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Research Article

New Properties of Dual Continuous K-g-Frames in Hilbert Spaces

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The concept of frames in Hilbert spaces continues to play a very interesting role in many kinds of applications. In this paper, we study the notion of dual continuous K-g-frames in Hilbert spaces. Also, we establish some new properties.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces was introduced by Duffin and Schaeffer in 1952 [1] to study some deep problems in nonharmonic Fourier series; after the fundamental paper [2] by Daubechies, Grossman, and Meyer, frames' theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [3].

Let X be a Banach space, (Ω, μ) a measure space, and function $f\colon \Omega \longrightarrow X$ a measurable function. The integral of the Banach-valued function f was defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions.

Let (Ω, μ) be a measure space, let U and V be two separable Hilbert spaces, $\{V_w\}_{w\in\Omega}$ is a sequence of subspaces of V, we denote $\mathscr{B}(U, V)$ as the collection of all bounded linear operators from U to V, and $\mathscr{B}(U, U)$ is abbreviated to $\mathscr{B}(U)$.

For $T \in \mathscr{B}(U, V)$, we use the notation $\mathscr{R}(T)$ and $\mathscr{N}(T)$ to denote, respectively, the range and the null space of T.

Definition 1 (see [4]). A sequence $\{\Lambda_\omega \in \mathscr{B}(U, V_\omega)\}_{\omega\in\Omega}$ is called a continuous g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ if there exists $0 < A < B < \infty$ such that

$$A\|x\|^2 \le \int_\Omega \left\|\Lambda_\omega x\right\|^2 \mathrm{d}\mu(\omega) \le B\|x\|^2, \quad \forall x \in U. \tag{1}$$

The numbers A and B are called lower and upper bounds of the continuous g-frame, respectively.

If $A = B = \lambda$, the frame is λ-tight.

If $A = B = 1$, it is called a normalized tight continuous g-frame or a Parseval continuous g-frame.

We call $\{\Lambda_\omega \in \mathscr{B}(U, V_\omega)\}_{\omega\in\Omega}$ a continuous g-Bessel sequence if the right-hand inequality of (1) holds.

Let $\Lambda = \{\Lambda_w\}_{\in\Omega}$ be a continuous g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.

The synthesis operator of Λ is defined by $T_\Lambda\colon \oplus_{\omega\in\Omega} V_w \longrightarrow U$, given by

$$T_\Lambda(\{y_w\}_{w\in\Omega}) = \int_\Omega \Lambda_\omega^* y_\omega \mathrm{d}\mu(w), \quad \forall \{y_w\}_{w\in\Omega} \in \bigoplus_{\omega\in\Omega} V_w, \tag{2}$$

where $\oplus_{\omega\in\Omega} V_w = \left\{(y_w)_{w\in\Omega}, y_\omega \in V_\omega; \|\int_\Omega |y_\omega|^2 \mathrm{d}\mu(\omega)\| < \infty\right\}$.

The existence of $\int_\Omega \Lambda_\omega^* y_\omega \mathrm{d}\mu(w)$ implies that T_Λ is well defined and bounded.

The adjoint operator of T_Λ which is given by

$$T_\Lambda^*\colon U \longrightarrow \bigoplus_{\omega\in\Omega} V_w; T_\Lambda^*(x) = \{\Lambda_\omega x\}_{w\in\Omega}, \quad \forall x \in U, \tag{3}$$

is said to be the analysis operator of $\{\Lambda_w\}_{\omega\in\Omega}$.

Definition 2 (see [5]). Let $K \in \mathscr{B}(U)$. A sequence $\{\Lambda_\omega \in \mathscr{B}(U, V_\omega)\}_{\omega\in\Omega}$ is called a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ if there exists $0 < A < B < \infty$ such that

$$A\left\|K^* x\right\|^2 \le \int_\Omega \left\|\Lambda_\omega x\right\|^2 \mathrm{d}\mu(\omega) \le B\|x\|^2, \quad \forall x \in U. \tag{4}$$

The numbers A and B are called lower and upper bounds of the continuous K-g-frame, respectively.

If $A\|K^* x\|^2 = \int_\Omega \|\Lambda_\omega x\|^2 \mathrm{d}\mu(\omega)$, the sequence $\{\Lambda_\omega x\}_{w\in\Omega}$ is called a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.

The A-tight continuous K-g-frame for U is said to be a Parseval continuous K-g-frame if $A = 1$.

Suppose that $K \in \mathscr{B}(U)$ and $\Lambda = \{\Lambda_\omega \in \mathscr{B}(U, V_\omega)\}_{\omega\in\Omega}$ is a Parseval continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ with synthesis operator T_Λ. Then, it is easy to check

$$KK^* x = T_\Lambda T_\Lambda x = \int_\Omega \Lambda_\omega^* \Lambda_\omega x \mathrm{d}\mu(\omega), \quad \forall x \in U. \tag{5}$$

Definition 3. Let $K \in \mathscr{B}(U)$ and $\{\Lambda_\omega\}_{\omega\in\Omega} \in \mathscr{B}(U, V_\omega)$ be a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$. A continuous K-g-Bessel sequence $\{\Gamma_\omega\}_{\omega\in\Omega}$ for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ is said to be a dual continuous K-g-Bessel sequence of $\{\Lambda_\omega\}_{\omega\in\Omega}$ if

$$Kx = \int_\Omega \Lambda_\omega^* \Gamma_\omega x \mathrm{d}\mu(w), \quad \forall x \in U. \tag{6}$$

Remark 1

(1) If $K = Id_U$, then a dual continuous K-g-Bessel sequence is just a dual continuous g-frame

(2) A dual continuous K-g-Bessel sequence is necessarily a continuous K^*-g-frame

(3) Denote the synthesis operator of $\{\Lambda_\omega\}_{\omega\in\Omega}$ and $\{\Gamma_\omega\}_{\omega\in\Omega}$ by T_Λ and T_Γ, respectively; then, (6) means that $K = T_\Lambda T_\Gamma^*$

Lemma 1 (see [6]). *Suppose that $T \in \mathscr{B}(U, V)$ has a closed range; then, there exists a unique operator $T^+ \in \mathscr{B}(U, V)$, called the pseudo-inverse of T, satisfying*

$$\begin{aligned} TT^+T &= T, \\ T^+TT^+ &= T^+, \\ (TT^+)^* &= TT^+, \\ (T^+T)^* &= T^+T, \\ (T^*)^+ &= (T^+)^* \mathscr{N}(T^+) = (\mathscr{R}(T))^\perp, \\ \mathscr{R}(T^+) &= (\mathscr{N}(T))^\perp. \end{aligned} \tag{7}$$

Lemma 2 (see [7]). *Let U, U_1, and U_2 be three Hilbert spaces; also, let $T \in \mathscr{B}(U_1, U)$ and $K \in \mathscr{B}(U_2, U)$. The following statements are equivalent:*

(1) $\mathscr{R}(T) \subset \mathscr{R}(K)$

(2) There exists $\lambda > 0$ such that $TT^* \le \lambda KK^*$

(3) There exists $\theta \in \mathscr{B}(U_1, U_2)$ such that $T = K\theta$

Moreover, if (1)–(3) are valid, then there exists a unique operator θ such that

(a) $\|\theta\|^2 = \inf\{\mu\colon TT^* \le \mu KK^*\}$

(b) $\mathscr{N}(T) = \mathscr{N}(\theta)$

(c) $\mathscr{R}(\theta) \subset \overline{\mathscr{R}(K^*)}$

In the following section, we set out to generalize some results of Xiang [8]; in other words, we characterize the concept of dual continuous K-g-frames in Hilbert spaces, and we give some properties. Our results extend, generalize, and improve many existing results.

2. Main Result

Proposition 1. *Suppose that $K \in \mathscr{B}(U)$ has a closed range, and let $\Lambda = \{\Lambda_\omega \in \mathscr{B}(U, V_\omega)\}_{\omega\in\Omega}$ to be a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ and $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ be a dual continuous K-g-Bessel sequence of $\{\Lambda_\omega\}_{\omega\in\Omega}$; then, $\{\Gamma_\omega K^*\}_{\omega\in\Omega}$ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.*

Proof. For each $x \in U$, we have $Kx = \int_\Omega \Lambda_\omega^* \Gamma_\omega x \mathrm{d}\mu(w)$, and then

$$\begin{aligned} \|Kx\|^2 &= \sup_{\|y\|=1} |\langle Kx, y\rangle|^2 \\ &= \sup_{\|y\|=1} \left|\int_\Omega \langle \Gamma_\omega x, \Lambda_\omega y\rangle \mathrm{d}\mu(\omega)\right|^2 \\ &\le \sup_{\|y\|=1} \int_\Omega \left\|\Gamma_\omega x\right\|^2 \mathrm{d}\mu(\omega) \int_\Omega \left\|\Lambda_\omega y\right\|^2 \mathrm{d}\mu(\omega) \\ &\le B_\Lambda \int_\Omega \left\|\Gamma_\omega x\right\|^2 \mathrm{d}\mu(\omega), \end{aligned} \tag{8}$$

which give

$$B_\Lambda^{-1} \|Kx\|^2 \le \int_\Omega \left\|\Gamma_\omega x\right\|^2 \mathrm{d}\mu(\omega) \le B_\Gamma \|x\|^2, \tag{9}$$

where B_Λ and B_Γ are the upper bounds of Λ and Γ, respectively.

Consequently,

$$B_\Lambda^{-1} \left\|KK^* x\right\|^2 \le \int_\Omega \left\|\Gamma_\omega K^* x\right\|^2 \mathrm{d}\mu(\omega) \le B_\Gamma \left\|K^* x\right\|^2 \le B_\Gamma \|K\|^2 \|x\|^2. \tag{10}$$

Assume that K^* has a closed range; hence, for each $y \in \mathscr{R}(K^*)$, we get $y = K^*(K^*)^+ y = K^+ K y$, by Lemma 1, and thus,

$$\|y\|^2 = \left\|K^+ K y\right\|^2 \le \left\|K^+\right\|^2 \|Ky\|^2. \tag{11}$$

Now,

$$B_\Lambda^{-1}\|K^+\|^{-2}\|K^*x\|^2 \le B_\Lambda^{-1}\|KK^*x\|^2 \le \int_\Omega \|\Gamma_\omega K^* x\|^2 \mathrm{d}\mu(\omega) \le B_\Gamma \|K\|^2\|x\|^2, \tag{12}$$

which gives $\{\Gamma_\omega K^*\}_{\omega\in\Omega}$ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ with bounds $B_\Lambda^{-1}\|K^+\|^{-2}$ and $B_\Gamma\|K\|^2$. □

Proposition 2. *Let $K \in \mathscr{B}(U)$; then, every continuous K-g-frame admits a dual continuous K-g-Bessel sequence.*

Proof. Let $\Lambda = \{\Lambda_\omega \in \mathscr{B}(U, V_\omega)\}_{\omega\in\Omega}$ be a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$, with bounds C_Λ and D_Λ; then, for each $x \in U$, we have

$$C_\Lambda\langle K^*x, K^*x\rangle = C_\Lambda\|K^*x\|^2 \le \int_\Omega \|\Lambda_\omega x\|^2 \mathrm{d}\mu(\omega) = \|T_\Lambda^* x\|^2 = \langle T_\Lambda^* x, T_\Lambda^* x\rangle, \tag{13}$$

which is equivalent to $C_\Lambda KK^* \le T_\Lambda T_\Lambda^*$; by Lemma 2, there exists $\theta \in \mathscr{B}(U, \oplus_{\omega\in\Omega}V_w)$ such that $K = T_\Lambda\theta$.

Let P_ω be the projection on $\oplus_{\omega\in\Omega}V_w$ that maps each element to its w-th component, i.e., $P_\omega(\{y_z\}_{z\in\Omega}) = \{u_z\}_{z\in\Omega}$, where $u_z = y_\omega$ if $z = \omega$, and $u_z = 0$ if not.

If we let $\Gamma_\omega = P_\omega\theta$, then

$$\begin{aligned}\int_\Omega \|\Gamma_\omega x\|^2 \mathrm{d}\mu(\omega) &= \int_\Omega \|P_\omega\theta x\|^2 \mathrm{d}\mu(\omega)\\ &= \int_\Omega \|(\theta x)_\omega\|^2 \mathrm{d}\mu(\omega) = \|\theta x\|^2\\ &\le \|\theta\|^2\|x\|^2, \quad \forall x \in U.\end{aligned} \tag{14}$$

Hence, $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$. Now,

$$Kx = T_\Lambda\theta x = \int_\Omega \Lambda_\omega^*(\theta x)_\omega \mathrm{d}\mu(\omega) = \int_\Omega \Lambda_\omega^*(P_\omega\theta)x\mathrm{d}\mu(\omega), \quad \forall x \in U, \tag{15}$$

showing that $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence. □

Proposition 3. *Suppose that $K \in \mathscr{B}(U)$ and $\Lambda = \{\Lambda_\omega \in \mathscr{B}(U, V_\omega)\}_{\omega\in\Omega}$ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$; then, there exists a unique dual continuous K-Bessel sequence $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ of Λ such that $\|T_\Gamma\| \le \|T_\theta\|$ for any dual continuous K-g-Bessel sequence $\Theta = \{\theta_\omega\}_{\omega\in\Omega}$ of Λ.*

Proof. Since $KK^* \le (1/C_\Lambda)T_\Lambda T_\Lambda^*$, by Lemma 2, it follows that there is a unique operator $R \in \mathscr{B}(U, \oplus_{\omega\in\Omega}V_w)$ such that $K = T_\Lambda R$ and

$$\|R\|^2 = \inf\left\{\lambda\colon \|K^*x\|^2 \le \lambda\|T_\Lambda^* x\|^2\right\}, \tag{16}$$

taking $\Gamma_\omega = P_\omega R$ for each $\omega \in \Omega$; then, as shown in Proposition 2, Γ is a continuous g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$, and furthermore, it is a dual continuous K-g-Bessel sequence of Λ.

Since $P_\omega T_\Gamma^* x = \Gamma_\omega x = P_\omega Rx$ for any $\omega \in \Omega$ and $x \in U$, $T_\Gamma^* = R$.

Now, let $\{\theta_\omega\}_{\omega\in\Omega}$ be any dual continuous K-g-Bessel sequence of $\{\Lambda_\omega\}_{\omega\in\Omega}$; then, $K = T_\Lambda T_\theta^*$, and $K^* = T_\theta T_\Lambda^*$ as a sequence. For each $x \in U$, we have

$$\|K^*x\|^2 = \|T_\theta T_\Lambda^* x\|^2 \le \|T_\theta\|^2\|T_\Lambda^* x\|^2. \tag{17}$$

It follows that $\|T_\theta\|^2 \ge \|R\|^2 = \|T_\Gamma^*\|^2 = \|T_\Gamma\|^2$.

Equivalently, $\|T_\theta\| \ge \|T_\Gamma\|$. □

Proposition 4. *Suppose that $K \in \mathscr{B}(U)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$; then, the dual continuous K-g-Bessel sequence of Λ is precisely the family $\{\Gamma_\omega\}_{\omega\in\Omega} = \{P_\omega\theta^*\}_{\omega\in\Omega}$, where $\theta \in \mathscr{B}(\oplus_{\omega\in\Omega}V_w, U)$ satisfies the condition $K = T_\Lambda\theta^*$.*

Theorem 1. *Let $K \in \mathscr{B}(U)$ and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$, where K has a closed range; then, $\{\Lambda_\omega(K^+)^*\}_{\omega\in\Omega}$ is a dual continuous K-g-Bessel sequence of $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$.*

Proof. It is easy to see that $\{\Lambda_\omega(K^+)^*\}_{\omega\in\Omega}$ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.

By Lemma 1, $y = K^*(K^*)^+y = K^*(K^+)^*y$ for every $y \in \mathscr{R}(K^*)$; thus, by (5),

$$Ky = KK^*(K^+)^*y = \int_\Omega \Lambda_\omega^*\Lambda_\omega(K^+)^*y\mathrm{d}\mu(\omega). \tag{18}$$

If $h \in (\mathscr{R}(K^*))^\perp = \mathscr{N}(K)$, then by Lemma 1 again, we obtain $h \in \mathscr{N}((K^*)^+)$, implying that $\int_\Omega \Lambda_\omega^*\Lambda_\omega(K^+)^*h\mathrm{d}\mu(\omega) = 0 = Kh$. Altogether, we have

$$Kx = \int_\Omega \Lambda_\omega^*\Lambda_\omega(K^+)^*x\mathrm{d}\mu(\omega), \quad \forall x \in U, \tag{19}$$

which ends the proof. □

Theorem 2. *Let $K \in \mathscr{B}(U)$ have a closed range and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$. For each $\omega \in \Omega$, let $\Gamma_\omega \in \mathscr{B}(U, V_\omega)$; then, $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ is a dual continuous K-g-Bessel sequence of Λ for upper bound B_Γ if and only if there exists $\varphi \in \mathscr{B}(U, \oplus_{\omega\in\Omega}V_w)$ such that $T_\Lambda\varphi = 0$ and $\Gamma_\omega = \Lambda_\omega(K^+)^* + P_\omega\varphi$ for each $\omega \in \Omega$. In this case, the continuous g-Bessel bound of Γ is $(\|K^+K\| + \|\varphi\|)^2$.*

Proof. First, assume that Γ is a dual continuous K-g-Bessel sequence of Λ, and we define φ: $U \longrightarrow \oplus_{\omega\in\Omega} V_w$ by $(\varphi x)_\omega = \Gamma_\omega x - \Lambda_\omega (K^+)^* x$ for each $x \in U$ and $\omega \in \Omega$; then, $\varphi \in \mathscr{B}(U, \oplus_{\omega\in\Omega} V_w)$. Indeed,

$$\begin{aligned}\|\varphi x\| &= \left\|\{(\varphi x)_\omega\}_{\omega\in\Omega}\right\| = \left\|\{\Gamma_\omega x\}_{\omega\in\Omega} - \left\{\Lambda_\omega (K^+)^* x\right\}_{\omega\in\Omega}\right\| \\ &\le \left\|\{\Gamma_\omega x\}_{\omega\in\Omega}\right\| + \left\|\left\{\Lambda_\omega (K^+)^* x\right\}_{\omega\in\Omega}\right\| \\ &= \left(\int_\Omega \left\|\Gamma_\omega x\right\|^2\right)^{(1/2)} + \left(\int_\Omega \left\|\Lambda_\omega (K^+)^* x\right\|^2\right)^{(1/2)} \\ &\le \sqrt{B_\Gamma}\|x\| + \left\|K^* (K^+)^* x\right\| \\ &\le \left(\sqrt{B_\Gamma} + \|K\|\left\|K^+\right\|\right)\|x\|,\end{aligned} \tag{20}$$

for each $x \in U$, and by Theorem 1, we have

$$\begin{aligned}T_\Lambda \varphi x &= \int_\Omega \Lambda_\omega^* (\varphi x)_\omega d\mu(\omega) = \int_\Omega \Lambda_\omega^* \left(\Gamma_\omega - \Lambda_\omega (K^+)^*\right) x d\mu(\omega) \\ &= \int_\Omega \Lambda_\omega^* \Gamma_\omega x d\mu(\omega) - \int_\Omega \Lambda_\omega^* \Lambda_\omega (K^+)^* x d\mu(\omega) = Kx - Kx = 0.\end{aligned} \tag{21}$$

Conversely, we show that $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ with bound $(\|K^+K\| + \|\varphi\|)^2$.

From Theorem 1, we have

$$\begin{aligned}\int_\Omega \Lambda_\omega^* \Gamma_\omega x d\mu(\omega) &= \int_\Omega \Lambda_\omega^* \left(\Lambda_\omega (K^+)^* + P_\omega \varphi\right) x d\mu(\omega) \\ &= \int_\Omega \Lambda_\omega^* (\Lambda_\omega K^+)^* x d\mu(\omega) + T_\Lambda \varphi x \\ &= \int_\Omega \Lambda_\omega^* (\Lambda_\omega K^+)^* x d\mu(\omega) = Kx, \quad \forall x \in U.\end{aligned} \tag{22}$$

□

Corollary 1. *Let $K \in \mathscr{B}(U)$ have a closed range and $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$, and let $\Gamma_\omega \in \mathscr{B}(U, \otimes_{\omega\in\Omega} V_w), \forall \omega \in \Omega$; then, $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a dual continuous K-g-Bessel sequence of Λ if and only if there exists a continuous g-Bessel sequence $\{\theta_\omega\}_{\omega\in\Omega}$ for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ such that*

$$\begin{aligned}&\Gamma_\omega = \Lambda_\omega (K^+)^* + \theta_\omega, \quad \forall \omega \in \Omega, \\ &\int_\Omega \Lambda_\omega^* \theta_\omega x d\mu(\omega) = 0, \quad \forall x \in U.\end{aligned} \tag{23}$$

Proof. Assume that $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a dual continuous K-g-Bessel sequence of $\{\Lambda_\omega\}_{\omega\in\Omega}$; then, from Theorem 2, there exists $\varphi \in \mathscr{B}(U, \oplus_{\omega\in\Omega} V_w)$ such that

$$\begin{aligned}&\Gamma_\omega = \Lambda_\omega (K^+)^* + P_\omega \varphi, \quad \forall \omega \in \Omega, \\ &\int_\Omega \Lambda_\omega^* (\varphi x)_\omega x = 0, \quad \forall x \in U.\end{aligned} \tag{24}$$

Taking $\theta_\omega = P_\omega \varphi, \forall \omega \in \Omega$, then clearly, $\{\theta_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ with bound $\|\varphi\|^2$, and

$$\int_\Omega \Lambda_\omega^* \theta_\omega x d\mu(\omega) = \int_\Omega \Lambda_\omega^* (\varphi x)_\omega x d\mu(\omega), \quad \forall x \in U. \tag{25}$$

Conversely, let $\{\Gamma_\omega\}_{\omega\in\Omega} = \{\Lambda_\omega (K^+)^* + \theta_\omega\}_{\omega\in\Omega}$, where $\{\theta_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ such that

$$\int_\Omega \Lambda_\omega^* \theta_\omega x d\mu(\omega) = 0. \tag{26}$$

By Theorem 1,

$$\begin{aligned}&\int_\Omega \Lambda_\omega^* \Gamma_\omega x d\mu(\omega) \\ &= \int_\Omega \Lambda_\omega^* \left(\Lambda_\omega (K^+)^* + \theta_\omega\right) x d\mu(\omega) \\ &= \int_\Omega \Lambda_\omega^* \left(\Lambda_\omega (K^+)^*\right) x d\mu(\omega) + \int_\Omega \Lambda_\omega^* \theta_\omega x d\mu(\omega) \\ &= \int_\Omega \Lambda_\omega^* \left(\Lambda_\omega (K^+)^*\right) x d\mu(\omega) = Kx.\end{aligned} \tag{27}$$

□

Theorem 3. *Let $K \in \mathscr{B}(U)$ have a closed range and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a Parseval continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$; then, $\{\Lambda_\omega (K^+)^*\}$ is the canonical dual continuous K-g-Bessel sequence of Λ.*

Proof. By Theorem 1, we know that $\{\Lambda_\omega (K^+)^*\}$ is a dual continuous K-g-Bessel sequence of Λ; to complete this proof, we need to prove by Proposition 3 that $\|T_{\Lambda'}\| \le \|T_\Gamma\|$ for any dual continuous K-g-Bessel sequence $\{\Gamma_\omega\}_{\omega\in\Omega}$ of Λ, where $T_{\Lambda'}$ is the synthesis operator of $\{\Lambda_\omega (K^+)^*\}$. By Theorem 2, there exists $\varphi \in \mathscr{B}(U, \oplus_{\omega\in\Omega} V_w)$ such that $T_\Lambda \varphi = 0$ and $\Gamma_\omega = \Lambda_\omega (K^+)^* + P_\omega \varphi$. A simple computation gives $T_\Gamma^* = T_{\Lambda'}^* + \varphi$, noting $T_{\Lambda'} = K^+ T_\Lambda$.

For each $x \in U$, we have

$$\begin{aligned}\left\|T_\Gamma^* x\right\|^2 &= \left\langle T_\Gamma^* x, T_\Gamma^* x\right\rangle = \left\langle T_{\Lambda'}^* x + \varphi x, T_{\Lambda'}^* x + \varphi x\right\rangle \\ &= \left\|T_{\Lambda'}^* x\right\|^2 + \left\langle T_{\Lambda'}^* x, \varphi x\right\rangle + \left\langle \varphi x, T_{\Lambda'}^* x\right\rangle + \|\varphi x\|^2 \\ &= \left\|T_{\Lambda'}^* x\right\|^2 + \|\varphi x\|^2 \ge \left\|T_{\Lambda'}^* x\right\|^2.\end{aligned} \tag{28}$$

Therefore, $\|T_{\Lambda'} x\| = \|T_{\Lambda'}^* x\| \le \|T_\Gamma^* x\| = \|T_\Gamma x\|$. □

The question of stability plays an important role in various fields of applied mathematics. The classical theorem of the stability of a base is due to Paley and Wiener. It is based on the fact that a bounded operator T on a Banach space is invertible if we have $\|I - T\| < 1$.

Lemma 3 (see [9]; Paley–Wiener). *Let $\{f_i\}_{i\in\mathbb{N}}$ be a basis of a Banach space X and $\{g_i\}_{i\in\mathbb{N}}$ a sequence of vectors in X. If there exists a constant $\lambda \in [0, 1)$ such that*

$$\left\|\sum_{i\in\mathbb{N}} c_i(f_i - g_i)\right\| \le \lambda \left\|\sum_{i\in\mathbb{N}} c_i f_i\right\|, \tag{29}$$

for all finite sequences $\{c_i\}_{i\in\mathbb{N}}$ of scalars, then $\{g_i\}_{i\in\mathbb{N}}$ is also a basis for X.

Proposition 5. *Let $K \in \mathscr{B}(U)$ have a closed range, $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ with bounds A_Λ and B_Λ, and $\{\Gamma_\omega\}_{\omega\in\Omega}$ be a dual continuous K-g-Bessel sequence of Λ with upper bound B_Λ. For each $\omega \in \Omega$, let $\theta_\omega \in B(U, V_\omega)$; if the following conditions are satisfied,*

(1) $\lambda = \int_\Omega \|\theta_\omega - \Lambda_\omega\|^2 d\mu(\omega) < \infty$ and
(2) $\beta = \int_\Omega \|K^+\|\|\Lambda_\omega - \theta_\omega\|\|\Gamma\| d\mu(\omega) < 1$,

then $\Theta = \{\theta_\omega\}_{\omega\in\Omega}$ is a $P_{L(\mathscr{R}(K))}$ continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ with continuous K-g-frame bounds $(B_\Gamma^{-1})\|K^+\|^{-2}(1-\beta^2)$ and $(\sqrt{\lambda} + \sqrt{B_\Lambda})^2$, where L is defined by $L\colon \mathscr{R}(K) \longrightarrow U$ such that $Lx = \int_\omega \theta_\omega^* \Gamma_\omega K^+ x d\mu(\omega)$ and $P_{L(\mathscr{R}(K))}$ denotes the orthogonal projection on $L(\mathscr{R}(K))$.

Proof. Define

$$\psi\colon \bigoplus_{\omega\in\Omega} V_w \longrightarrow U, \psi(\{y_\omega\}_{\omega\in\Omega}) = \int_\omega \theta_\omega^* y_\omega \mathrm{d}\mu(\omega). \tag{30}$$

Then, (1) implies that ψ is well defined and bounded with $\|\psi\| \le \sqrt{\lambda} + \sqrt{B_\Lambda}$.

Clearly, ψ is linear.

Thus,

$$\int_\omega \theta_\omega^* y_\omega \mathrm{d}\mu(\omega) = \|\psi^*(x)\|^2 \le \|\psi\|^2\|x\|^2 \le \left(\sqrt{\lambda} + \sqrt{B_\Lambda}\right)^2\|x\|^2. \tag{31}$$

Therefore, $\{\theta_\omega\}_{\omega\in\Omega}$ is a continuous K-g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.

For any $x \in \mathscr{R}(K)$, we have

$$\begin{aligned}\|x - Lx\| &= \left\|\int_\omega \Lambda_\omega^* \Gamma_\omega K^+ x \mathrm{d}\mu(\omega) - \int_\omega \theta_\omega^* \Gamma_\omega K^+ x \mathrm{d}\mu(\omega)\right. \\ &= \left\|\int_\omega (\Lambda_\omega^* - \theta_\omega^*)\Gamma_\omega K^+ x \mathrm{d}\mu(\omega)\right. \\ &\le \left\|\int_\omega \|K^+\|\|\Lambda_\omega - \theta_\omega\|\|\Gamma_\omega\|\|x\| \mathrm{d}\mu(\omega) = \beta\|x\|.\end{aligned} \tag{32}$$

Hence,

$$(1-\beta)\|x\| \le \|Lx\|, \quad \forall x \in \mathscr{R}(K). \tag{33}$$

From this, we conclude that the operator $L\colon \mathscr{R}(K) \longrightarrow L(\mathscr{R}(K))$ is invertible with $\|L^{-1}\| \le (1/1-\beta)$. It is trivial to show that $L(\mathscr{R}(K))$ is closed.

For any $x \in U$, we have

$$P_{L(\mathscr{R}(K))}Kx = LL^{-1}P_{L(\mathscr{R}(K))}Kx = \int_\Omega \theta_\omega^* \Gamma_\omega K^+ L^{-1} P_{L(\mathscr{R}(K))} Kx \mathrm{d}\mu(\omega). \tag{34}$$

Therefore,

$$\begin{aligned}\left\|K^* P_{L(\mathscr{R}(K))}x\right\|^2 &= \sup_{\|y\|=1}\left|\left\langle K^* P_{L(\mathscr{R}(K))}x, y\right\rangle\right|^2 \\ &= \sup_{\|y\|=1}\left|\int_\omega \left\langle \theta_\omega^* \Gamma_\omega K^+ L^{-1} P_{L(\mathscr{R}(K))}Kx, y\right\rangle\right|^2 \\ &= \sup_{\|y\|=1}\left|\int_\Omega \left\langle \Gamma_\omega K^+ L^{-1} P_{L(\mathscr{R}(K))}Kx, \theta_\omega y\right\rangle\right|^2 \\ &\le \int_\Omega \left\|\theta_\omega x\right\|^2 \mathrm{d}\mu(\omega) \sup_{\|y\|=1}\int_\Omega \left\|\Gamma_\omega K^+ L^{-1} P_{L(\mathscr{R}(K))}Ky\right\|^2 \mathrm{d}\mu(\omega)n \\ &\le B_\Gamma \left\|K^+\right\|^2 \left\|L^{-1}\right\| \|K\|^2 \int_\Omega \left\|\theta_\omega x\right\|^2 \mathrm{d}\mu(\omega)n \\ &\le B_\Gamma \left\|K^+\right\|^2 \|K\|^2 \frac{1}{(1-\beta)^2}\left\|\int_\Omega \left\|\theta_\omega x\right\|^2 \mathrm{d}\mu(\omega)\right.\end{aligned} \tag{35}$$

It follows that

$$B_\Gamma^{-1}\left\|K^+\right\|^{-2}\|K\|^{-2}\left\|K^* P_{L(\mathscr{R}(K))}x\right\|^2 \le \int_\Omega \left\|\theta_\omega x\right\|^2 \mathrm{d}\mu(\omega). \tag{36}$$

This completes the proof. □

3. Continuous *K*-g-Frame Sequences and Continuous g-Frame Sequences

Theorem 4. *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$; then, Λ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ if and only if $\mathscr{R}(K) \subset \overline{span}(\{\{\Lambda_\omega^*(V_\omega)\}\}_{\omega\in\Omega})$.*

Proof. Suppose that Λ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$; then, $\forall x \in U$, we have

$$\begin{aligned} C_\Lambda\langle KK^*x, x\rangle &= C_\Lambda\|K^*x\|^2 \\ &\le \int_\Omega \|\Lambda_\omega x\| d\mu(\omega) = \|T_\Lambda^* x\|^2 \\ &= \langle T_\Lambda T_\Lambda^* x, x\rangle. \end{aligned} \tag{37}$$

From Lemma 2, we have $\mathscr{R}(K) \subset \mathscr{R}(T_\Lambda)$ since $\mathscr{R}(T_\Lambda) \subseteq \overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$. Conversely, let $\mathscr{R}(K) \subset \overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$.

Since $U = \overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) \oplus (\overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^\perp$, we have

$$\begin{aligned} &\forall x \in U, \\ &x = y + z; \\ &y \in \overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}), \\ &z \in (\overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^\perp. \end{aligned} \tag{38}$$

For any $\omega \in \Omega, \Lambda_\omega^*(\Lambda_\omega z) \in \overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$.

For all $x \in U$, $\|\Lambda_\omega x\|^2 = \|\Lambda_\omega y\|^2$, so

$$\int_\Omega \|\Lambda_\omega x\|^2 d\mu(\omega) = \int_\Omega \|\Lambda_\omega y\|^2 d\mu(\omega) \le B_\Lambda\left(\|y\|^2 + \|z\|^2\right) = B_\Lambda\|x\|^2. \tag{39}$$

On the contrary, since $\mathscr{R}(K) \subset \overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$, $z \in (\mathscr{R}(K))^\perp$; then, $\langle K^*z, h\rangle = \langle z, Kh\rangle = 0, h \in U$; hence, $K^*z = 0$.

Then,

$$\int_\Omega \|\Lambda_\omega x\|^2 d\mu(\omega) = \int_\Omega \|\Lambda_\omega y\|^2 d\mu(\omega) \ge C_\Lambda\|K^*y\|^2 = C_\Lambda\|K^*x\|^2. \tag{40}$$

□

Proposition 6. *Let $K \in \mathscr{B}(U)$ have a closed range and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$; then, $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for $\mathscr{R}(K)$ with respect to $\{V_\omega\}_{\omega\in\Omega}$.*

Proof. We just prove the lower continuous g-frame inequality. From Proposition 2, there exists a continuous g-Bessel sequence $\Gamma = \{\Gamma_\omega\}_{\omega\in\Omega}$ for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ such that $Kx = \int_\Omega \Lambda_\omega^*\Gamma_\omega x, \forall x \in U$.

For every $y \in \mathscr{R}(K)$, by Lemma 1,

$$y = KK^+y = \int_\Omega \Lambda_\omega^*\Gamma_\omega K^+ y d\mu(\omega) = \int_\Omega \Lambda_\omega^*\left(\Gamma_\omega K^+ P_{\mathscr{R}(K)}\right) y d\mu(\omega). \tag{41}$$

Take $\theta_\omega = \Gamma_\omega K^+ P_{\mathscr{R}(K)}$ for every $\omega \in \Omega$.

Then, we have

$$\int_\Omega \|\theta_\omega x\|^2 d\mu(\omega) = \int_\Omega \|\Gamma_\omega K^+ P_{\mathscr{R}(K)}\|^2 d\mu(\omega) \le B_\Gamma\|K^+\|^2\|x\|^2. \tag{42}$$

So, $\{\theta_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.

For any $y, z \in \mathscr{R}(K)$, we have

$$\begin{aligned} \langle y, z\rangle &= \left\langle \int_\Omega \Lambda_\omega^*\theta_\omega y d\mu(\omega), z\right\rangle \\ &= \int_\Omega \langle \Lambda_\omega^*\theta_\omega y, z\rangle d\mu(\omega) \\ &= \int_\Omega \left\langle \Gamma_\omega K^+ P_{\mathscr{R}(K)} y, \Lambda_w z\right\rangle d\mu(\omega) \\ &= \int_\Omega \left\langle y, \left(\Gamma_\omega K^+ P_{\mathscr{R}(K)}\right)^*, \Lambda_\omega z\right\rangle d\mu(\omega) \\ &= \left\langle y, \int_\Omega \left(\Gamma_\omega K^+ P_{\mathscr{R}(K)}\right)^* \Lambda_\omega z\right\rangle d\mu(\omega) \\ &= \left\langle y, \int_\Omega \theta_\omega \Lambda_\omega z d\mu(\omega)\right\rangle, \end{aligned} \tag{43}$$

and then $z = \int_\Omega \theta_\omega \Lambda_\omega z d\mu(\omega)$.

Hence,

$$\begin{aligned} \|z\|^2 &= \sup_{\|y\|=1} |\langle z, y\rangle|^2 \\ &= \sup_{\|y\|=1} \left|\left\langle \int_\Omega \theta_\omega \Lambda_\omega z d\mu(\omega), y\right\rangle\right|^2 \\ &\le \left(\sup_{\|y\|=1} \int_\Omega \|\theta_\omega y\|^2 d\mu(\omega)\right)\left(\int_\Omega \|\Lambda_\omega z\|^2 d\mu(\omega)\right) \\ &\le B_\Gamma\|K^+\|^2\left(\int_\Omega \|\Lambda_\omega z\|^2 d\mu(\omega)\right). \end{aligned} \tag{44}$$

Then, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for $\mathscr{R}(K)$ with respect to $\{V_\omega\}_{\omega\in\Omega}$. □

Proposition 7. *Let $K \in \mathscr{B}(U)$ have a closed range and $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous K-g-frame for $\mathscr{R}(K)$ with respect to $\{V_\omega\}_{\omega\in\Omega}$. If $\overline{span}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) \subset \mathscr{R}(K)$, then $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.*

Proof. For each $x \in U$, we have $x = y + z$ such that $y \in \mathscr{R}(K)$ and $z \in (\mathscr{R}(K))^\perp$. So, $\|\Lambda_\omega x\|^2 = \|\Lambda_\omega y\|^2$.

Since $\overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) \subset \mathscr{R}(K)$ and $\Lambda_\omega^*(\Lambda_\omega e) \in \overline{\text{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}), \forall e \in U$,

$$\int_\Omega \|\Lambda x\|^2 d\mu(\omega) = \int_\Omega \|\Lambda y\|^2 d\mu(\omega) \le B_\Lambda\|x\|^2 \le B_\Lambda\left(\|y\|^2 + \|z\|^2\right) = B_\Lambda\|x\|^2. \tag{45}$$

Let $\Phi = \{\phi_\omega\}_{\omega\in\Omega}$ be a dual continuous g-frame of $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$.

So,

$$\int_\Omega \Lambda_\omega^* \phi_\omega y \mathrm{d}\mu(\omega) = \int_\Omega \phi_\omega^* \Lambda_\omega y \mathrm{d}\mu(\omega). \tag{46}$$

Hence,

$$\begin{aligned}
\|K^* x\|^2 &= \|K^*(y+z)\| = \|K^* y\|^2 = |\langle K^* y, K^* y\rangle| \\
&= |\langle KK^* y, y\rangle| = \left|\left\langle \int_\Omega \phi_\omega^* \Lambda_\omega y \mathrm{d}\mu(\omega), KK^* y\right\rangle\right| \\
&= \left|\int_\Omega \langle \Lambda_\omega y, \phi_\omega KK^* y\rangle \mathrm{d}\mu(\omega)\right| \\
&\le \left(\int_\Omega \|\Lambda_\omega y\|^2 \mathrm{d}\mu(\omega)\right)^{(1/2)} \left(\int_\Omega \|\phi_\omega KK^* y\|^2 \mathrm{d}\mu(\omega)\right)^{(1/2)} \\
&\le \sqrt{B_\phi}\|KK^* y\| \left(\int_\Omega \|\Lambda_\omega y\|^2\right)^{(1/2)} \\
&\le \sqrt{B_\phi}\|K\|\|K^* x\| \left(\int_\Omega \|\Lambda_\omega y\|^2 \mathrm{d}\mu(\omega)\right)^{(1/2)}.
\end{aligned} \tag{47}$$

Then, $B_\phi^{-1}\|K\|^{-2}\|K^* x\|^2 \le \int_\Omega \|\Lambda_\omega y\|^2$.

On the contrary, we have $\int_\Omega \|\Lambda_\omega y\|^2 = \int_\Omega \|\Lambda_\omega x\|^2$.

Then, $B_\phi^{-1}\|K\|^{-2}\|K^* x\|^2 \le \int_\Omega \|\Lambda_\omega x\|^2$.

So, Λ is a continuous K-g-frame for U. □

Theorem 5. *For every $\omega \in \Omega$, let $\Lambda_\omega \in \mathscr{B}(U, V_w)$; then, the following statements are equivalent:*

(1) $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g-frame sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.

(2) $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for $\overline{span}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$, and there exists a continuous g-Bessel sequence $\{\Gamma_\omega\}_{\omega\in\Omega}$ for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ such that

$$\left(\frac{K^*}{\overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* x = \int_\Omega \Lambda_\omega^* \Gamma_\omega x, \quad \forall x \in U. \tag{48}$$

(3) $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for $\overline{span}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$, and there exists a continuous g-Bessel sequence $\{\Gamma_\omega\}_{\omega\in\Omega}$ for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ such that

$$K^* x = \int_\Omega \Gamma_\omega^* \Lambda_\omega x \mathrm{d}\mu(\omega), \quad \forall x \in \overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}). \tag{49}$$

Proof. (1) $\Longrightarrow$ (2):

We have

$$\frac{K^*}{\overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})} : \overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) \longrightarrow U \tag{50}$$

then their adjoint

$$\left(\frac{K^*}{\overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* : U \longrightarrow \overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}). \tag{51}$$

By the definition on the continuous K-g-frame sequence, there exists $C_\Lambda > 0$ such that

$$C_\Lambda \|K^* x\|^2 \le \|T_\Lambda^* x\|^2, \quad \forall x \in \overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}). \tag{52}$$

This implies

$$C_\Lambda \left(\frac{K^*}{\overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* \left(\frac{K^*}{\overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right) \le T_\Lambda T_\Lambda^*. \tag{53}$$

From Lemma 2, there exists $R \in \mathscr{B}(U, \oplus_{\omega\in\Omega} V_w)$ such that

$$\left(\frac{K^*}{\overline{span}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* = T_\Lambda R. \tag{54}$$

For every $\omega \in \Omega$, denote $\Gamma_\omega = P_\omega R$; then, we have

$$\begin{aligned}
\int_\Omega \|\Gamma_\omega x\|^2 \mathrm{d}\mu(\omega) &= \int_\Omega \|P_\omega R x\|^2 \mathrm{d}\mu(\omega) \\
&= \int_\Omega \|(Rx)_\omega\|^2 \mathrm{d}\mu(\omega) \\
&= \|Rx\|^2 \\
&\le \|R\|^2 \|x\|^2.
\end{aligned} \tag{55}$$

Hence, $\{\Gamma_\omega\}_{\omega\in\Omega}$ is a continuous g-Bessel sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$.

On the contrary, we have for all $x \in U$,

$$\begin{aligned}
\left(\frac{K^*}{\overline{\mathrm{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* x &= T_\Lambda R x \\
&= \int_\Omega \Lambda_\omega^* (Rx)_\omega \mathrm{d}\mu(\omega) \\
&= \int_\Omega \Lambda_\omega^* (P_\omega R x) \mathrm{d}\mu(\omega) \\
&= \int_\Omega \Lambda_\omega^* \Gamma_\omega x \mathrm{d}\mu(\omega).
\end{aligned} \tag{56}$$

(2) $\Longrightarrow$ (3):

For every $x \in U$ and $y \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$, by (48), we have

$$\left\langle \left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* x, y\right\rangle = \int_\Omega \langle \Lambda_\omega^*\Gamma_\omega x, y\rangle \mathrm{d}\mu(\omega) = \left\langle x, \int_\Omega \Gamma_\omega^*\Lambda_\omega y \mathrm{d}\mu(\omega)\right\rangle. \tag{57}$$

So,

$$\langle x, K^* y\rangle = \left\langle x, \frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})y}\right\rangle = \left\langle x, \int_\Omega \Gamma_\omega^*\Lambda_\omega y \mathrm{d}\mu(\omega)\right\rangle. \tag{58}$$

Hence, $K^* y = \int_\Omega \Gamma_\omega^*\Lambda_\omega y \mathrm{d}\mu(\omega)$.

(3) $\Longrightarrow$ (1):

Suppose that (49) holds; to prove (1), we just prove the lower bound inequality on continuous K-g-frames.

For every $y \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$, we have

$$\begin{aligned}\|K^* y\|^2 &= \sup_{\|z\|=1} |\langle K^* y, z\rangle|^2 = \sup_{\|z\|=1} \left|\left\langle \int_\Omega \Gamma_\omega^*\Lambda_\omega y \mathrm{d}\mu(\omega), z\right\rangle\right|^2 \\ &= \sup_{\|z\|=1} \left|\left\langle \int_\Omega \Lambda_\omega y \mathrm{d}\mu(\omega), \Gamma_\omega z\right\rangle\right|^2 \\ &\le \int_\Omega \|\Lambda_\omega y\|^2 \mathrm{d}\mu(\omega) \sup_{\|z\|=1} \int_\Omega \|\Gamma_\omega z\|^2 \mathrm{d}\mu(\omega) \\ &\le B_\Gamma \int_\Omega \|\Lambda_\omega y\|^2 \mathrm{d}\mu(\omega).\end{aligned} \tag{59}$$

So, $B_\Lambda^{-1}\|K^* y\|^2 \le \int_\Omega \|\Lambda_\omega y\|^2 \mathrm{d}\mu(\omega), \forall y \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$. □

Theorem 6. *A sequence $\{\Lambda_\omega\}_{\omega\in\Omega} \in \mathscr{B}(U, \oplus_{\omega\in\Omega} V_w)$ is a continuous K-g-frame sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ if and only if there exists $T \in \mathscr{B}(\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}), \oplus_{\omega\in\Omega} V_w)$ such that*

$$\frac{\Lambda_\omega}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})} = P_\omega T, \quad \forall \omega \in \Omega,$$

$$\mathscr{R}\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right) \subset \mathscr{R}(T^*). \tag{60}$$

Proof. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous K-g-frame sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$, and let $T = T_\Lambda^*$.

Hence, $\Lambda_\omega/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})x = P_\omega(T_\Lambda^* x) = P_\omega(Tx)$ for every $x \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ and any $\omega \in \Omega$. So, $\Lambda_\omega/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) = P_\omega T$.

From (53), we have

$$C_\Lambda\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* \left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right) \le T^* T. \tag{61}$$

By Lemma 2, it follows that $\mathscr{R}((K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^*) \subset \mathscr{R}(T^*)$.

Conversely, $\Lambda_\omega/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) = P_\omega T$ for any $\omega \in \Omega$.

Let $x \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$; we have

$$\begin{aligned}\int_\Omega \|\Lambda_\omega x\|^2 \mathrm{d}\mu(\omega) &= \int_\Omega \left\|\frac{\Lambda_\omega}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})x}\right\|^2 \mathrm{d}\mu(\omega) \\ &= \int_\Omega \|P_\omega T\|^2 \mathrm{d}\mu(\omega) \\ &= \|Tx\|^2 \\ &\le \|T\|^2 \|x\|^2.\end{aligned} \tag{62}$$

So, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g-frame sequence for $\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$.

Since $\mathscr{R}((K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^*) \subset \mathscr{R}(T^*)$, by Lemma 2, there exists $\lambda > 0$ such that

$$\left(\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^*\right) \frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})} \le \lambda T^* T. \tag{63}$$

Hence, for every $x \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$, we have

$$\lambda^{-1}\|K^* x\|^2 = \lambda^{-1}\left\|\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)x\right\|^2 \le \|Tx\|^2 = \int_\Omega \|\Lambda x\|^2 \mathrm{d}\mu\omega, \tag{64}$$

and then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous K-g-frame sequence for $\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$ with bounds λ^{-1} and $\|T\|^2$. □

In general, a continuous K-g-frame $\{\Lambda_\omega\}_{\omega\in\Omega}$ for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ is not a continuous g-frame for $\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$.

Proposition 8. *Let $\Lambda = \{\Lambda_\omega\}_{\omega\in\Omega}$ be a continuous K-g-frame sequence for U with respect to $\{V_\omega\}_{\omega\in\Omega}$. Suppose that $K^*/\overline{span}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) \neq 0$ and that it has a closed range; then, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for $\mathscr{R}(K^*/\overline{span}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^*$ with respect to $\{V_\omega\}_{\omega\in\Omega}$.*

Proof. Since $K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ has a closed range, its pseudo-inverse exists; from Lemma 1, every $x \in \mathscr{R}(K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^*$ can be written as

$$x = \left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^+ \left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)x. \tag{65}$$

Hence,

$$\|x\|^2 \leq \left\|\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^+\right\|^2 \left\|\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})x}\right\|^2. \tag{66}$$

Noting that $\mathscr{R}(K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^* \subset \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$, we have

$$\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)x = K^*x. \tag{67}$$

So,

$$\begin{aligned}\|x\|^2 &\leq \left\|\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^+\right\|^2 \|K^*x\|^2 \\ &\leq \frac{1}{A_\Lambda}\left\|\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^+\right\|^2 \int_\Omega \|\Lambda_\omega x\|^2 d\mu(\omega).\end{aligned} \tag{68}$$

Since $K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}) \neq 0$, its pseudo-inverse

$$\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^+ \neq 0, \tag{69}$$

and then

$$A_\Lambda\left\|\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^+\right\|^{-2} \|x\|^2 \leq \int_\Omega \|\Lambda_\omega x\|^2 d\mu(\omega). \tag{70}$$

On the contrary, we have

$$\int_\Omega \|\Lambda_\omega x\|^2 d\mu(\omega) \leq B_\Lambda \|x\|^2, \quad \forall x \in \mathscr{R}\left(\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^*\right). \tag{71}$$

So, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for $\mathscr{R}((K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^*)$ with respect to $\{V_\omega\}_{\omega\in\Omega}$ with bounds $C_\Lambda\|(K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^+\|^{-2}$ and B_Λ. □

In the following, we give a necessary and sufficient condition under which a tight continuous K-g-frame sequence $\{\Lambda_\omega\}_{\omega\in\Omega}$ for U with respect to $\{V_\omega\}_{\omega\in\Omega}$ is a tight continuous g-frame for $\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$.

Proof. Theorem 7Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a C-tight continuous g-frame for U with respect to $\{V_\omega\}_{\omega\in\Omega}$; then, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a D-tight continuous g-frame for $\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$ if and only if $K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ admits a left inverse $C/D(K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^*$. □

Proof. First, we suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a D-tight continuous g-frame for $\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$ with respect to $\{V_\omega\}_{\omega\in\Omega}$.

Hence,

$$\begin{aligned} D\|x\|^2 &= \int_\Omega \|\Lambda_\omega x\|^2 d\mu(\omega), \quad K4\forall x \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}), \\ C\|K^*x\|^2 &= \int_\Omega \|\Lambda_\omega x\|^2 d\mu(\omega), \quad \forall x \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}). \end{aligned} \tag{72}$$

So, $D\|x\|^2 = C\|K^*x\|^2$ for every $x \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$. Then,

$$\begin{aligned} D\langle x, x\rangle &= C\langle K^*x, K^*x\rangle \\ &= C\left\langle \left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* \left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)x, x\right\rangle. \end{aligned} \tag{73}$$

This implies that

$$\frac{C}{D}\left(\frac{K}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}^*\right)^* \left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right) = Id_{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}. \tag{74}$$

Hence, $(C/D)(K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega}))^*$ is a left inverse of $K^*/\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$; conversely, we have

$$x = \frac{C}{D}\left(\frac{K^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^*. \tag{75}$$

So,

$$\begin{aligned}\int_{\Omega}\|\Lambda_{\omega}x\|^{2}\mathrm{d}\mu(\omega) &= C\|K^{*}x\|^{2} = D\frac{C}{D}\|K^{*}x\|^{2}\\ &= D\frac{C}{D}\left\langle\left(\frac{K^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)x,\left(\frac{K^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)x\right\rangle\\ &= D\|x\|^{2}.\end{aligned}\tag{76}$$

It follows that $\Lambda=\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a D-tight continuous g-frame for $\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})$ with respect to $\{V_{\omega}\}_{\omega\in\Omega}$. □

Theorem 7. *Let $K_1,K_2\in\mathscr{B}(U)$ and $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a C-tight continuous K_1-g-frame sequence for U with respect to $\{V_{\omega}\}_{\omega\in\Omega}$; then, $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a C-tight continuous K_2-g-frame sequence for U with respect to $\{V_{\omega}\}_{\omega\in\Omega}$ if and only if*

$$\mathscr{R}\left(\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)^{*}\right)\subset\mathscr{R}\left(\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)^{*}\right).\tag{77}$$

Proof. First, we suppose that $\Lambda=\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a C-tight continuous K_2-g-frame sequence for U with respect to $\{V_{\omega}\}_{\omega\in\Omega}$; then, there exists $D>0$ such that

$$D\|K_2^{*}x\|^{2}\le\int_{\Omega}\|\Lambda_{\omega}x\|^{2}\mathrm{d}\mu(\omega),\quad\forall x\in\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega}).\tag{78}$$

We have $\Lambda=\{\Lambda_{\omega}\}_{\omega\in\Omega}$ is a C-tight continuous K_1-g-frame sequence for U with respect to $\{V_{\omega}\}_{\omega\in\Omega}$; then, $\|K_2^{*}x\|^{2}\le(C/D)\|K_1^{*}x\|^{2}$ for every $x\in\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})$.

Hence,

$$\begin{aligned}\left\langle\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)x,\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)x\right\rangle &= \langle K_2^{*}x,K_2^{*}x\rangle\\ &\le\frac{C}{D}\langle K_1^{*}x,K_1^{*}x\rangle\\ &= \frac{C}{D}\left\langle\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)x,\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)x\right\rangle.\end{aligned}\tag{79}$$

Therefore,

$$\begin{aligned}&\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)^{*}\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)\\ &\le\frac{C}{D}\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right).\end{aligned}\tag{80}$$

By Lemma 2, we have

$$\mathscr{R}\left(\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)^{*}\right)\subset\mathscr{R}\left(\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)^{*}\right).\tag{81}$$

Conversely, suppose $\mathscr{R}((K_2^{*}/\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega}))^{*})\subset\mathscr{R}((K_1^{*}/\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})^{*})$.

By Lemma 2, there exists $\lambda>0$ such that

$$\begin{aligned}&\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)^{*}\left(\frac{K_2^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)\\ &\le\lambda\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right)\left(\frac{K_1^{*}}{\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})}\right),\end{aligned}\tag{82}$$

so for every $x\in\overline{\operatorname{span}}(\{\Lambda_{\omega}^{*}(V_{\omega})\}_{\omega\in\Omega})$, we have

$$\begin{aligned}\|K_2^*x\|^2 &= \left\langle \left(\frac{K_2^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)^* \left(\frac{K_2^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)x, x\right\rangle \\ &\le \lambda\left(\frac{K_1^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right)\left(\frac{K_1^*}{\overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})}\right) \\ &= \lambda\|K_1^*x\|^2 = \frac{\lambda}{c}\int_\Omega \|\Lambda_\omega x\|^2 \mathrm{d}\mu(\omega).\end{aligned} \tag{83}$$

Then, for all $x \in \overline{\operatorname{span}}(\{\Lambda_\omega^*(V_\omega)\}_{\omega\in\Omega})$,

$$\frac{c}{\lambda}\|K_2^*x\|^2 \le \int_\Omega \|\Lambda_\omega x\|^2 \mathrm{d}\mu(\omega) \le C\|K_1^*\|^2\|x\|^2. \tag{84}$$

This completes the proof. □

4. Conclusion

In this article, we have proved some properties of dual continuous K-g-frames in Hilbert spaces. These results are extensions of the related results announced in [8]. The presented theorems extend, generalize, and improve many existing results in the literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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