Research Article

Some Properties of Curvature Tensors and Foliations of Locally Conformal Almost Kähler Manifolds

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We investigate a class of locally conformal almost Kähler structures and prove that, under some conditions, this class is a subclass of almost Kähler structures. We show that a locally conformal almost Kähler manifold admits a canonical foliation whose leaves are hypersurfaces with the mean curvature vector field proportional to the Lee vector field. The geodesibility of the leaves is also characterized, and their minimality coincides with the incompressibility of the Lee vector field along the leaves.

1. Introduction

The study of manifolds whose metric is locally conformal to an almost Kähler metric is considered as one of the most interesting studies in the field of differential geometry (see [1] for details and references therein). This is because of its richness in the theory that is applicable in physics, algebraic geometry, symplectic geometry, etc. To our knowledge, locally conformal (almost) Kähler structures were first studied by Libermann [2] in the 1950s. In 1966, Gray [3] also contributed to the study by considering (almost) Hermitian manifolds whose metric is conformal to a local (almost) Kähler metric. However, globally conformal (almost) Kähler manifolds share the same topological properties with locally conformal (almost) Kähler manifolds [4]. It is therefore provocative to consider those almost Hermitian manifolds whose metric is locally conformal to an almost Kähler metric. The difference between locally conformal Kähler manifolds and locally conformal almost Kähler manifolds is the condition of integrability of an almost complex structure. This is equivalent to an almost complex structure being parallel with respect to a globally defined connection or the vanishing of a Nijenhuis tensor. Therefore, the geometric

properties which do not depend on the almost complex structure will apply to both of these manifolds.

Libermann defined a locally conformal (almost) Kähler metric as a metric g at which in the neighborhood of each point of an almost Hermitian manifold, it is locally conformal to an (almost) Kähler metric.

In this paper, we investigate some properties of curvature tensors and foliations of locally conformal almost Kähler manifolds. For the foliations, we pay attention to the ones that arise naturally when the Lee form is nowhere vanishing. The paper is organized as follows. In Section 2, we recall the definition of locally conformal almost Kähler structures supported by an example. In Section 3, we deal with curvature tensors. The latter generalizes those found by Olszak in [5]. Under some conditions, we prove that a class of locally conformal almost Kähler structures is a subclass of almost Kähler structures. Section 4 is devoted to the canonical foliations that arise for the nonvanishing Lee form. We prove that these foliations are Riemannian if and only if the Lee vector field is autoparallel. We also prove that the locally conformal almost Kähler manifolds contain leaves with mean curvature vector field proportional to the Lee vector field, and their geodesibility coincides with the killing

condition of the Lee vector field. The latter is incompressible along the leaves if and only if the leaves are minimal.

2. Locally Conformal Almost Kähler Metrics

Let M be a $2n$-dimensional almost Hermitian manifold with the metric g and the almost complex structure J satisfying

$$
\begin{aligned}
J^2 &= -\mathbb{I}, \\
g(JX, JY) &= g(X, Y),
\end{aligned}
\tag{1}
$$

for any vector fields X and Y tangent to M, where $\mathbb{I}$ stands for the identity transformation of tangent bundle TM. Then, for any vector fields X and Y, the tensor

$$
\Omega(X, Y) = g(X, JY)
\tag{2}
$$

defines the fundamental 2-form of M which is nondegenerate and gives an almost symplectic structure on M. If Ω is closed, i.e., $d\Omega = 0$, then (M, J, g) is called an almost Kähler manifold [6].

Now, let (M, J, g) be a $2n$-dimensional almost Hermitian manifold. Such a manifold is said to be a locally conformal almost Kähler manifold [4] if there is an open covering $\{U_t\}_{t \in I}$ of M and a family $\{f_t\}_{t \in I}$ of $\mathbb{C}^\infty$-functions $f_t: U_t \longrightarrow \mathbb{R}$ such that, for any $t \in I$, the metric form

$$
g_t = \exp(-f_t)g_{|_{U_t}}
\tag{3}
$$

is an almost Kähler metric.

If the structures (J, g_t) defined in (3) are Kähler, then (M, J, g) is called locally conformal Kähler. Moreover, a locally conformal almost Kähler manifold M is almost Kähler if and only if $df_t = 0$.

The Lee form is important because it characterizes locally conformal almost Kähler manifolds. Locally conformal almost Kähler manifolds were characterized by Vaisman in [4]. This is stated as follows: an almost Hermitian manifold (M, J, g) is a locally conformal almost Kähler manifold if and only if there exists 1-form ω such that

$$
d\Omega = \omega \wedge \Omega, \quad d\omega = 0.
\tag{4}
$$

Example 1. We consider the 4-dimensional manifold $M^4 = \{p \in \mathbb{R}^4 | x_1 \neq 0, x_2 > 0\}$, where $p = (x_1, x_2, y_1, y_2)$ are the standard coordinates in $\mathbb{R}^4$. The vector fields,

$$
\begin{aligned}
X_i &= x_2 \frac{\partial}{\partial x_i}, \\
Y_i &= \frac{1}{x_2^3} \frac{\partial}{\partial y_i}, \quad i = 1, 2,
\end{aligned}
\tag{5}
$$

are linearly independent at each point of M. Let g be the Riemannian metric on M defined by $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol, $g(X_i, Y_j) = 0$. That is, the form of the metric becomes

$$
g = \frac{1}{x_2^2}(dx_1^2 + dx_2^2) + x_2^6(dy_1^2 + dy_2^2).
\tag{6}
$$

Let J be the $(1,1)$-tensor field defined by $JX_1 = Y_1$, $JX_2 = -Y_2$, $JY_2 = X_2$, and $JY_1 = -X_1$. Thus, (J, g) defines an almost Hermitian structure on M^4. The nonzero component of fundamental 2-form J is

$$
\Omega\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right) = -\frac{1}{x_2^2} \text{ and } \Omega\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right) = \frac{1}{x_2^2},
\tag{7}
$$

and we have

$$
\Omega = \frac{1}{x_2^2}\{-dx_1 \wedge dy_1 + dx_2 \wedge dy_2\}.
\tag{8}
$$

Its differential gives

$$
d\Omega = \frac{2}{x_2^3} dx_1 \wedge dy_1 \wedge dx_2.
\tag{9}
$$

By letting

$$
\omega = -\frac{1}{x_2} dx_2,
\tag{10}
$$

we have

$$
d\Omega = 2\omega \wedge \Omega.
\tag{11}
$$

It is easy to see that $d\omega = 0$, and the dual vector field B is given by

$$
B = -\frac{1}{x_2^2} X_2.
\tag{12}
$$

Let us consider the open neighborhood U of M given by $U = \{p \in M^4 | x_2 > 0\}$, and there exists a differentiable function f on U such that $\omega = df$, where $f = -\ln(x_2)$. By the aforementioned characterization given in (4), (M^4, J, g) is a locally conformal almost Kähler manifold.

Next, we wish to study the relationship of the Levi-Civita connections induced by the locally conformal Kähler metrics g_t and g.

Throughout this paper, $\Gamma(\Xi)$ will denote the $\mathcal{F}(M)$-module of differentiable sections of a vector bundle Ξ.

Let ∇ and ∇^t be the Levi-Civita connections associated with the metrics g and g_t, respectively. As is well known, they are connected by

$$
\nabla_X^t Y = \nabla_X Y - \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X, Y)B\},
\tag{13}
$$

for all $X, Y \in \Gamma(TM)$.

3. Curvature Relations of Locally Conformal Almost Kähler Metrics

Let (M, J, g) be a $2n$-dimensional almost Hermitian manifold. Here, we keep the formalism of local transformations and other formulas defined in the previous section. For the

Riemann curvature R of a metric g, we use the following convention:

$$R(X, Y, Z, W) = g(R(X, Y)Z, W), \qquad (14)$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad (15)$$

for any vector field X, Y, and Z on M.

Let $\{E_i\}_{1 \le i \le 2n}$ be the orthonormal basis with respect to g. The Ricci curvature tensor ρ and the scalar curvature τ are given by

$$\rho(X, Y) = \sum_{i=1}^{2n} R(E_i, X, Y, E_i),$$

$$\tau = \sum_{i=1}^{2n} \rho(E_i, E_i). \qquad (16)$$

Now, we consider the Ricci $*$-curvature tensor ρ^* and the scalar $*$-curvature τ^* defined by

$$\rho^*(X, Y) = \sum_{i=1}^{2n} R(E_i, X, JY, JE_i), \tau^* = \sum_{i=1}^{2n} \rho^*(E_i, E_i). \qquad (17)$$

Similarly, the curvatures corresponding to the metric g_t will be denoted by R^t, ρ^t, τ^t, ρ^{t*}, and τ^{t*}, respectively.

Lemma 1. *Let (M, J, g) be a locally conformal almost Kähler manifold. Then, the curvature tensors R^t and R with respect to the metrics g_t and g, respectively, are related as*

$$R^t(X, Y)Z = R(X, Y)Z + \frac{1}{2}\left\{(\nabla_Y \omega)Z + \frac{1}{2}\omega(Y)\omega(Z)\right\}X$$
$$- \frac{1}{2}\left\{(\nabla_X \omega)Z + \frac{1}{2}\omega(X)\omega(Z)\right\}Y$$
$$+ \frac{1}{2}g(Y, Z)\left\{\nabla_X B + \frac{1}{2}\omega(X)B\right\}$$
$$- \frac{1}{2}g(X, Z)\left\{\nabla_Y B + \frac{1}{2}\omega(Y)B\right\}$$
$$- \frac{\|B\|^2}{4}\{g(Y, Z)X - g(X, Z)Y\}, \qquad (18)$$

where $\|B\|^2 = g(B, B)$.

Proof. Using the convention in (15) for the curvature tensors R^t and R and relation (13) and for any vector fields X, Y, and Z tangent to M, the expressions

$$\nabla_X^t \nabla_Y^t Z = \nabla_X \nabla_Y Z - \frac{1}{2}\omega(X)\nabla_Y Z - \frac{1}{2}\omega(\nabla_Y Z)$$
$$+ \frac{1}{2}g(X, \nabla_Y Z)B - \frac{1}{2}X(\omega(Y))Z$$
$$- \frac{1}{2}\omega(Y)\nabla_X Z + \frac{1}{4}\omega(X)\omega(Y)Z$$
$$+ \frac{1}{4}\omega(Y)\omega(Z)X - \frac{1}{4}\omega(Y)g(X, Z)B$$
$$- \frac{1}{2}X(\omega(Z))Y - \frac{1}{2}\omega(Z)\nabla_X Y$$
$$+ \frac{1}{4}\omega(X)\omega(Z)Y + \frac{1}{4}\omega(Y)\omega(Z)X$$
$$- \frac{1}{4}\omega(Z)g(X, Y)B + \frac{1}{2}X(g(Y, Z))B$$
$$+ \frac{1}{2}g(Y, Z)\nabla_X B - \frac{1}{4}\|B\|^2 g(Y, Z)X. \qquad (19)$$

It is worth noting that

$$\nabla_{[X,Y]}^t Z = \nabla_{[X,Y]} Z - \frac{1}{2}\omega([X, Y])Z - \frac{1}{2}\omega(Z)[X, Y]$$
$$+ \frac{1}{2}g([X, Y], Z)B. \qquad (20)$$

Putting pieces (19) and (20) together, one obtains relation (18). This completes the proof.

Next, from the above lemma, we define $(0, 2)$-tensor field P by

$$P(X, Y) = (\nabla_X \omega)Y + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}\|B\|^2 g(X, Y), \quad (21)$$

and this trace is given by

$$\text{trace}P = \text{div}B - \frac{1}{2}(1 - n)\|B\|^2. \qquad (22)$$
$\square$

Lemma 2. *The $(0, 2)$-tensor field P is symmetric.*

Proof. For any vector fields X and Y tangent to M and since ω is closed, we have

$$P(X,Y) = (\nabla_Y \omega)X + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}\|B\|^2 g(X,Y)$$

$$= Y(\omega(X)) - \omega(\nabla_Y X) + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}\|B\|^2 g(X,Y)$$

$$= Y(\omega(X)) - \omega([Y,X]) - \omega(\nabla_X Y) + \frac{1}{2}\omega(X)\omega(Y)$$

$$- \frac{1}{4}\|B\|^2 g(X,Y)$$

$$= (\nabla_X \omega)Y + \frac{1}{2}\omega(X)\omega(Y) - \frac{1}{4}\|B\|^2 g(X,Y),$$

(23)

which completes the proof.

The Lie derivative g with respect to the vector field B gives, for any vector fields X and Y,

$$(L_B g)(X,Y) = X(g(B,Y)) - g([B,X],Y) - g(X,[B,Y])$$

$$= (\nabla_X \omega)Y + (\nabla_Y \omega)X = 2(\nabla_X \omega)Y.$$

(24)

The last equality of (24) follows from the fact that smooth 1-form ω is closed. □

Lemma 3. *The dual vector field B of ω preserves the metric g if and only if the Lee form ω is ∇-parallel.*

The Riemannian curvatures are related by, for any X, Y, Z, and W tangent to M,

$$\exp(f_t)R^t(X,Y,Z,W) = R(X,Y,Z,W) + \frac{1}{2}\{g(X,W)P(Y,Z)$$

$$- g(Y,W)P(X,Z)\} + \frac{1}{2}\{g(Y,Z)P(X,W)$$

$$- g(X,Z)P(Y,W)\}.$$

(25)

Let $\{E_i\}$ be the orthonormal basis with respect to g. Then, we have

$$g(E_i, E_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(26)

Let $E_i^t = \exp(f_t)^{(1/2)} E_i$, for any $i = 1, 2, \ldots, 2n$. Therefore, we have the following.

Lemma 4. *The frame $\{E_i^t\}_{1 \leq i \leq 2n}$ is the orthonormal basis with respect to g_t.*

The following identities generalize the ones given in [9, p.216].

Lemma 5. *The Ricci curvature tensors ρ^t and ρ with respect to g_t and g, respectively, are related by*

$$\rho^t(X,Y) = \rho(X,Y) + (n-1)P(X,Y) + \frac{1}{2}g(X,Y)\text{trace}P.$$

(27)

Proof. Using Lemma 4 and for any vector fields X and Y tangent to M, one has

$$\rho^t(X,Y) = \sum_{i=1}^{2n} R^t(E_i^t, X, Y, E_i^t) = \sum_{i=1}^{2n} \exp(f_t)R^t(E_i, X, Y, E_i)$$

$$= \sum_{i=1}^{2n} R(E_i, X, Y, E_i) + \frac{1}{2}\left\{\sum_{i=1}^{2n} g(E_i, E_i)P(X,Y)\right.$$

$$\left. - \sum_{i=1}^{2n} g(X, E_i)P(E_i, Y)\right\}$$

$$+ \frac{1}{2}\left\{\sum_{i=1}^{2n} g(X,Y)P(E_i, E_i) - \sum_{i=1}^{2n} g(E_i, Y)P(X, E_i)\right\}$$

$$= \rho(X,Y) + (n-1)P(X,Y) + \frac{1}{2}g(X,Y)\text{trace}P,$$

(28)

which completes the proof.

Also, corresponding Ricci $*$-curvatures are related by

$$\rho^{t*}(X,Y) = \rho^*(X,Y) + \frac{1}{2}\{P(X,Y) + P(JX, JY)\}.$$

(29)

□

Corollary 1. *The scalar curvatures τ^t and τ are related by*

$$\exp(-f_t)\tau_t = \tau + (2n-1)\left\{\text{div}B - \frac{1}{2}(1-n)\|B\|^2\right\}.$$

(30)

Proof. Using Lemma 4 and the scalar curvature τ^t, we have

$$\tau^t = \sum_{i=1}^{2n} \rho^t(E_i^t, E_i^t) = \exp(f_t)\sum_{i=1}^{2n} \rho^t(E_i, E_i).$$

(31)

Then, applying equation (26) to (31), we get

$$\exp(-f_t)\tau^t = \sum_{i=1}^{2n} \rho(E_i, E_i) + (n-1)\sum_{i=1}^{2n} P(E_i, E_i) + n\text{trace}P$$

$$= \tau + (2n-1)\text{trace}P$$

$$= \tau + (2n-1)\left\{\text{div}B - \frac{1}{2}(1-n)\|B\|^2\right\}.$$

(32)

Therefore,

$$\exp(-f_t)\tau^t = \tau + (2n-1)\left\{\text{div}B - \frac{1}{2}(1-n)\|B\|^2\right\},$$

(33)

which completes the proof.

Now, if we consider a relation between scalar $*$-curvature τ^{t*} and τ^*, we get the following. □

Corollary 2. *The scalar $*$-curvatures τ^{t*} and τ^* are related by*

$$\exp(-f_t)\tau^{t*} = \tau^* + \text{div}B + (n-1)\|B\|^2.$$

(34)

Proof. The scalar $*$-curvature τ^{t^*} is given by

$$\tau^{t^*} = \sum_{i=1}^{2n} \rho^{t^*}\left(E_i^t, E_i^t\right) = \exp\left(f_t\right)\sum_{i=1}^{2n} \rho^{t^*}\left(E_i, E_i\right). \qquad (35)$$

Now, applying relation (29) to (35), we compute

$$\begin{aligned}
\exp\left(-f_t\right)\tau^{t^*} &= \sum_{i=1}^{2n} \rho^{t^*}\left(E_i, E_i\right) \\
&= \sum_{i=1}^{2n} \rho^*\left(X, Y\right) + \frac{1}{2}\sum_{i=1}^{2n}\{P\left(E_i, E_i\right) + P\left(JE_i, jE_i\right)\} \\
&= \tau^* + \mathrm{div}B + (n-1)\|B\|^2.
\end{aligned} \qquad (36)$$

Hence,

$$\exp\left(-f_t\right)\tau^{t^*} = \tau^* + \mathrm{div}B + (n-1)\|B\|^2, \qquad (37)$$

as required.

Gray in [7] considered some curvature identities for Hermitian and almost Hermitian manifolds. Let $\mathscr{L}$ be the class of almost Hermitian manifolds as defined in [7]. Then, the manifold under consideration is an element of class $\mathscr{L}$. Now, consider as in [7] the curvature operator R^t of a locally conformal almost Kähler manifold M:

$$\begin{aligned}
(1)\, R^t\left(X, Y, Z, W\right) &= R^t\left(X, Y, JZ, JW\right), \\
(2)\, R^t\left(X, Y, Z, W\right) - R^t\left(JX, JY, Z, W\right) &= R^t\left(JX, Y, JZ, W\right), \\
&\quad + R^t\left(JX, Y, Z, JW\right), \\
(3)\, R^t\left(X, Y, Z, W\right) &= R^t\left(JX, JY, JZ, JW\right),
\end{aligned} \qquad (38)$$

for any X, Y, Z, and W tangent to M. Item (1) is called a Kähler identity if M is a locally conformal Kähler manifold (see [7] for more details and references therein).

We will focus, throughout the rest of this note, on item (1). Using further notations as in [7], we denoted by $\mathscr{L}_i$ the subclass of manifolds whose curvature operator R^t satisfies identity (i). Here, (i) may be either item (1), (2), or (3) above. As in [7], it is easy to see that

$$\mathscr{L}_1 \subseteq \mathscr{L}_2 \subseteq \mathscr{L}_3 \subseteq \mathscr{L}. \qquad (39)$$

Therefore, we have the following result. $\qquad\square$

Lemma 6. *If a locally conformal almost Kähler manifold is in class $\mathscr{L}_1$, then the equality holds*

$$\tau^* - \tau = 2(n-1)\mathrm{trace}P. \qquad (40)$$

Proof. The proof follows from a straightforward calculation using the fact that, for any vector fields X and Y tangent to M, we have

$$\begin{aligned}
\rho^t\left(X, Y\right) &= \sum_{i=1}^{2n} \exp\left(f_t\right)R^t\left(E_i, X, Y, E_i\right) \\
&= \sum_{i=1}^{2n} \exp\left(f_t\right)R^t\left(E_i, X, JY, JE_i\right) = \rho^{*t}\left(X, Y\right),
\end{aligned} \qquad (41)$$

which leads to

$$\begin{aligned}
\left(\rho^* - \rho\right)\left(X, Y\right) &= \left(n - \frac{3}{2}\right)P\left(X, Y\right) + \frac{1}{2}g\left(X, Y\right)\mathrm{trace}P \\
&\quad - \frac{1}{2}P\left(JX, JY\right).
\end{aligned} \qquad (42)$$

This completes the proof.
Relation (41) leads to

$$\begin{aligned}
\tau^t &= \sum_{i=1}^{2n}\sum_{j=1}^{2n} R^t\left(E_j, E_i, E_i, E_j\right) \\
&= \sum_{i=1}^{2n}\sum_{j=1}^{2n} R^t\left(E_j, E_i, JE_i, JE_j\right) = \tau^{t^*}.
\end{aligned} \qquad (43)$$

$\qquad\square$

Theorem 1. *Let (M, J, g) be a $2n$-dimensional compact locally conformal almost Kähler manifold with $n > 1$ and contained in $\mathscr{L}_1$. If*

$$\tau^* = \tau, \qquad (44)$$

then (M, J, g) is an almost Kähler manifold.

Proof. By Lemma 6, we have $\tau^* - \tau = 2(n-1)\mathrm{trace}P$, with $\mathrm{trace}P = \mathrm{div}B - (1/2)(1-n)\|B\|^2$. Taking into account this, integrating relation (40), and using Green's theorem, we have $0 = \int_M\{\tau^* - \tau\} = (n-1)^2\int_M\|B\|^2$. Hence, under our assumption, we obtain $B = 0$. Therefore, $\omega = 0$ identically on M. Hence, (M, J, g) is an almost Kähler manifold.

As an example for this theorem, we have compact flat locally almost Kähler manifolds. Compact flat manifolds have been detailed in [8] and references therein. $\qquad\square$

4. Lee Form and Canonical Foliations

Let (M, J, g) be a locally conformal almost Kähler manifold, and assume that the Lee form ω is never vanishing on M. Then, $\omega = 0$ defines on M an integrable distribution, and hence a foliation $\mathscr{F}$, on M (see [9] for more details and references therein).

Let $D := \ker\omega$ be the distribution on M and $D^\perp$ be the distribution spanned by the vector field B. Then, we have the following decomposition:

$$TM = D \oplus D^\perp, \qquad (45)$$

where $\oplus$ denotes the orthogonal direct sum. By decomposition (45), any $X \in \Gamma(TM)$ is written as

$$X = QX + Q^\perp X, \tag{46}$$

where Q and $Q^\perp$ are the projection morphisms of TM into D and $D^\perp$, respectively. Here, it is easy to see that $Q^\perp X = (1/\|B\|^2)\omega(X)B$, and

$$X = QX + \frac{1}{\|B\|^2}\omega(X)B. \tag{47}$$

Let $\mathcal{F}$ be a foliation on a locally conformal almost Kähler manifold (M, J, g) of codimension 1. The metric g is said to be bundle-like for the foliation $\mathcal{F}$ if the induced metric on the transversal distribution $D^\perp$ is parallel with respect to the intrinsic connection on $D^\perp$. This is true if and only if the Levi-Civita connection ∇ of (M, J, g) satisfies (see [10, 11] for more details)

$$g\left(\nabla_{Q^\perp Y} QX, Q^\perp Z\right) + g\left(\nabla_{Q^\perp Z} QX, Q^\perp Y\right) = 0, \tag{48}$$

for any $X, Y, Z \in \Gamma(TM)$. A foliation $\mathcal{F}$ is said to be Riemannian on (M, J, g) if the Riemannian metric g on M is bundle-like for $\mathcal{F}$.

Let $\mathcal{F}^\perp$ be the orthogonal complementary foliation generated by B. Now, we provide necessary and sufficient conditions for the metric on an locally conformal almost Kähler manifold to be bundle-like for foliations $\mathcal{F}$ and $\mathcal{F}^\perp$.

Theorem 2. *Let (M, J, g) be a locally conformal almost Kähler manifold, and let $\mathcal{F}$ be a foliation on M of codimension 1. Then, the following assertions are equivalent:*

(i) The foliation $\mathcal{F}$ is Riemannian.

(ii) The Lee vector field B is autoparallel with respect to ∇, that is,

$$\nabla_B B = B(\ln(\|B\|))B. \tag{49}$$

Proof. For any $X, Y, Z \in \Gamma(TM)$, we have $Q^\perp Y = (1/\|B\|^2)\omega(Y)B$ and $Q^\perp Z = (1/\|B\|^2)\omega(Z)B$, and the left-hand side of (48) gives

$$g\left(\nabla_{Q^\perp Y} QX, Q^\perp Z\right) + g\left(\nabla_{Q^\perp Z} QX, Q^\perp Y\right) = \frac{2}{\|B\|^2}\omega(Y)\omega(Z)\omega(\nabla_B QX), \tag{50}$$

for which the equivalence follows.

Let M' be a leaf of the distribution D. Since M' is a submanifold of M and for any $X, Y \in \Gamma(TM')$, we have

$$\nabla_X Y = \nabla'_X Y + \alpha(X, Y), \tag{51}$$

$$\nabla_X B = -A_B X + \nabla'^\perp_X B, \tag{52}$$

where ∇' and α are the Levi-Civita connection and the second fundamental form of M', respectively. Here, A_B is the shape operator with respect to B. On the contrary, we have $g(\nabla_X B, B) = X(\omega(B)) - g(\nabla_X B, B)$; hence,

$$g\left(\nabla'^\perp_X B, B\right) = \frac{1}{2}X(\omega(B)), \tag{53}$$

for any $X \in \Gamma(TM')$. Therefore, Weingarten formula (52) becomes

$$\nabla_X B = -A_B X + \frac{1}{2}X(\omega(B))B. \tag{54}$$
□

Proposition 1. *Let (M, J, g) be a locally conformal almost Kähler manifold. Then, the mean curvature vector field H' of the leaves of the integrable distribution D defined in (45) is given by*

$$H' = \frac{1}{2n-1}\left(\mathrm{div}_{|_{M'}} B\right)B. \tag{55}$$

Moreover, these leaves are totally geodesic hypersurfaces of M if and only if the dual vector field B of ω preserves their metrics.

Proof. Let M' be a leaf of the integrable distribution D. Using (51) and (54), the second fundamental form of M' gives

$$\alpha(X, Y) = g(A_B X, Y)B = g(\nabla_X B, Y)B, \tag{56}$$

for any $X, Y \in \Gamma(TM')$. Fixing a local orthonormal frame $\{e_1, \ldots, e_{2n-1}\}$ in TM', one has

$$H = \frac{1}{2n-1}\sum_{i=1}^{2n-1}\alpha(e_i, e_i) = \frac{1}{2n-1}\left(\mathrm{div}_{|_{M'}} B\right)B. \tag{57}$$

The last assertion follows, and this completes the proof. Therefore, we have the following result. □

Corollary 4. *Let (M, J, g) be a locally conformal almost Kähler manifold. Then, the leaves M' of the distribution D in (45) are minimal if and only if the dual vector field B is incompressible along M'.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Acknowledgments

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