C}[y] \longrightarrow \mathbb{C}[x]$ that identifies $y = x^n$. Then, we claim that, as (left) D_Y-modules,

$$f_*\mathbb{C}[x] \cong \mathbb{C}[y] \oplus \mathbb{C}\left[y, y^{-1}\right]x \oplus \ldots \oplus \mathbb{C}\left[y, y^{-1}\right]x^{n-1} \subset \mathbb{C}\left[x, y^{-1}\right]. \tag{39}$$

We use the description of the direct image in [1, Lemma 2.3]; it is thus a question of finding the D_Y-module generated by $\mathbb{C}[x] \subset \mathbb{C}[x, y^{-1}]$.

The extension of function fields $K = \mathbb{C}(y) \subset \mathbb{C}(x)$ is a Galois extension with Galois group, the cyclic group of order n. Abelian groups only have one-dimensional irreducible representations. Hence, by the theory in [1], we know that $f_*\mathbb{C}[x]$ splits as a sum of rank 1 simple D_Y-modules.

But we may of course also easily see this using the action of D_Y. The relation between D_X and D_Y is described by noting that

$$\partial_x = nx^{n-1}\partial_y. \tag{40}$$

This implies that all the factors in (39) above are D_Y-modules:

$$\partial_y\left(y^{-k}x\right) = \left(-k + \frac{1}{n}\right)y^{-k-1}x. \tag{41}$$

Hence, $\mathrm{D}_Y x = \mathbb{C}[y, y^{-1}]x$, and this module, as rank 1 and torsionfree $\mathbb{C}[y]$-module, is irreducible.

Data Availability

No data were used to support this study.

Disclosure

This research is part of the author's presentation at Stockholm University Licentiate seminar: http://su.diva-portal.org/smash/record.jsf?pid=diva2%3A611460&dswid=−9965.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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