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Research Article

Derivations of a Sullivan Model and the Rationalized G-Sequence

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Let $G_{k,n}(\mathbb{C})$ for $2 \le k < n$ denote the Grassmann manifold of k-dimensional vector subspaces of $\mathbb{C}^n$. In this paper, we compute, in terms of the Sullivan models, the rational evaluation subgroups and, more generally, the G-sequence of the inclusion $G_{2,n}(\mathbb{C}) \rightarrowtail G_{2,n+r}(\mathbb{C})$ for $r \ge 1$.

1. Introduction

Throughout this paper, we rely on the theory of minimal Sullivan models in rational homotopy theory for which [1] is our standard reference. Let X be a based CW-complex; the nth Gottlieb group of X (or the nth evaluation subgroup [2] of $\pi_n(X)$), denoted by $G_n(X)$, consists of those elements $a \in \pi_n(X)$ for which there is a continuous map $H\colon X \times S^n \longrightarrow X$ such that the following diagram commutes:

$$\begin{tikzcd} X \vee S^n \arrow[r, "id_X \vee h"] \arrow[d] & X \vee X \arrow[d, "\nabla"] \\ X \times S^n \arrow[r, "H"] & X, \end{tikzcd} \tag{1}$$

where $h\colon S^n \longrightarrow X$ is a representative of a and ∇ is the folding map. Let $f\colon X \longrightarrow Y$ be a based map of simply connected finite CW-complexes. In [3], the evaluation at the base point of X gives the *evaluation map* ω: (Map) (X, Y; f) $\longrightarrow$ Y, where (Map) (X, Y; f) is the component of f in the space of mappings from X to Y. The image of the homomorphism induced in homotopy groups

$$\omega_\sharp\colon \pi_*(\mathrm{Map}(X, Y; f)) \longrightarrow \pi_*(\mathrm{Y}) \tag{2}$$

is called the nth *evaluation subgroup of p*, and it is denoted by $G_n(Y, X; f)$. Moreover, if $f = id_X$, the space (Map) (X, Y; f) is the monoid $\mathrm{aut}_1(X)$ of self-equivalences of X homotopic to the identity of X; then, $ev\colon \mathrm{aut}_1(X) \longrightarrow X$ is the evaluation map, and the image of the induced homomorphism

$$ev_\sharp\colon \pi_*(\mathrm{aut}_1(X)) \longrightarrow \pi_*(X) \tag{3}$$

is $G_n(X)$, i.e., the nth Gottlieb group. Moreover, in [4], Woo and Lee studied the relative evaluation subgroups $G_n^{\mathrm{rel}}(X, Y; p)$ and proved that they fit in a sequence

$$\cdots \longrightarrow G_{n+1}^{\mathrm{rel}}(X, Y; f) \longrightarrow G_n(X) \longrightarrow G_n(X, Y; f) \longrightarrow \cdots \tag{4}$$

called the G-sequence of f. Finally, in [3], Smith and Lupton identify the homomorphism induced on rational homotopy groups by the evaluation map ω: $(\mathrm{Map}(X, Y; f)) \longrightarrow \mathrm{Y}$, in terms of a map of complexes of derivations constructed directly from the Sullivan minimal model of f. In [5], the authors use a map of complexes of derivations of minimal Sullivan models of mapping spaces to compute rational relative Gottlieb groups of the inclusion $G_{2,n}(\mathbb{C}) \rightarrowtail G_{2,n+1}(\mathbb{C})$ between complex Grassmannians. In this paper, we generalize their work to compute rational relative Gottlieb groups of the inclusion $G_{k,n}(\mathbb{C}) \rightarrowtail G_{k,n+r}(\mathbb{C})$, $r \ge 1$ between complex Grassmannians for $2 \le k < n$.

2. Preliminaries

Here, we fix terminology and recall some standard facts on differential graded algebras. All vector spaces and algebras are taken over a field $\mathbb{Q}$ of rational numbers.

Definition 1. A graded algebra A is a sum $A = \oplus_{n\geq 0} A^n$,where A^n is a vector space, together with an associative multiplication $A^i \otimes A^j \longrightarrow A^{i+j}, x\otimes y \mapsto xy$, and has $1 \in A^0$. It is graded commutative if, for any homogeneous elements x and y,

$$xy = (-1)^{|x||y|} yx, \tag{5}$$

where $|x| = i$ for $x \in A^i$. If A is a graded algebra equipped with a linear differential map d: $A^n \longrightarrow A^{n+1}$ such that $d^{\circ}d = 0$ and

$$d(xy) = (dx)y + (-1)^{|x|} x(dy), \tag{6}$$

then (A, d) is called a differential-graded algebra and d is called a differential. Moreover, if A is also a graded commutative algebra, then (A, d) is a commutative differential graded algebra (cdga). It is said to be connected if $A^0 \cong \mathbb{Q}$.

Definition 2. Let $V = \oplus_{i\geq 0} V^i$ with $V^{\text{even}}\colon = \oplus_{i\geq 0} V^{2i}$ and $V^{\text{odd}}\colon = \oplus_{i\geq 1} V^{2i-1}$. A commutative-graded algebra A is called free commutative if $A = \wedge V = S(V^{\text{even}}) \otimes E(V^{\text{odd}})$, where $S(V^{\text{even}})$ is the symmetric algebra on V^{even} and $E(V^{\text{odd}})$ is the exterior algebra on V^{odd}.

Definition 3. A Sullivan algebra is a commutative differential-graded algebra $(\wedge V, d)$, where $V = \cup_{k\geq 0} V(k)$ and $V(0) \subset V(1)\cdots$ such that $dV(0) = 0$ and $dV(k) \subset \wedge V(k-1)$. It is called minimal if $dV \subset \wedge^{\geq 2} V$.

If (A, d) is a cdga whose cohomology is connected and finite dimensional in each degree, then there always exists a quasi-isomorphism from a Sullivan algebra $(\wedge V, d)$ to (A, d) [1]. To each simply connected space, Sullivan associates a cdga $A_{\text{PL}}(X)$ of rational polynomial differential forms on X that uniquely determines the rational homotopy type of X [6]. A minimal Sullivan model of X is a minimal Sullivan model of $A_{\text{PL}}(X)$. More precisely, $H^*(\wedge V, d) \cong H^*(X; \mathbb{Q})$ as graded algebras and $V \cong \pi_*(X) \otimes \mathbb{Q}$ as graded vector spaces.

3. Derivations of a Sullivan Model and the G-Sequence

Let (A, d) be a commutative differential-graded algebra. A derivation θ of degree k is a linear mapping θ: $A^n \longrightarrow A^{n-k}$ such that $\theta(ab) = \theta(a)b + (-1)^{k|a|} a\theta(b)$. Denote by $\text{Der}_k A$ the vector space of all derivation of degree k and $\text{Der}A = \oplus_k \text{Der}_k A$. The commutator bracket induces a graded Lie algebra structure on $\text{Der}A$. Moreover, $(\text{Der}A, \delta)$ is a differential graded Lie algebra [6], with the differential δ defined in the usual way by

$$\delta\theta = d \circ \theta + (-1)^{k+1} \theta \circ d. \tag{7}$$

Let $(\wedge V, d)$ be a Sullivan algebra, where V is spanned by $\{v_1, \ldots, v_k\}$. Then, $\text{Der}\wedge V$ is spanned by $\theta_1, \ldots, \theta_k$, where θ_i is the unique derivation of $\wedge V$ defined by $\theta_i(v_j) = \delta_{ij}$. The derivation θ_i will be denoted by $(v_i, 1)$. Moreover, an element $v \in V \cong \pi_*(X) \otimes \mathbb{Q}$ is a Gottlieb element of $\pi_*(X) \otimes \mathbb{Q}$ if and only if there is a derivation θ of $\wedge V$ satisfying $\theta(v) = 1$ and such that $\delta\theta = 0$,see page 392 in [1].

Let ϕ: $(A, d) \longrightarrow (B, d)$ be a morphism of cdga's. A ϕ-derivation of degree k is a linear mapping θ: $A^n \longrightarrow B^{n-k}$ for which $\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|} \phi(a)\theta(b)$.

We consider only derivations of positive degree. Denote by $\text{Der}_n(A, B; \phi)$ the vector space of ϕ-derivations of degree n for $n > 0$ and by $\text{Der}(A, B; \phi) = \oplus_n \text{Der}_n(A, B; \phi)$ the $\mathbb{Z}$-graded vector space of all ϕ-derivations. The differential-graded vector space of ϕ-derivations is denoted by $(\text{Der}(A, B; \phi), \partial)$, where the differential ∂ is defined by $\partial\theta = d_B \circ \theta + (-1)^{k+1} \theta^{\circ} \text{d}_{\text{A}}$. In case $A = B$ and $\phi = 1_B$, then $(\text{Der}(B, B; 1), \partial)$ is just the usual differential-graded Lie algebra of derivations on the cdga B [3]. Whenever $A = (\wedge V, d)$ is a Sullivan algebra, we note that there is an isomorphism of graded vector spaces:

$$\text{Der}(A, B; \phi) \cong \text{Hom}(V, B). \tag{8}$$

If $\{v_i\}$ is a basis of V, then the vector space $\text{Der}(A, B; \phi)$ is spanned by the unique ϕ-derivation θ denoted by (v_i, b_i) such that $\theta_i(v_i) = b_i$, where $b_i \in B$ and $\theta_i(v_j) = 0$ for $i \neq j$. Moreover, in [3], precomposition with ϕ gives a chain complex map ϕ^*: $\text{Der}(B, B; 1) \longrightarrow \text{Der}(A, B; \phi)$ and postcomposition with the augmentation ε: $B \longrightarrow \mathbb{Q}$ gives a chain complex map ε_*: $\text{Der}(A, B; \phi) \longrightarrow \text{Der}(A, \mathbb{Q}; \varepsilon)$. The evaluation subgroup of ϕ is defined as follows:

$$G_n(A, B; \phi) = \text{Im}\{H(\varepsilon_*)\colon H_n(\text{Der}(A, B; \phi)) \longrightarrow H_n(\text{Der}(A, \mathbb{Q}; \varepsilon))\}. \tag{9}$$

In case $A = B$ and $\phi = 1_B$, we get the Gottlieb group of (B, d) defined as follows:

$$G_n(B) = \text{Im}\{H(\varepsilon_*)\colon H_n(\text{Der}(B, B; 1)) \longrightarrow H_n(\text{Der}(B, \mathbb{Q}; \varepsilon))\}. \tag{10}$$

In particular, $G_n(B) \cong G_n(X_{\mathbb{Q}})$, if B is the minimal Sullivan model of a simply connected space X (see Proposition 29.8 in [1]).

Definition 4 (see [3, 7]). Let ϕ: $A \longrightarrow B$ be a map of differential-graded vector spaces. A differential-graded vector space, $\text{Rel}_*(\phi)$, called the mapping cone of ϕ is defined as follows. $\text{Rel}_n(\phi) = A_{n-1} \oplus B_n$ with the differential $\delta(a, b) = (-d_A(a), \phi(a) + d_B(b))$. There are inclusion and projection chain maps J: $B_n \longrightarrow \text{Rel}_n(\phi)$ and P: $\text{Rel}_n(\phi) \longrightarrow A_{n-1}$ defined by $J(w) = (0, w)$ and $P(a, b) = a$. These yields a short exact sequence of chain complexes

$$0 \longrightarrow B_* \xrightarrow{J} \text{Rel}_*(\phi) \xrightarrow{P} A_{*-1} \longrightarrow 0, \tag{11}$$

and a long exact homology sequence of ϕ

$$\cdots \longrightarrow H_{n+1}(\mathrm{Rel}(\phi)) \xrightarrow{H} (P)\ H_n(A) \xrightarrow{H} (\phi)\ H_n(B) \xrightarrow{H} (J)\ H_n(\mathrm{Rel}(\phi)) \longrightarrow \cdots \tag{12}$$

whose connecting homomorphism is $H(\phi)$.

Following [3], we consider a commutative diagram of differential-graded vector spaces:

$$\begin{array}{ccc} \mathrm{Der}(B,B;1) & \xrightarrow{\phi^*} & \mathrm{Der}(A,B;\phi) \\ \varepsilon_* \downarrow & & \downarrow \varepsilon_* \\ \mathrm{Der}(B,\mathbb{Q};\varepsilon) & \xrightarrow{\bar{\phi}^*} & \mathrm{Der}(A,\mathbb{Q};\varepsilon), \end{array} \tag{13}$$

where ε is the augmentation of either A or B, which leads to the following homology ladder for $n \geq 2$:

$$\begin{array}{ccccccc} \cdots \longrightarrow H_{n+1}(\mathrm{Rel}(\bar{\phi}^*)) & \xrightarrow{H(P)} & H_n(\mathrm{Der}(B,B;1)) & \xrightarrow{H(\phi^*)} & H_n(\mathrm{Der}(A,B;\phi)) & \longrightarrow \cdots \\ H(\varepsilon_*,\varepsilon_*) \downarrow & & H(\varepsilon_*) \downarrow & & H(\varepsilon_*) \downarrow & \\ \cdots \longrightarrow H_{n+1}(\mathrm{Rel}(\bar{\phi}^*)) & \xrightarrow{H(\bar{P})} & H_n(\mathrm{Der}(B,\mathbb{Q};\varepsilon)) & \xrightarrow{H(\bar{\phi}^*)} & H_n(\mathrm{Der}(A,\mathbb{Q};\varepsilon)) & \longrightarrow \cdots \end{array} \tag{14}$$

The nth relative evaluation subgroup of ϕ is defined as follows:

$$G_n^{rel} = \mathrm{Im}\left\{H(\varepsilon_*,\varepsilon_*): H_n(\mathrm{Rel}(\phi^*)) \longrightarrow H_n\left(\mathrm{Rel}\left(\overline{\phi}^*\right)\right)\right\}. \tag{15}$$

The G-sequence of the map $\phi: A \longrightarrow B$ is given by

$$\cdots \longrightarrow^{H(\bar{J})} G_{n+1}^{rel}(A,B;\phi) \longrightarrow^{H(\bar{P})} G_n(B) \xrightarrow{H} (\overline{\phi}^*)\ G_n(A,B;\phi) \longrightarrow^{H(\bar{J})} \cdots \tag{16}$$

which ends in $G_2(A,B;\phi)$. Moreover, Theorem 3.5 in [3] can be applied to the Sullivan model $\phi: (\wedge V, d) \longrightarrow (B,d)$ of the map $f: X \longrightarrow Y$.

4. The Inclusion $G_{k,n}(\mathbb{C}) \rightarrowtail G_{k,n+1}(\mathbb{C})$

The complex Grassmannian $G_{k,n}(\mathbb{C})$ is a simply connected homogeneous space as $G_{k,n}(\mathbb{C}) \cong U(n)/(U(k) \times U(n-k))$ for $1 \leq k < n$, where $U(n)$ is the unitary group. It is a symplectic manifold of dimension $2m$, where $m = k(n-k)$. As the complex Grassmannian $G_{k,n}(\mathbb{C})$ is simply connected, so we may associate a minimal Sullivan model.

The method to compute a Sullivan model of the homogeneous space $G_{k,n}(\mathbb{C})$ is given in detail in [8, 9].

Following [9], a Sullivan model of $G_{k,n}(\mathbb{C})$ for $1 \leq k < n$ is given by

$$\left(\wedge\left(b_2, b_4, \ldots, b_{2k}, x_2, x_4, \ldots, x_{2(n-k)}, y_1, y_3, \ldots, y_{2n-1}\right), d\right), \tag{17}$$

with

$$db_i = 0 = dx_j, dy_{2p-1} = \sum_{p_1+p_2=p} b_{2p_1} x_{2p_2}, 1 \leq p \leq n. \tag{18}$$

Lemma 1. *The minimal Sullivan model of $G_{k,n}(\mathbb{C})$ for $2 \leq k < n$ is given by*

$$\left(\wedge\left(b_2, \ldots, b_{2k}, y_{2(n-k)+1}, \ldots, y_{2n-1}\right), d\right), \tag{19}$$

where $db_i = 0$ and $dy_{2(n-k)+1} \in \wedge(b_2, \ldots, b_{2k})$.

Proof. Consider the Sullivan model

$$\left(\wedge\left(b_2, b_4, \ldots, b_{2k}, x_2, x_4, \ldots, x_{2(n-k)}, y_1, y_3, \ldots, y_{2n-1}\right), d\right), \tag{20}$$

of $G_{k,n}(\mathbb{C})$ for $2 \leq k < n$:

$$\begin{aligned} dy_1 &= b_2 + x_2, \\ dy_3 &= b_4 + x_4 + b_2x_2, \\ &\vdots \\ dy_{2n-1} &= b_{2k}x_{2(n-k)}. \end{aligned} \tag{21}$$

The model is not minimal as the linear part is not zero. To find its minimal Sullivan model, we make a change of variable $t_2 = b_2 + x_2$ and replace x_2 by $t_2 - b_2$ wherever it appears in the differential. This gives an isomorphic Sullivan algebra

$$\left(\wedge\left(b_2, t_2, b_4, \ldots, b_{2k}, x_4, \ldots, x_{2(n-k)}, y_1, y_3, \ldots, y_{2n-1}\right), d\right), \tag{22}$$

where

$$\begin{aligned} dy_1 &= t_2, \\ dy_3 &= b_4 + x_4 + b_2(t_2 - b_2), \\ &\vdots \\ dy_{2n-1} &= b_{2k}x_{2(n-k)}. \end{aligned} \tag{23}$$

As the ideal generated by y_1 and t_2 is acyclic, the above Sullivan algebra is quasi-isomorphic to

$$\left(\wedge\left(b_2, b_4, \ldots, b_{2k}, x_4, \ldots, x_{2(n-k)}, y_3, \ldots, y_{2n-1}\right), d\right), \tag{24}$$

where

$$\begin{aligned} dy_3 &= b_4 + x_4 - b_2^2, \\ &\vdots \\ dy_{2n-1} &= b_{2k}x_{2(n-k)}. \end{aligned} \tag{25}$$

One continues in this fashion and makes another change of variable $t_4 = b_4 + x_4 - b_2^2$ and replaces x_4 by $t_4 - b_4 + b_2^2$ wherever it appears in the differential and does so until they reach a change of variable of the form

$$\begin{aligned} t_{2(n-k)} &= b_{2(n-k)} + x_{2(n-k)} + \alpha \text{ for } n = 2k \\ \text{or } t_{2(n-k)} &= x_{2(n-k)} + \beta \text{ for } n > 2k, \end{aligned} \tag{26}$$

where $\alpha \in \wedge(b_2, \ldots, b_{2(k-1)}), \beta \in \wedge(b_2, \ldots, b_{2k})$ and replace

$$x_{2(n-k)} = \begin{cases} t_{2(n-k)} - b_{2k} + \alpha, & \text{for } n = 2k, \\ t_{2(n-k)} + \beta, & \text{for } n > 2k, \end{cases} \tag{27}$$

wherever it appears in the differential. This gives an isomorphic Sullivan algebra:

$$\left(\wedge\left(b_2, \ldots, b_{2k}, y_{2(n-k)-1}, y_{2(n-k)+1}, \ldots, y_{2n-1}\right), d\right), \tag{28}$$

where

$$\begin{aligned} dy_{2(n-k)-1} &= t_{2(n-k)}, \\ &\vdots \\ dy_{2n-1} &= b_{2k}x_{2(n-k)}. \end{aligned} \tag{29}$$

As the ideal generated by $t_{2(n-k)}$ and $y_{2(n-k)-1}$ is acyclic, we get the minimal Sullivan model:

$$\left(\wedge\left(b_2, \ldots, b_{2k}, y_{2(n-k)+1}, \ldots, y_{2n-1}\right), d\right), \tag{30}$$

with $db_i = 0$ and $dy_{2(n-k)+1} \in \wedge(b_2, \ldots, b_{2k})$.

In the same way, by Lemma 1, the minimal Sullivan model of $G_{k,n+r}(\mathbb{C})$ for $2 \le k < n + r$ and $r \ge 1$ is given by

$$\left(\wedge\left(a_2, \ldots, a_{2k}, z_{2(n+r-k)+1}, \ldots, z_{2(n+r)-1}\right), d\right), \tag{31}$$

where $da_i = 0$ and $dz_{2(n+r-k)+1} \in \wedge(a_2, \ldots, a_{2k})$. We establish the following results.

Theorem 1. *Let* $B = (\wedge(b_2, \ldots, b_{2k}, y_{2(n-k)+1}, \ldots, y_{2n-1}), d)$. *Then,* $G_n(B) = \langle [y^*_{2(n-k)+1}], \ldots, [y^*_{2n-1}] \rangle$.

Proof. Let $(y_{2(n-t)+1}, 1)$ denote the derivation $\theta_{2(n-t)+1}$ for $t \in \{1, \ldots, k\}$ such that $\theta_{2(n-t)+1}(y_{2(n-t)+1}) = 1$ and zero on other generators. Then,

$$\delta\theta_{2(n-t)+1}\left(y_{2(n-t)+1}\right) = 0. \tag{32}$$

Moreover, the generators $\theta_{2(n-t)+1}$ cannot be boundaries for degree reasons. Therefore, $[\theta_{2(n-t)+1}]$ are nonzero homology classes in $H_*(\mathrm{Der}(B, B; 1))$. Furthermore, $\varepsilon_*(\theta_{2(n-t)+1}) = y^*_{2(n-t)+1}$.

As $G_{k,n}(\mathbb{C})$ is a simply connected finite CW-complex, then $G_{\text{even}}(B) = 0$ (see Proposition 28.8 in [1]. Thus, $G_n(B) = \langle [y^*_{2(n-k)+1}], \ldots, [y^*_{2n-1}] \rangle$.

Theorem 2. *Given the inclusion* $G_{k,n}(\mathbb{C}) \rightarrowtail G_{k,n+r}(\mathbb{C}), r \ge 1$ *for* $2 \le k < n$ *and* $\phi\colon (\wedge V, d) \longrightarrow (B, d)$ *for its Sullivan model, then* $G_*(\wedge V, B; \phi) = \langle [z^*_{2(n+r-k)+1}], \ldots, [z^*_{2(n+r)-1}] \rangle$.

Proof. The vector space $\mathrm{Der}(\wedge V, B; \phi)$ is generated by the derivations $\theta_{2(n+r-t)+1} = (z_{2(n+r-t)+1}, 1)$ for $t \in \{1, \ldots, k\}$. The differential is given by

$$\delta\theta_{2(n+r-t)+1}\left(z_{2(n+r-t)+1}\right) = 0. \tag{33}$$

Hence, $[\theta_{2(n+r-t)+1}]$ are the nonzero homology classes in $H_*(\mathrm{Der}(\wedge V, B; \phi))$. Moreover, $H(\varepsilon_*)([\theta_{2(n+r-t)+1}]) = [z^*_{2(n+r-t)+1}]$, where $z^*_{2(n+r-t)+1} \in \mathrm{Der}(\wedge V, \mathbb{Q}, \varepsilon)$. Thus, $G_*(\wedge V, B; \phi) = \langle [z^*_{2(n+r-k)+1}] \ldots, [z^*_{2(n+r)-1}] \rangle$.

Theorem 3. *Given the inclusion* $G_{k,n}(\mathbb{C}) \rightarrowtail G_{k,n+r}(\mathbb{C}), r \ge 1$ *for* $2 \le k < n$ *and* $\phi\colon (\wedge V, d) \longrightarrow (B, d)$ *for its Sullivan model, then* $G^{rel}_*(\wedge V, B; \phi) = \langle [(y^*_{2(n-k)+1}, 0)], [(0, z^*_{2(n+r)-1})] \rangle$.

Proof. Define the derivations $\alpha_{2(n-t)+1} = (y_{2(n-t)+1}, 1)$ for $t \in \{1, \ldots, k\}$ in $\mathrm{Der}(B, B; 1)$ and $\theta_{2(n+r-t)+1} = (z_{2(n+r-t)+1}, 1)$ in $\mathrm{Der}(\wedge V, B; \phi)$. Then,

$$\phi^*\left(\alpha_{2(n-t)+1}\right) = \begin{cases} \theta_{2(n+r-t)+1}, & \text{if } 2(n-t)+1 = 2(n+r-t)+1, \\ 0, & \text{if } 2(n-t)+1 \ne 2(n+r-t)+1, \end{cases} \tag{34}$$

such that

$$D\left(\alpha_{2(n-t)+1}, 0\right) = \begin{cases} \left(0, \theta_{2(n+r-t)+1}\right), & \text{if } 2(n-t)+1 = 2(n+r-t)+1, \\ (0,0), & \text{if } 2(n-t)+1 \neq 2(n+r-t)+1, \end{cases} \tag{35}$$

and $D(0, \theta_{2(n+r-t)+1}) = (0,0)$. Thus, $[(\alpha_{2(n-t)+1}, 0)]$ if $2(n-t)+1 \neq 2(n+r-t)+1$ and $[(0, \theta_{2(n+r-t)+1})]$ are the nonzero homology classes in $H_*(\mathrm{Rel}(\phi^*))$. Moreover, $H(\varepsilon_*, \varepsilon_*)([(\alpha_{2(n-t)+1}, 0)]) = [(y^*_{2(n-t)+1}, 0)]$, for $2(n-t)+1 \neq 2(n+r-t)+1$, and in the same way, $H(\varepsilon_*, \varepsilon_*)([(0, \theta_{2(n+r-t)+1})]) = [(0, z^*_{2(n+r-t)+1})]$. A straightforward calculation shows that $[(y^*_{2(n-t)+1}, 0)]$ for $2(n-t)+1 \neq 2(n+r-t)+1$ and $[(0, z^*_{2(n+r-t)+1})]$ span $H(\varepsilon_*, \varepsilon_*)$.

The G-sequence reduces to

$$\begin{aligned} &0 \longrightarrow G_{2(n+r)-1}(\wedge V, B; \phi) \underset{\simeq}{\xrightarrow{H(J)}} G^{\mathrm{rel}}_{2(n+r)-1}(\wedge V, B; \phi) \longrightarrow 0, \\ &0 \longrightarrow G_{2(n-t)+1=2(n+r-t)+1}(B) \underset{\simeq}{\xrightarrow{H(\phi^*)}} G_{2(n-t)+1=2(n+r-t)+1}(\wedge V, B; \phi) \longrightarrow 0, \\ &0 \longrightarrow G^{\mathrm{rel}}_{2(n-k)+1}(\wedge V, B; \phi) \underset{\simeq}{\xrightarrow{H(P)}} G_{2(n-k)+1}(B) \longrightarrow 0, \end{aligned} \tag{36}$$

and it is exact.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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