Research Article

New Properties of Dual Continuous K-g-Frames in Hilbert Spaces

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Received 30 December 2020; Revised 10 March 2021; Accepted 28 March 2021; Published 12 April 2021

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces was introduced by Duffin and Schaeffer in 1952 [1] to study some deep problems in nonharmonic Fourier series; after the fundamental paper [2] by Daubechies, Grossman, and Meyer, frames’ theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [3].

The concept of frames in Hilbert spaces continues to play a very interesting role in many kinds of applications. In this paper, we study the notion of dual continuous K-g-frames in Hilbert spaces. Also, we establish some new properties.

Definition 1 (see [4]). A sequence \( \{ \Lambda_w \} \subset B(U, V_w) \) is called a continuous g-frame for \( U \) with respect to \( \{ V_w \} \) if there exists \( 0 < A < B < \infty \) such that

\[
A\|x\|^2 \leq \int_{\Omega} \|A_w x\|^2 d\mu(w) \leq B\|x\|^2, \quad \forall x \in U.
\]  

The numbers \( A \) and \( B \) are called lower and upper bounds of the continuous g-frame, respectively.

If \( A = B = \lambda \), the frame is \( \lambda \)-tight.

If \( A = B = 1 \), it is called a normalized tight continuous g-frame or a Parseval continuous g-frame.

We call \( \{ \Lambda_w \} \subset B(U, V_w) \) a continuous g-Bessel sequence if the right-hand inequality of (1) holds.

Let \( \Lambda = \{ \Lambda_w \} \subset \oplus_{w \in \Omega} V_w \) be a continuous g-frame for \( U \) with respect to \( \{ V_w \} \).

The synthesis operator of \( \Lambda \) is defined by

\[
T_\Lambda : \oplus_{w \in \Omega} V_w \longrightarrow \mathcal{B}(U, V) = B(U, V) = B(U, \oplus_{w \in \Omega} V_w),
\]

where

\[
\mathcal{B}(U, V) = \{ (y_w)_{w \in \Omega} \in \oplus_{w \in \Omega} V_w : \|y_w\|_{V_w} < \infty \}.
\]

The existence of \( \int_{\Omega} \Lambda_w^* y_w d\mu(w) \) implies that \( T_\Lambda \) is well defined and bounded.

The adjoint operator of \( T_\Lambda \) which is given by

\[
T_\Lambda^* : U \longrightarrow \oplus_{w \in \Omega} V_w : T_\Lambda^*(x) = \{ \Lambda_w x \}_{w \in \Omega}, \quad \forall x \in U,
\]

is said to be the analysis operator of \( \{ \Lambda_w \} \).

The concept of frames in Hilbert spaces continues to play a very interesting role in many kinds of applications. In this paper, we study the notion of dual continuous K-g-frames in Hilbert spaces. Also, we establish some new properties.
Definition 2 (see [5]). Let \( K \in \mathcal{B}(U) \). A sequence \( \{A_w \in \mathcal{B}(U, V_w)\}_{w \in \Omega} \) is called a continuous \( K \)-g-frame for \( U \) with respect to \( \{V_w\}_{w \in \Omega} \) if there exists \( 0 < A < B < \infty \) such that

\[
A \|K^*x\|^2 \leq \int_\Omega \|A_w x\|^2 d\mu(\omega) \leq B\|x\|^2, \quad \forall x \in U.
\]

The numbers \( A \) and \( B \) are called lower and upper bounds of the continuous \( K \)-g-frame, respectively.

If \( A\|K^*x\|^2 = \int_\Omega \|A_w x\|^2 d\mu(\omega) \), the sequence \( \{A_w x\}_{w \in \Omega} \) is called a continuous \( K \)-g-frame for \( U \) with respect to \( \{V_w\}_{w \in \Omega} \).

The \( A \)-tight continuous \( K \)-g-frame for \( U \) is said to be a Parseval continuous \( K \)-g-frame if \( A = 1 \).

Suppose that \( K \in \mathcal{B}(U) \) and \( \Lambda = \{A_w \in \mathcal{B}(U, V_w)\}_{w \in \Omega} \) is a Parseval continuous \( K \)-g-frame for \( U \) with respect to \( \{V_w\}_{w \in \Omega} \) with synthesis operator \( T_\Lambda \). Then, it is easy to check

\[
KK^* x = T_\Lambda^* T_\Lambda x = \int_\Omega A_w^* A_w x d\mu(\omega), \quad \forall x \in U.
\]

Lemma 1 (see [6]). Suppose that \( T \in \mathcal{B}(U, V) \) has a closed range; then, there exists a unique operator \( T^* \in \mathcal{B}(U, V) \), called the pseudo-inverse of \( T \), satisfying

\[
(TT^*)^* = TT^*,
\]

\[
(TT^*)^* = TT^* = (TT^*)^* = T^* T = T^* T^* = T^* T^*.
\]

Lemma 2 (see [7]). Let \( U, U_1 \), and \( U_2 \) be three Hilbert spaces; also, let \( T \in \mathcal{B}(U, U) \) and \( K \in \mathcal{B}(U_1, U) \). The following statements are equivalent:

1. \( \mathcal{R}(T) \subset \mathcal{R}(K) \)
2. There exists \( \lambda > 0 \) such that \( TT^* \leq \lambda KK^* \)
3. There exists \( \theta \in \mathcal{B}(U_1, U_2) \) such that \( T = K\theta \)

Moreover, if (1)–(3) are valid, then there exists a unique operator \( \theta \) such that

\[
(\theta \theta^*) = \mathcal{R}(\theta) \subset \mathcal{R}(K).
\]

In the following section, we set out to generalize some results of Xiang [8]; in other words, we characterize the concept of dual continuous \( K \)-frames in Hilbert spaces, and we give some properties. Our results extend, generalize, and improve many existing results.

2. Main Result

Proposition 1. Suppose that \( K \in \mathcal{B}(U) \) has a closed range, and let \( \Lambda = \{A_w \in \mathcal{B}(U, V_w)\}_{w \in \Omega} \) be a continuous \( K \)-g-frame for \( U \) with respect to \( \{V_w\}_{w \in \Omega} \) and \( \Gamma = \{\Gamma_w\}_{w \in \Omega} \) be a dual continuous \( K \)-Bessel sequence of \( \{A_w\}_{w \in \Omega} \); then, \( \{\Gamma_w^*\}_{w \in \Omega} \) is a continuous \( K \)-g-frame for \( U \) with respect to \( \{V_w\}_{w \in \Omega} \).

Proof. For each \( x \in U \), we have \( Kx = \int_\Omega A_w^* \Gamma_w x d\mu(\omega) \), and then

\[
\|Kx\|^2 = \sup_{\|y\|=1} \|\langle Kx, y \rangle\|^2 = \sup_{\|y\|=1} \int_\Omega \|\Gamma_w x, A_w y\|^2 d\mu(\omega)
\]

\[
\leq \sup_{\|y\|=1} \int_\Omega \|\Gamma_w x\|^2 d\mu(\omega) \int_\Omega \|A_w y\|^2 d\mu(\omega)
\]

\[
\leq B_\Lambda \int_\Omega \|\Gamma_w x\|^2 d\mu(\omega),
\]

which give

\[
B_\Lambda^{-1} \|Kx\|^2 \leq \int_\Omega \|\Gamma_w x\|^2 d\mu(\omega) \leq B_I \|x\|^2,
\]

where \( B_\Lambda \) and \( B_I \) are the upper bounds of \( \Lambda \) and \( \Gamma \), respectively.

Consequently,

\[
B_\Lambda^{-1} \|K^*x\|^2 \leq \int_\Omega \|\Gamma_w K^* x\|^2 d\mu(\omega) \leq B_I \|K^* x\|^2 \leq B_I \|K\|^2 \|x\|^2.
\]

Assume that \( K^* \) has a closed range; hence, for each \( y \in \mathcal{R}(K^*) \), we get \( y = K^* (K^*)^* y = K^* Ky \), by Lemma 1, and thus,

\[
\|y\|^2 = \|K^* Ky\|^2 \leq \|K^*\|^2 \|Ky\|^2.
\]

Now,
\[ B_{r_{1}} \|K\|^2 \|K^* x\|^2 \leq B_{r_{2}} \|K K^* x\|^2 \leq \int_{\Omega} \|\Gamma_{\omega} K^* x\|^2 \, d\mu(\omega) \leq B_{r_{1}} \|K\|^2 \|x\|^2, \]

which gives \([\Gamma_{\omega} K^*]_{\omega \in \Omega}\) is a continuous K-g-frame for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\) with bounds \(B_{r_{1}}\) and \(B_{r_{2}}\).

**Proposition 2.** Let \(K \in \mathcal{B}(U)\); then, every continuous K-frame admits a dual continuous K-g-Bessel sequence.

\[
C_{\lambda} \langle K^* x, K^* x \rangle = C_{\lambda} \|K^* x\|^2 \leq \int_{\Omega} \|\Lambda_{\omega} x\|^2 \, d\mu(\omega) = \|T_{\omega} x\|^2 = \langle T_{\omega}^* x, T_{\omega} x \rangle,
\]

which is equivalent to \(C_{\lambda} K^* K \leq T_{\omega} T_{\omega}^*\) by Lemma 2, there exists \(\theta \in \mathcal{B}(U, \Theta \setminus V_{\omega})\) such that \(K = T_{\omega} \theta\).

Let \(P_{\omega}\) be the projection on \(\Theta \setminus V_{\omega}\) that maps each element to its \(\omega\)-th component, i.e., \(P_{\omega} \{y_{z}\}_{z \in \Omega} = \{u_{z}\}_{z \in \Omega}\), where \(u_{z} = y_{\omega}\) if \(z = \omega\), and \(u_{z} = 0\) if not.

If \(\lim_{\omega} \Gamma_{\omega} = P_{\omega} \theta\), then

\[
\int_{\Omega} \|\Gamma_{\omega} x\|^2 \, d\mu(\omega) = \int_{\Omega} \|P_{\omega} \theta x\|^2 \, d\mu(\omega)
\]

\[
= \int_{\Omega} \|\theta (x)_{\omega}\|^2 \, d\mu(\omega) = \|\theta x\|^2
\]

\[
\leq \|\theta\|^2 \|x\|^2, \quad \forall x \in U.
\]

Hence, \([\Gamma_{\omega}]_{\omega \in \Omega}\) is a continuous g-Bessel sequence for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\). Now,

\[
K x = T_{\omega} \theta x = \int_{\Omega} \Lambda_{\omega} (\theta x)_{\omega} \, d\mu(\omega) = \int_{\Omega} \Lambda_{\omega} (P_{\omega} \theta x) \, d\mu(\omega), \quad \forall x \in U,
\]

showing that \([\Gamma_{\omega}]_{\omega \in \Omega}\) is a continuous g-Bessel sequence.

**Proposition 3.** Suppose that \(K \in \mathcal{B}(U)\) and \(\Lambda = \{\Lambda_{\omega} \subset \mathcal{B}(U, V_{\omega})\}_{\omega \in \Omega}\) is a continuous K-g-frame for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\), then, there exists a unique dual continuous K-g-Bessel sequence \(\Gamma = \{\Gamma_{\omega}\}_{\omega \in \Omega}\) of \(\Lambda\) such that \(\|T_{\omega}\| \leq \|T_{\omega}\|\) for any dual continuous K-g-Bessel sequence \(\Theta = \{\theta_{\omega}\}_{\omega \in \Omega}\) of \(\Lambda\).

**Proof.** Since \(KK^* \leq (1/C_{\lambda}) T_{\omega} T_{\omega}^*\), by Lemma 2, it follows that there is a unique operator \(R \in \mathcal{B}(U, \Theta \setminus V_{\omega})\) such that \(K = T_{\omega} R\) and

\[
\|R\|^2 = \inf \left\{ \lambda : \|K^* x\|^2 \leq \lambda \|T_{\omega} x\|^2 \right\},
\]

taking \(\Gamma_{\omega} = P_{\omega} R\) for each \(\omega \in \Omega\); then, as shown in Proposition 2, \(\Gamma_{\omega}\) is a continuous g-Bessel sequence for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\), and furthermore, it is a dual continuous K-g-Bessel sequence of \(\Lambda\).

Since \(P_{\omega} T_{\omega} x = \Gamma_{\omega} x = P_{\omega} R x\) for any \(\omega \in \Omega\) and \(x \in U\), \(T_{\omega}^* = R\).

**Proposition 4.** Suppose that \(K \in \mathcal{B}(U)\) and \(\Lambda = \{\Lambda_{\omega} \subset \mathcal{B}(U, V_{\omega})\}_{\omega \in \Omega}\) is a continuous K-g-frame for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\), then, the dual continuous K-g-Bessel sequence of \(\Lambda\) is precisely the family \(\{T_{\omega}^* x\}_{\omega \in \Omega} = \{P_{\omega} B^*\}_{\omega \in \Omega}\) where \(\theta \in \mathcal{B}(U, \Theta \setminus V_{\omega}, U)\) satisfies the condition \(K = T_{\omega}^*\).

**Theorem 1.** Let \(K \in \mathcal{B}(U)\) and \(\Lambda = \{\Lambda_{\omega} \subset \mathcal{B}(U, V_{\omega})\}_{\omega \in \Omega}\) be a Parseval continuous K-g-frame for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\), where \(K\) has a closed range; then, \(\Lambda_{\omega}(K^*)^\dagger \subset \mathcal{B}(U, \Theta \setminus V_{\omega}, U)\) is a dual continuous K-g-Bessel sequence of \(\Lambda\) for every \(\omega \in \Omega\).

**Proof.** It is easy to see that \(\Lambda_{\omega}(K^*)^\dagger \subset \mathcal{B}(U, V_{\omega})\) is a continuous K-frame for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\).

By Lemma 1, \(y = K^*(K^*)^\dagger y = K^* (K^*)^\dagger y\) for every \(y \in \mathcal{R}(K)\); thus, by (S),

\[
K y = K K^* (K^*)^\dagger y = \int_{\Omega} \Lambda_{\omega} \Lambda_{\omega}(K^*)^\dagger y \, d\mu(\omega).
\]

If \(h \in \mathcal{R}(K^*)^\dagger = \mathcal{K}(K)\), then by Lemma 1 again, we obtain \(h \in \mathcal{N}(K^*)^\dagger\), implying that \(\int_{\Omega} \Lambda_{\omega} \Lambda_{\omega}(K^*)^\dagger h \, d\mu(\omega) = 0 = Kh\). Altogether, we have

\[
K x = \int_{\Omega} \Lambda_{\omega} \Lambda_{\omega}(K^*)^\dagger x \, d\mu(\omega), \quad \forall x \in U,
\]

which ends the proof.

**Theorem 2.** Let \(K \in \mathcal{B}(U)\) have a closed range and \(\Lambda = \{\Lambda_{\omega} \subset \mathcal{B}(U, V_{\omega})\}_{\omega \in \Omega}\) be a Parseval continuous K-g-frame for \(U\) with respect to \([V_{\omega}]_{\omega \in \Omega}\). For each \(\omega \in \Omega\), let \(\Gamma_{\omega} \in \mathcal{B}(U, V_{\omega})\) then, \(\Gamma = \{\Gamma_{\omega}\}_{\omega \in \Omega}\) is a dual continuous K-g-Bessel sequence of \(\Lambda\) for upper bound \(B_{\Gamma}\) if and only if there exists \(\varphi \in \mathcal{B}(U, \Theta \setminus V_{\omega})\) such that \(T_{\omega} \varphi = 0\) and \(\Gamma_{\omega} = \Lambda_{\omega}(K^*)^\dagger + P_{\omega} \varphi\) for each \(\omega \in \Omega\).

In this case, the continuous g-Bessel bound of \(\Gamma\) is \((\|K^* K\| + \|\varphi\|^2)^\frac{1}{2}\).
Proof. First, assume that $\Gamma$ is a dual continuous $K$-$g$-Bessel sequence of $\Lambda$, and we define $\varphi : U \rightarrow \mathcal{B}(V) \ast V_\omega$ by $(\varphi x)_\omega = \Gamma_\omega x - \Lambda_\omega (K^*) x$ for each $x \in U$ and $\omega \in \Omega$; then, $\varphi \in \mathcal{B}(U, \otimes \omega \in \mathcal{B}(V_\omega))$. Indeed,
\[
\|\varphi x\| = \|\{\varphi x\}_{\omega \in \Omega}\| = \|\{\Gamma_\omega x\}_{\omega \in \Omega} - \{\Lambda_\omega (K^*) x\}_{\omega \in \Omega}\|
\leq \|\{\Gamma_\omega x\}_{\omega \in \Omega}\| + \|\{\Lambda_\omega (K^*) x\}_{\omega \in \Omega}\|
\leq \left(\int \|\Gamma_\omega x\|^2\right)^{1/2} + \left(\int \|\Lambda_\omega (K^*) x\|^2\right)^{1/2}
\leq \sqrt{\mathcal{B}_T} \|x\| + \|K^* (K^*) \| \|x\|
\leq \left(\sqrt{\mathcal{B}_T} + \|K\|\right) \|x\|,
\]
for each $x \in U$, and by Theorem 1, we have
\[
T_A \varphi x = \int_\Omega \Lambda_\omega^* (\varphi x)_\omega d\mu(\omega) = \int_\Omega \Lambda_\omega^* (\Gamma_\omega x - \Lambda_\omega (K^*) x) d\mu(\omega)
\leq \int_\Omega \Lambda_\omega^* \Gamma_\omega x d\mu(\omega) - \int_\Omega \Lambda_\omega^* \Lambda_\omega (K^*) x d\mu(\omega) = Kx - Kx = 0.
\]
Conversely, we show that $\{\Gamma_\omega\}_{\omega \in \Omega}$ is a continuous $g$-$Bessel$ sequence for $U$ with respect to $\{V_\omega\}_{\omega \in \Omega}$ with bound $(\|K\| + \|\varphi\|)^2$.

From Theorem 1, we have
\[
\int_\Omega \Lambda_\omega^* \Gamma_\omega x d\mu(\omega) = \int_\Omega \Lambda_\omega^* (\Gamma_\omega x)_\omega d\mu(\omega)
= \int_\Omega \Lambda_\omega^* (\Lambda_\omega (K^*) x + P_\omega \varphi) d\mu(\omega)
\leq \int_\Omega \Lambda_\omega^* (\Lambda_\omega (K^*) x) d\mu(\omega) + T_A^* \varphi x
\leq \int_\Omega \Lambda_\omega^* (\Lambda_\omega (K^*) x) d\mu(\omega) + T_A^* \varphi x = Kx, \forall x \in U.
\]
\[
(20)
\]
\[
(21)
\]

**Theorem 3.** Let $K \in \mathcal{B}(U)$ have a closed range and $\Lambda = \{\Lambda_\omega\}_{\omega \in \Omega}$ be a Parseval continuous $K$-$g$-frame for $U$ with respect to $\{V_\omega\}_{\omega \in \Omega}$, then, $\{\Lambda_\omega (K^*)^*\}$ is the canonical dual continuous $K$-$g$-Bessel sequence of $\Lambda$.

Proof. By Theorem 1, we know that $\{\Lambda_\omega (K^*)^*\}$ is a dual continuous $K$-$g$-Bessel sequence of $\Lambda$; to complete this proof, we need to prove that $\|T_A^* x\| \leq \|T_A x\|$ for any dual continuous $K$-$g$-Bessel sequence $\{\Gamma_\omega\}_{\omega \in \Omega}$, where $T_A^*$ is the synthesis operator of $\Lambda$. By Theorem 2, there exists $\varphi \in \mathcal{B}(U, \otimes \omega \in \mathcal{B}(V_\omega))$ such that $T_A^* \varphi = 0$ and $\Gamma_\omega = \Lambda_\omega (K^*)^* + P_\omega \varphi$. A simple computation gives $T_A^* x = T_A^* \varphi$, noting $T_A^* = K^* T_A$.

For each $x \in U$, we have
\[
\|T_A^* x\|^2 = (T_A^* x, T_A^* x) = (T_A^* x + \varphi x, T_A^* x + \varphi x)
= \|T_A^* x\|^2 + (T_A^* x, \varphi x) + (\varphi x, T_A^* x) + \|\varphi x\|^2
\leq \|T_A^* x\|^2 + \|\varphi x\|^2 \geq \|T_A^* x\|^2.
\]
\[
(28)
\]
Therefore, $\|T_A^* x\| = \|T_A^* x\| \leq \|T_A^* x\| = \|T_A x\|$. 
\[
(29)
\]

The question of stability plays an important role in various fields of applied mathematics. The classical theorem of the stability of a base is due to Paley and Wiener. It is based on the fact that a bounded operator $T$ on a Banach space is invertible if we have $\|T - I\| < 1$.

**Lemma 3** (see [9]: Paley–Wiener). Let $\{f_i\}_{i \in \mathbb{N}}$ be a basis of a Banach space $X$ and $\{g_i\}_{i \in \mathbb{N}}$ a sequence of vectors in $X$. If there exists a constant $\lambda \in [0, 1)$ such that
\[
\int \Lambda_\omega^* \theta_\omega x d\mu(\omega) = \int \Lambda_\omega^* (\varphi x)_\omega d\mu(\omega), \forall x \in U.
(25)
\]
\[
\left\| \sum_{i \in \mathbb{N}} c_i (f_i - g_i) \right\| \leq \lambda \left\| \sum_{i \in \mathbb{N}} c_i f_i \right\|, \tag{29}
\]
for all finite sequences \( \{c_i\}_{i \in \mathbb{N}} \) of scalars, then \( \{g_i\}_{i \in \mathbb{N}} \) is also a basis for \( X \).

**Proposition 5.** Let \( K \in \mathcal{B}(U) \) have a closed range, \( \{\Lambda_w\}_{w \in \Omega} \) be a continuous \( K \)-g-frame for \( U \) with respect to \( \{\mathcal{V}_w\}_{w \in \Omega} \), then \( \mathcal{V}_w \) is a dual continuous \( K \)-g-frame for \( U \) with upper bound \( B \Lambda \), and \( \{\Gamma_w\}_{w \in \Omega} \) is a dual continuous \( K \)-g-frame for \( U \) with lower bound \( \Lambda \).

**Proof.** Define

\[
\psi: \mathcal{V}_w \rightarrow \mathcal{V}_w \setminus \{\mathcal{V}_w\}_{w \in \Omega} = \int_{\omega} \theta_w y_\omega d\mu(\omega). \tag{30}
\]

Then, (1) implies that \( \psi \) is well defined and bounded with \( \|\psi\| \leq \sqrt{\lambda} + \sqrt{B} \Lambda \).

Clearly, \( \psi \) is linear. Thus,

\[
\|K^* P_{L(\mathcal{R}(K))} x\|^2 = \sup_{y \in \mathcal{V}_w} \left| \langle K^* P_{L(\mathcal{R}(K))} x, y \rangle \right|^2
\]

\[
= \sup_{y \in \mathcal{V}_w} \left| \int_{\omega} \Gamma_w y_\omega d\mu(\omega) \right|^2
\]

\[
= \sup_{y \in \mathcal{V}_w} \left| \int_{\omega} \Gamma_w K^* L^{-1} P_{L(\mathcal{R}(K))} K x, y \right|^2
\]

\[
\leq \int_{\omega} \|\theta_w y_\omega\|^2 d\mu(\omega) \sup_{y \in \mathcal{V}_w} \left\| \Gamma_w K^* L^{-1} P_{L(\mathcal{R}(K))} K y \right\|^2 \int_{\omega} d\mu(\omega)n
\]

\[
\leq B_T \|K^*\|^2 \|L^{-1}\| \|K\|^2 \int_{\omega} \|\theta_w x\|^2 d\mu(\omega)n
\]

\[
\leq B_T \|K^*\|^2 \|L^{-1}\| \|K\|^2 \frac{1}{(1 - \beta)} \int_{\omega} \|\theta_w x\|^2 d\mu(\omega)
\]

It follows that

\[
B_T^{-1} \|K^*\|^2 - \|L^{-1}\| \|K\|^2 \|K^* P_{L(\mathcal{R}(K))} x\|^2 \leq \int_{\omega} \|\theta_w x\|^2 d\mu(\omega).
\tag{36}
\]

This completes the proof.

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**3. Continuous \( K \)-g-Frame Sequences and Continuous \( g \)-Frame Sequences**

**Theorem 4.** Let \( \Lambda = \{\Lambda_w\}_{w \in \Omega} \) be a \( g \)-frame for \( U \) with respect to \( \{\mathcal{V}_w\}_{w \in \Omega} \); then, \( \Lambda \) is a continuous \( K \)-g-frame for \( U \) with respect to \( \{\mathcal{V}_w\}_{w \in \Omega} \) if and only if \( \mathcal{R}(K) \subset \text{span}(\{\Lambda_w^*(\mathcal{V}_w)\}_{w \in \Omega}) \).
Proof. Suppose that \( \Lambda \) is a continuous \( K \)-g-frame for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \); then, \( \forall x \in U \), we have
\[
C_\Lambda (K^* x, x) = C_\Lambda \| K^* x \|^2 \leq \int_\Omega \| A_\omega x \|^2 d\mu(\omega) = \| T_\Lambda x \|^2 \tag{37}
\]

From Lemma 2, we have \( \mathcal{S}(K) \subset \mathcal{S}(T_\Lambda) \) since \( \mathcal{S}(T_\Lambda) \leq \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \). Conversely, let \( \mathcal{S}(K) \subset \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \). Then
\[
U = \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \oplus \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega})^\perp,
\]
we have
\[
\forall x \in U, \quad x = y + z, \quad y \in \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}), \quad z \in \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega})^\perp. \tag{38}
\]

For any \( \omega \in \Omega, A_\omega^*(\Lambda^* z) \in \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \). For all \( x \in U \), \( \| A_\omega x \|^2 = \| A_\omega \|^2 \), so
\[
\int_\Omega \| A_\omega x \|^2 d\mu(\omega) = \int_\Omega \| A_\omega y \|^2 d\mu(\omega) \leq B_\lambda \| y \|^2 + \| z \|^2 = B_\lambda \| x \|^2. \tag{39}
\]

On the contrary, since \( \mathcal{S}(K) \subset \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \), \( z \in \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega})^\perp \); then, \( \langle K^* z, h \rangle = \langle z, Kh \rangle = 0 \), \( h \in U \); hence, \( K^* z = 0 \).

Then,
\[
\int_\Omega \| A_\omega x \|^2 d\mu(\omega) = \int_\Omega \| A_\omega y \|^2 d\mu(\omega) \geq C_\lambda \| K^* y \|^2 = C_\lambda \| K^* x \|^2. \tag{40}
\]

Proposition 6. Let \( K \in \mathcal{S}(U) \) have a closed range and \( \Lambda = \{A_\omega\}_{\omega \in \Omega} \) be a continuous \( K \)-g-frame for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \); then, \( \Lambda = \{A_\omega\}_{\omega \in \Omega} \) is a continuous g-frame for \( \mathcal{S}(K) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \).

Proof. We just prove the lower continuous g-frame inequality. From Proposition 2, there exists a continuous g-Bessel sequence \( \Gamma = \{\Gamma_\omega\}_{\omega \in \Omega} \) for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \) such that \( K x = \int \Gamma_\omega \Gamma_\omega^* y d\mu(\omega), \forall x \in U \).

For every \( y \in \mathcal{S}(K) \), by Lemma 1,
\[
y = K^* y = \int \Lambda_\omega^* \Gamma_\omega^* K^* y d\mu(\omega) = \int \Lambda_\omega^* (\Gamma_\omega K^* P_{\mathcal{S}(K)}) y d\mu(\omega). \tag{41}
\]

Thus, \( \theta_\omega = \Gamma_\omega K^* P_{\mathcal{S}(K)} \) for every \( \omega \in \Omega \).

Then, we have
\[
\int_\Omega \| \theta_\omega x \|^2 d\mu(\omega) \leq \int_\Omega \| \Gamma_\omega K^* P_{\mathcal{S}(K)} x \|^2 d\mu(\omega) \leq B_\Gamma \| K^* \|^2 \| x \|^2. \tag{42}
\]

Hence, \( \| z \|^2 = \sup_{y \in U} \| \langle z, y \rangle \|^2 \),
\[
= \sup_{y \in U} \left| \left| \int \theta_\omega z d\mu(\omega), y \right|^2 \right| \leq \left( \sup_{y \in U} \int \| \theta_\omega y \|^2 d\mu(\omega) \right) \left( \int_\Omega \| A_\omega z \|^2 d\mu(\omega) \right) 
\]
\[
\leq B_\Gamma \| K^* \|^2 \left( \int_\Omega \| A_\omega z \|^2 d\mu(\omega) \right). \tag{44}
\]

Then, \( \{A_\omega\}_{\omega \in \Omega} \) is a continuous g-frame for \( \mathcal{S}(K) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \). \( \Box \)

Proposition 7. Let \( K \in \mathcal{S}(U) \) have a closed range and \( \Lambda = \{A_\omega\}_{\omega \in \Omega} \) be a continuous \( K \)-g-frame for \( \mathcal{S}(K) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \). If \( \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \subset \mathcal{S}(K) \), then \( \Lambda = \{A_\omega\}_{\omega \in \Omega} \) is a continuous \( K \)-g-frame for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \).

Proof. For each \( x \in U \), we have \( x = y + z \) such that \( y \in \mathcal{S}(K) \) and \( z \in \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega})^\perp \). So, \( \| A_\omega x \|^2 = \| A_\omega y \|^2 \).

Since \( \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \subset \mathcal{S}(K) \) and \( A_\omega^*(\Lambda^* e) \in \overline{\text{span}}(\{A_\omega^*(V_\omega)\}_{\omega \in \Omega}) \), \( \forall e \in U \),
\[
\int_\Omega \| A_\omega x \|^2 d\mu(\omega) = \int_\Omega \| A_\omega y \|^2 d\mu(\omega) \leq B_\lambda \| y \|^2 \leq B_\lambda \left( \| y \|^2 + \| z \|^2 \right) = B_\lambda \| x \|^2. \tag{45}
\]
Let $\Phi = \{\phi_\omega\}_{\omega \in \Omega}$ be a dual continuous g-frame of $\Lambda = \{\Lambda_\omega\}_{\omega \in \Omega}$.

So,

$$\int_\Omega \Lambda_\omega^* \phi_\omega y \, d\mu (\omega) = \int_\Omega \phi_\omega^* \Lambda_\omega y \, d\mu (\omega).$$  \hspace{1cm} (46)

Hence,

$$\|K^* x\|^2 = \|K^* (y + z)\| = \|K^* y\|^2 = \|K^* y, K^* y\| = \left| \left\langle K^* y, y \right\rangle \right| = \left| \left\langle \int_\Omega \phi_\omega^* \Lambda_\omega y \, d\mu (\omega), K^* y \right\rangle \right| = \int_\Omega \left\langle \Lambda_\omega \phi_\omega y, K^* y \right\rangle \, d\mu (\omega)
\leq \left( \int_\Omega \|\Lambda_\omega y\|^2 \, d\mu (\omega) \right)^{1/2} \left( \int_\Omega \|\phi_\omega K^* y\|^2 \, d\mu (\omega) \right)^{1/2}
\leq B_\phi |K| \|K^* x\| \left( \int_\Omega \|\Lambda_\omega y\|^2 \, d\mu (\omega) \right)^{1/2}.$$

Then, $B_\phi^{-1} \|K\|^{-2} \|K^* x\|^2 \leq \int_\Omega \|\Lambda_\omega y\|^2.$

On the contrary, we have $\int_\Omega \|\Lambda_\omega y\|^2 = \int_\Omega \|\Lambda_\omega x\|^2.$

Then, $B_\phi \|K\|^{-2} \|K^* x\|^2 \leq \int_\Omega \|\Lambda_\omega x\|^2.$

So, $\Lambda$ is a continuous K-g-frame for $U.$ \hfill $\square$

**Theorem 5.** For every $\omega \in \Omega,$ let $\Lambda_\omega \in \mathcal{B}(U, V_\omega);$ then, the following statements are equivalent:

1. $\Lambda = \{\Lambda_\omega\}_{\omega \in \Omega}$ is a continuous K-g-frame sequence for $U$ with respect to $\{V_\omega\}_{\omega \in \Omega}.$

2. $\Lambda = \{\Lambda_\omega\}_{\omega \in \Omega}$ is a continuous g-Bessel sequence for $\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}$ with respect to $\{V_\omega\}_{\omega \in \Omega},$ and there exists a continuous g-Bessel sequence $\{\Gamma_\omega\}_{\omega \in \Omega}$ for $U$ with respect to $\{V_\omega\}_{\omega \in \Omega}$ such that

$$\left( \frac{K^* \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}}{\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}} \right)^* = \int_\Omega \Lambda_\omega^* \Gamma_\omega x, \ \forall x \in U.$$

(48)

3. $\Lambda = \{\Lambda_\omega\}_{\omega \in \Omega}$ is a continuous g-Bessel sequence for $\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}$ with respect to $\{V_\omega\}_{\omega \in \Omega},$ and there exists a continuous g-Bessel sequence $\{\Gamma_\omega\}_{\omega \in \Omega}$ for $U$ with respect to $\{V_\omega\}_{\omega \in \Omega}$ such that

$$K^* x = \int_\Omega \Gamma_\omega^* \Lambda_\omega x \, d\mu (\omega), \ \forall x \in \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}.$$  \hspace{1cm} (49)

**Proof.** (1) $\implies$ (2):

We have

$$\frac{K^* \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}}{\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}} \longrightarrow U$$

then their adjoint

$$\left( \frac{K^* \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}}{\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}} \right)^* : U \longrightarrow \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}.$$

By the definition on the continuous K-g-frame sequence, there exists $C_\Lambda > 0$ such that

$$C_\Lambda \|K^* x\|^2 \leq \|T^*_\Lambda x\|^2, \ \forall x \in \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}.$$  \hspace{1cm} (52)

This implies

$$C_\Lambda \left( \frac{K^* \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}}{\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}} \right)^* \left( \frac{K^* \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}}{\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}} \right) \leq T_\Lambda T^*_\Lambda.$$  \hspace{1cm} (53)

From Lemma 2, there exists $R \in \mathcal{B}(U, \Phi \omega \omega V_\omega)$ such that

$$\left( \frac{K^* \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}}{\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}} \right)^* = T_\Lambda R.$$  \hspace{1cm} (54)

For every $\omega \in \Omega,$ denote $\Gamma_\omega = P_\omega R;$ then, we have

$$\int_\Omega \|\Gamma_\omega x\|^2 \, d\mu (\omega) = \int_\Omega \|P_\omega Rx\|^2 \, d\mu (\omega)
\leq \int_\Omega \|P_\omega R\|^2 \|x\|^2 \, d\mu (\omega)
= \|Rx\|^2 \leq \|R\|^2 \|x\|^2.$$  \hspace{1cm} (55)

Hence, $\{\Gamma_\omega\}_{\omega \in \Omega}$ is a continuous g-Bessel sequence for $U$ with respect to $\{V_\omega\}_{\omega \in \Omega}.$

On the contrary, we have for all $x \in U,$

$$\left( \frac{K^* \text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}}{\text{span} \{\Lambda_\omega^* (V_\omega)\}_{\omega \in \Omega}} \right)^* = T_\Lambda Rx$$

$$= \int_\Omega \Lambda_\omega^* (Rx) \, d\mu (\omega)
= \int_\Omega \Lambda_\omega^* (P_\omega Rx) \, d\mu (\omega)
= \int_\Omega \Lambda_\omega^* \Gamma_\omega x \, d\mu (\omega).$$  \hspace{1cm} (56)

(2) $\implies$ (3):
For every \( x \in U \) and \( y \in \overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega}) \), by (48), we have

\[
\left\langle \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega})} \right)^* x, y \right\rangle = \int_\Omega \left\langle \Lambda^*_\omega \Gamma^*_{\omega x}, y \right\rangle d\mu(\omega) = \left\langle x, \int_\Omega \Gamma^*_{\omega x} \Lambda^*_\omega y d\mu(\omega) \right\rangle.
\]

(57)

So,

\[
\langle x, K^* y \rangle = \left\langle x, \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega})} y \right\rangle = \left\langle x, \int_\Omega \Gamma^*_{\omega x} \Lambda^*_\omega y d\mu(\omega) \right\rangle.
\]

(58)

Hence, \( K^* y = \int_\Omega \Gamma^*_{\omega x} \Lambda^*_\omega y d\mu(\omega) \).

(3) \implies (1):

Suppose that (49) holds; to prove (1), we just prove the lower bound inequality on continuous K-g-frames.

For every \( y \in \overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega}) \), we have

\[
\|K^* y\|^2 = \sup_{l \in I} \left| \left\langle K^* y, z \right\rangle \right|^2 = \sup_{l \in I} \left| \left\langle \int_\Omega \Lambda^*_{\omega l} \gamma d\mu(\omega), z \right\rangle \right|^2
\]

\[
= \sup_{l \in I} \left| \left\langle \int_\Omega \Lambda^*_{\omega l} \gamma d\mu(\omega), \Gamma^*_{\omega l} z \right\rangle \right|^2 \leq \int_\Omega \|\Lambda^*_{\omega l} \gamma\|^2 d\mu(\omega) \sup_{l \in I} \left| \left\langle \int_\Omega \Gamma^*_{\omega l} z d\mu(\omega) \right\rangle \right|^2 \leq 2 \int_\Omega \|\Lambda^*_{\omega l} \gamma\|^2 d\mu(\omega).
\]

(59)

So, \( B^2 \|K^* y\|^2 \leq \int_\Omega \|\Lambda^*_{\omega l} \gamma\|^2 d\mu(\omega), \forall y \in \overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega}) \).

\( \square \)

**Theorem 6.** A sequence \( \{\Lambda^*_\omega\}_{\omega \in \Omega} \in \mathcal{B}(U, \oplus_{\omega \in \Omega} V_w) \) is a continuous K-g-frame sequence for \( U \) with respect to \( \{V_w\}_{\omega \in \Omega} \) if and only if there exists \( T \in \mathcal{B}(\overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega}), \oplus_{\omega \in \Omega} V_w) \) such that

\[
\frac{\Lambda^*_\omega}{\overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega})} = P_{\omega T}, \quad \forall \omega \in \Omega,
\]

\[
\mathcal{R}\left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega})} \right) \subset \mathcal{R}(T^*).
\]

(60)

**Proof.** Let \( \{\Lambda^*_\omega\}_{\omega \in \Omega} \) be a continuous K-g-frame sequence for \( U \) with respect to \( \{V_w\}_{\omega \in \Omega} \), and let \( T = T^*_\lambda \).

\[
\lambda^{-1} \|K^* x\|^2 = \lambda^{-1} \left\| \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_w)\}_{\omega \in \Omega})} x \right\|^2 \leq \|T x\|^2 = \int \|\Lambda x\|^2 d\omega x.
\]

(64)
and then \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a continuous \( K \)-g-frame sequence for 
\( \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \) with bounds \( \lambda^{-1} \) and \( \|T\|^2 \).

In general, a continuous \( K \)-g-frame \( \{\Lambda_\omega\}_{\omega \in \Omega} \) for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \) is not a continuous g-frame for, \( \text{span}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \).

**Proposition 8.** Let \( \Lambda = \{\Lambda_\omega\}_{\omega \in \Omega} \) be a continuous \( K \)-g-frame sequence for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \). Suppose that \( K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) \( \neq 0 \) and that it has a closed range; then, \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a continuous g-frame for \( \mathcal{S}(K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}))^* \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \).

**Proof.** Since \( K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) has a closed range, its pseudo-inverse exists; from Lemma 1, every \( x \in \mathcal{S}(K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}))^* \) can be written as

\[
x = \left( \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \right)^+ \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right) x.
\]

(65)

Hence,

\[
\|x\|^2 \leq \left\| \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^+ \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right) x \right\|^2.
\]

(66)

Noting that \( \mathcal{S}(K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}))^* \subset \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \), we have

\[
\left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^+ x = K^* x.
\]

(67)

So,

\[
\|x\|^2 \leq \left\| \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^+ \right\|^2 \|K^* x\|^2
\]

\[
\leq \frac{1}{\Lambda} \left\| \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^+ \right\|^2 \int_{\Omega} \|A_\omega x\|^2 d\mu(\omega).
\]

(68)

Since \( K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \neq 0 \), its pseudo-inverse

\[
\left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^* \neq 0,
\]

(69)

and then

\[
\frac{C}{D} \left( \frac{K}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^* \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^+ = I_{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})}.
\]

(70)

Hence, \( (C/D)(K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}))^* \) is a left inverse of \( K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \); conversely, we have

\[
\Lambda \left\| \left( \frac{K}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^+ \right\|^2 \leq \int_{\Omega} \|A_\omega x\|^2 d\mu(\omega).
\]

(71)

On the contrary, we have

\[
\int_{\Omega} \|A_\omega x\|^2 d\mu(\omega) \leq B_\Lambda \|x\|^2, \quad \forall x \in \mathcal{S}(\left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^*).
\]

(72)

So, \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a continuous g-frame for \( \mathcal{S}(\left( K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \right)^* \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \) with bounds \( C \|K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})\|^2 \) and \( B_\Lambda \).

In the following, we give a necessary and sufficient condition under which a tight continuous \( K \)-g-frame sequence \( \{\Lambda_\omega\}_{\omega \in \Omega} \) for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \) is a tight continuous g-frame for \( \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \).

**Theorem 7.** Let \( \{\Lambda_\omega\}_{\omega \in \Omega} \) be a \( C \)-tight continuous g-frame for \( U \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \); then, \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a \( D \)-tight continuous g-frame for \( \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \) if and only if \( K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) admits a left inverse \( C/D(K^*/\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}))^* \).

**Proof.** First, we suppose that \( \{\Lambda_\omega\}_{\omega \in \Omega} \) is a \( D \)-tight continuous g-frame for \( \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \) with respect to \( \{V_\omega\}_{\omega \in \Omega} \). Hence,

\[
D\|x\|^2 = \int_{\Omega} \|A_\omega x\|^2 d\mu(\omega), \quad K\forall x \in \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}).
\]

(73)

\[
C\|K^* x\|^2 = \int_{\Omega} \|A_\omega x\|^2 d\mu(\omega), \quad \forall x \in \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}).
\]

So, \( D\|x\|^2 = C\|K^* x\|^2 \) for every \( x \in \overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega}) \).

Then,

\[
D(x, x) = C(K^* x, K^* x)
\]

(74)

This implies that

\[
\frac{C}{D} \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^* \left( \frac{K^*}{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})} \right)^+ = I_{\overline{\text{span}}(\{\Lambda^*_\omega(V_\omega)\}_{\omega \in \Omega})}.
\]

(75)
So,

\[
\int_{\Omega} \| \Lambda_\omega x \|^2 d\mu(\omega) = \int_{\Omega} \| \Lambda_\omega x \|^2 d\mu(\omega) = D \frac{C}{D} \| K^* x \|^2 = D \| K^* x \|^2.
\]

It follows that \( \Lambda = \{ \Lambda_\omega \}_{\omega \in \Omega} \) is a \( D \)-tight continuous g-frame for \( \text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega}) \) with respect to \( \{ V_\omega \}_{\omega \in \Omega} \).

\[ \text{Theorem 7.} \quad \text{Let} \ K_1, K_2 \in \mathcal{R}(U) \text{ and} \ \{ \Lambda_\omega \}_{\omega \in \Omega} \text{ be a } C \text{-tight continuous } K_1 \text{-g-frame sequence for } U \text{ with respect to} \ \{ V_\omega \}_{\omega \in \Omega}; \text{then,} \ \{ \Lambda_\omega \}_{\omega \in \Omega} \text{ is a } C \text{-tight continuous } K_2 \text{-g-frame sequence for } U \text{ with respect to} \ \{ V_\omega \}_{\omega \in \Omega} \text{ if and only if} \]

\[
\mathcal{R} \left( \frac{K^*_2}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^* \subset \mathcal{R} \left( \frac{K^*_1}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^*.
\]

\[ \text{Proof.} \quad \text{First, we suppose that} \ \Lambda = \{ \Lambda_\omega \}_{\omega \in \Omega} \text{ is a } C \text{-tight continuous } K_2 \text{-g-frame sequence for } U \text{ with respect to} \ \{ V_\omega \}_{\omega \in \Omega}; \text{then, there exists } D > 0 \text{ such that} \]

\[
D \| K^*_2 x \|^2 \leq \int_{\Omega} \| \Lambda_\omega x \|^2 d\mu(\omega), \quad \forall x \in \text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega}).
\]

Therefore,

\[
\frac{K^*_2}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^* \subset \mathcal{R} \left( \frac{K^*_2}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^*
\]

Conversely, suppose \( \mathcal{R} \left( \frac{K^*_1}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^* \subset \mathcal{R} \left( \frac{K^*_1}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^* \).

By Lemma 2, there exists \( \lambda > 0 \) such that

\[
\lambda \left( \frac{K^*_1}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^* \subset \mathcal{R} \left( \frac{K^*_1}{\text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega})} \right)^*.
\]

so for every \( x \in \text{span}(\{ \Lambda_\omega^* (V_\omega) \}_{\omega \in \Omega}) \), we have
\[ \|K^*_2 x\|^2 = \left\langle \left( \frac{K^*_1}{\text{span}(\Lambda^*_1(V_{\omega}))_{\omega \in \Omega}} \right), \left( \frac{K^*_1}{\text{span}(\Lambda^*_1(V_{\omega}))_{\omega \in \Omega}} \right), x, x \right\rangle \]
\[ \leq \lambda^2 \left\langle \frac{K^*_1}{\text{span}(\Lambda^*_1(V_{\omega}))_{\omega \in \Omega}} \right), \left( \frac{K^*_1}{\text{span}(\Lambda^*_1(V_{\omega}))_{\omega \in \Omega}} \right), x, x \right\rangle \]
\[ = \lambda^2 \left\| K^*_1 x \right\|_2^2 = \frac{\lambda^2}{c} \int_{\Omega} \| \Lambda^*_1 x \|^2 \, d\mu(\omega). \] (83)

Then, for all \( x \in \text{span}(\{\Lambda^*_1(V_{\omega})\}_{\omega \in \Omega}) \),
\[ \frac{c}{\lambda^2} \|K^*_2 x\|^2 \leq \int_{\Omega} \|\Lambda^*_1 x\|^2 \, d\mu(\omega) \leq C \|K^*_1 x\|^2 \|x\|^2. \] (84)

This completes the proof. \( \square \)

4. Conclusion

In this article, we have proved some properties of dual continuous \( K \)-g-frames in Hilbert spaces. These results are extensions of the related results announced in [8]. The presented theorems extend, generalize, and improve many existing results in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

References