Research Article

On the Determinants of the Square-Type Stirling Matrix and Bell Matrix

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We study determinants of the square-type Stirling matrix \( S^* \) and the square-type Bell matrix \( B^* \). For this purpose, we prove that \( S^* \) and \( B^* \) have LU factorizations \( S^* = L_S U_S \) and \( B^* = L_B U_B \) where the diagonal entries of \( U_S \) are \( k^{k-1} \), while those of \( U_B \) are \( k! \) (\( k \geq 1 \)).

1. Introduction

The Stirling numbers \( s_{m,n} \) (\( m, n \geq 0 \)) of second kind count the number of ways to partition an \( m \) element set into \( n \) subsets. The Stirling matrix \( S = [s_{m,n}] \) satisfies a recurrence rule \( s_{m+1,n+1} = s_{m,n} + (n+1)s_{m,n+1} \) [1]. The sum \( \sum_{k=0}^{n} s_{m,k} \) of the \( m^{th} \) row of \( \bar{S} \) is called the \( m^{th} \) Bell number \( B(m) \), so \( \{B(m) \mid m \geq 0\} = \{1, 1, 2, 5, 15, \ldots \} \). A triangular matrix \( B = [b_{i,j}] \) having Bell numbers on both border and holding \( b_{i+1,j+1} = b_{ij} + b_{i+1,j} \) (\( i, j \geq 1 \)) is called the Bell matrix [2].

Since every entries in the first row and column of \( \bar{S} \) are zeros except \( s_{0,0} = 1 \), we denote by \( S \) the Stirling matrix deleted in the first row and column from \( \bar{S} \).

\[
S = \begin{bmatrix}
1 \\
1 & 1 \\
1 & 3 & 1 \\
1 & 7 & 6 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\] (1)

\[
B = \begin{bmatrix}
1 \\
1 & 2 \\
2 & 3 & 5 \\
5 & 7 & 10 & 15 \\
15 & 20 & 27 & 37 & 52 \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\( i, j \geq 1. \)
Let

\[
S^* = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 6 & 10 \\
1 & 7 & 25 & 65 \\
1 & 15 & 90 & 350 \\
\vdots & & & \\
1 & 2 & 5 & 15 \\
1 & 3 & 10 & 37 \\
2 & 7 & 27 & 114 \\
5 & 20 & 87 & 409 \\
\vdots & & & 
\end{bmatrix},
\]

be a square-type Stirling matrix and a square-type Bell matrix. In the work, we study the determinants of \(S^*\) and \(B^*\) by finding their LU factorizations. In fact, we prove that \(S^*\) has an LU factorization \(S^* = L_SU_S\), where \(L_S = S\) and \(U_S\) has diagonal entries \(1, 2, 3^2, 4^3, \ldots\) (Theorem 2), and \(B^*\) has an LU factorization \(B^* = L_BU_B\) where \(L_B = SP\) along with the Pascal matrix \(P\) and \(U_B\) has diagonal entries \(1, 1, 2!, 3!, \ldots\) (Theorem 12). This consideration is motivated by the square-type Pascal matrix

\[
P^* = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 3 & 6 \\
\vdots & & 
\end{bmatrix},
\]

where its LU factorization is \(P^* = PPP^T\) [3] so \(\det P^* = 1\). Note that \(\det B^*_n\) was discussed in [4] by means of Hankel transformation. Our feature in the work is to study \(\det B^*_n\) by recurrence rules of Bell numbers over an LU factorization of \(B^*_n\). For our notations, with a matrix \(M, M_k\) denotes the size \(k \times k\), and let \(r_i(M_k)\) and \(c_j(M_k)\) be the \(i^{th}\) row and \(j^{th}\) column of \(M_k\). Write \((x, \ldots, x, \eta, b, \ldots)\) and \((a, b, \ldots, x, \ldots, x)\) simply by \((x'; a, b, \ldots)\) and \((a, b, \ldots; x')\) with the \(t\) copies \(x'\) of \(x\). Therefore, \((x'; r_i(M_k))\) means a row matrix having \(t\) \(x\)'s followed by a row matrix \(r_i(M_k)\), and similarly, \((x'; c_j(M_k))\) means a column matrix

\[
\begin{bmatrix}
x \\
\vdots \\
x \\
\end{bmatrix}
\begin{bmatrix}
c_j(M_k)
\end{bmatrix}
\]

diagonal entries \(a, b, \ldots\).

### 2. Square-Type Stirling Matrix

For \(i, j \geq 1\), let \(r_i(S)\) and \(c_j(S)\) (resp., \(r_j(S^{-1})\) and \(c_j(S^{-1})\)) be the \(i^{th}\) row and \(j^{th}\) column of \(S\) (resp., \(S^{-1}\)). Since \(S\) and \(S^{-1}\) are lower triangular matrices, \(r_i(S)\) and \(r_j(S^{-1})\) can be considered as of size \(1 \times i\), while \(c_j(S)\) and \(c_j(S^{-1})\) are of size \(i \times 1\). But, if necessary, like the case of multiplication \(r_i(S)c_j(S)\), we may regard \(r_i(S)\) filled with infinitely many zeros after the first \(i\) entries.

**Lemma 1.** Let \(S = [s_{ij}]\), \(S^{-1} = [s_{ij}']\), and \(S^* = [s_{ij}^*]\) for \(i, j \geq 1\). Let \(r_i(S^*)\) and \(c_j(S^*)\) be the \(i^{th}\) row and \(j^{th}\) column of \(S^*\).

(1) In \(S\), \(s_{i+1,j+1} = s_{ij} + (j + 1)s_{i, j+1}\) and \(c_{j+1} = 0; c_j(S)) + (j + 1)(0; c_{j+1}(S)).

(2) In \(S^{-1}\), \(s_{i-1,j+1} = s_{ij} - is_{i+1,j}^\prime\). Thus, \(s_{i+1,j}^\prime = (-1)^{i}j!\)

(3) In \(S^\prime\), \(s_{i,j}^\prime = s_{i+1,j+1}^\prime + j!s_{i-1,j+1}^\prime\). \(c_j(S^\prime) = c_j(S)\) \(+ j!s_{i,j}^\prime\).

**Proof.** The recurrence in (1) is well known [5]. The column \(c_j(S)\) is

\[
c_j(S) = \left(0, 0, s_{j+1,j+1}, s_{j+2,j+1}, s_{j+3,j+1}, \ldots\right)^T
\]

\[
= \left(0, s_{j,j} + (j + 1)s_{j+1,j+1} + (j + 1)s_{j+2,j+1} + \ldots\right)^T
\]

\[
= \left(0, s_{j,j} + (j + 1)\right) + (j + 1)(0; c_{j+1}(S)).
\]

The inverse \(S^{-1}\) is the signed Stirling matrix of first kind [6] satisfying the recurrence \(s_{i+1,j+1}^\prime = s_{ij}^\prime - is_{i+1,j}^\prime\). From

\[
I = S^{-1}_n S_n = [r_i(S^{-1})c_j(S_n)], \quad \text{we have} \quad 0 = r_i(S^{-1})c_j(S) = \sum_{j=1}^i s_{i,j}^\prime \quad \text{for} \ i \geq 2, \text{since} \ c_j(S) \text{is composed of all 1s}. \quad \text{Moreover, simple computation of} \ S^{-1} \text{shows} \ s_{i,j}^\prime (1 \leq i \leq 5) \text{equals} 1, -1, 2, -3!, 4!, \text{respectively}. \text{Hence, if we assume} \ s_{i+1,i}^\prime = (-1)^i i! \text{for some} \ i, \text{then} \ s_{i+2,i+1}^\prime = s_{i+1,i+1}^\prime - (i + 1)s_{i+1,i+1}^\prime = -(i + 1)(-1)^i i! = (-1)^{i+1} (i + 1)i!.
\]

Comparing

\[
S^* = \begin{bmatrix}
s_{1,1} & s_{1,2} & s_{1,3} \\
s_{2,1} & s_{2,2} & s_{2,3} \\
s_{3,1} & s_{3,2} & s_{3,3} \\
\vdots & & & \ddots
\end{bmatrix}
\]

with \(S = [s_{ij}]\), it is easy to see \(s_{ij}^* = s_{i+1,j-1}^\prime + j!s_{i-1,j}^\prime\). Now, for the \(j^{th}\) column \(c_j(S^\prime)\), we have
\[ c_j(S^*) = c_{j-1}(S^*) + \sum_{i=0}^{k} j_i(0; c_j(S^*)) \]
\[ = c_{j-1}(S^*) + j(0; c_{j-1}(S^*)) + \sum_{i=0}^{k} j^2(0^2; c_{j-1}(S^*)) + j^3(0^3; c_{j-1}(S^*)) \]
\[ = \cdots = \sum_{i=0}^{k-1} j_i(0; c_{j-1}(S^*)) \]
\[ = \cdots = \sum_{i=0}^{\infty} j_i(0; c_{j-1}(S^*)) \]

**Theorem 1.** For all \( 0 \leq t < j \),
\[ r_j(S^{-1})(\bar{0}^{-1}; c_j(S^*)) = 0 \]
\[ \text{when } t = 2, \text{ by Lemma 1 (3), we have } S^{-1}S = I. \]

**Proof.** Note that \( c_j(S^*) = (s_{j,j}, s_{j,j+1}, s_{j,j+2}, \ldots)^T = (s_{j,1}, s_{j+1, j}, s_{j+2,j}, \ldots)^T \) and \( c_j(S) = (\bar{0}^{-1}; c_j(S^*)) \). Hence, we have
\[ r_j(S^{-1})(\bar{0}^{-1}; c_j(S^*)) = r_j(S^{-1})(\bar{0}^{-1}; c_{j-1}(S^*) + j(0; c_j(S^*)]) \]
\[ = r_j(S^{-1})(\bar{0}^{-1}; c_{j-1}(S^*) + j_r(S^{-1})(\bar{0}^{-1}; c_j(S^*)) \]
\[ = r_j(S^{-1})c_{j-1}(S) + j_r(S^{-1})c_j(S) = 0. \]

Thus, by assuming \( r_j(S^{-1})(\bar{0}^{-1}; c_j(S^*)) = 0 \) for \( 1 \leq t < j - 2 \), we have
\[ r_j(S^{-1})(\bar{0}; c_j(S^*)) = r_j(S^{-1})(\bar{0}; c_{j-1}(S^*) + j(0; c_j(S^*)]) = 0. \]

**Theorem 2.** \( S^* \) has an LU factorization \( SX \), where \( X \) is an upper triangular matrix having diagonal entries \( \bar{r}^{-1} \) (\( i \geq 1 \)). Therefore, \( \det S^*_k = \prod_{i=1}^{k} \bar{r}^{-1} \).

\[ x_{i,j-1} = r_j(S^{-1})c_{i-1}(S^*) = r_j(S^{-1})(c_{i-2}(S^*) + (i-1)(0; c_{i-1}(S^*))) \]
\[ = x_{i,j-2} + (i-1)r_j(S^{-1})(0; c_{i-2}(S^*)) = 0, \]
which shows \( X \) is an upper triangular matrix. Now, for \( x_{i,j} \), we have
\[ x_{i,j} = r_j(S^{-1})c_j(S^*) \]
\[ = r_j(S^{-1})(c_{j-1}(S^*) + i(0; c_{j-1}(S^*))) + \cdots + i^{j-2}(0^{j-2}; c_{j-1}(S^*)) + i^{j-1}(0^{j-1}; c_j(S^*)) \]
\[ = r_j(S^{-1})c_{j-1}(S^*) + i_r(S^{-1})(0; c_{j-1}(S^*)) + \cdots + i^{j-2}r_j(S^{-1})(0^{j-2}; c_{j-1}(S^*)) + i^{j-1}r_j(S^{-1})(0^{j-1}; c_j(S^*)) \]
\[ = i^{-1}r_j(S^{-1})(\bar{0}^{-1}; 1) = i^{-1}. \]
Indeed,
\[ S^* = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 3 & 6 & 10 \\
1 & 7 & 25 & 65 \\
1 & 15 & 90 & 350
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 2 & 5 & 9 \\
1 & 3 & 1 & 3^2 & 37 \\
1 & 7 & 6 & 1 & 4^3
\end{bmatrix}. \]

(11)

3. Bell Matrix with the Pascal Matrix

Let \( r_i(P) \) and \( c_j(P) \) be the \( i \)th row and \( j \)th column of the Pascal matrix \( P = [p_{ij}] \), \( i, j \geq 1 \). Well-known recurrence rules of \( r_i(P) \) and \( c_j(P) \) are

\[
j c_{j+1}(P_k) = j \left( \binom{j}{0} \left( \binom{j+1}{1} \left( \binom{j+2}{2} \cdots \binom{k-1}{j} \right) \right) + \binom{j+1}{j+1} \left( \binom{j+2}{j+1} \cdots \binom{k-1}{j+1} \right) \right)
\]

\[
= j \left( \binom{j}{0} \left( \binom{j+1}{1} \left( \binom{j+2}{2} \cdots \binom{k-1}{j} \right) \right) + \binom{j+1}{j+1} \left( \binom{j+2}{j+1} \cdots \binom{k-1}{j+1} \right) \right)
\]

\[
= di[0, \ldots, j-1; j, \ldots, k-1] \left( 0, 0^{j-1}, \binom{j-1}{j-1}, \binom{j}{j-1}, \binom{j+1}{j-1}, \cdots, \binom{k-2}{j-1} \right)
\]

(13)

Clearly, \( r_1(P) di[2^1, 2, 1] = (1, 2, 1) \). Assume \( r_k(P) P_k = r_k(P) di[2^{k-1}, \ldots, 1] \) for some \( k \). Note that \( r_1(P) \) is the set of coefficients of \( (x+1)^{i-1} \) and \( r_i(P) P_i \) equals \( r_i(P^2) \) which is the set of coefficients of \( (2x+1)^{i-1} \) expanded in descending order. Thus, (12) with \( (2x+1)^k = 2x(2x+1)^{k-1} + (2x+1)^{k-1} \) implies

\[
r_{k+1}(P) P_{k+1} = \text{the set of coeff. of } (2x+1)^k
\]

\[
= \text{the set of coeff. of } 2x(2x+1)^{k-1} + \text{the set of coeff. of } (2x+1)^{k-1}
\]

\[
= 2(r_k(P) P_k; 0) + (0; r_k(P) P_k)
\]

\[
= (2r_k(P) di[2^{k-1}, \ldots, 1]; 0) + (0; r_k(P) di[2^{k-1}, \ldots, 1])
\]

(14)

\[
= (r_k(P); 0) \begin{bmatrix}
2^k & 2^k \\
\vdots & \ddots & \ddots \\
1 & \cdots & 1
\end{bmatrix} + (0; r_k(P)) \begin{bmatrix}
2^k \\
\vdots \\
1
\end{bmatrix}
\]

\[
= ((r_k(P); 0) + (0; r_k(P))) di[2^k, \ldots, 1]
\]

Finally, for any \( 1 \leq j \leq k \), \( di[0, 1, \ldots, k-1] \) \( (0; c_{j_1} P_k) \) and \( r_k(P) di[2^{k-1}, \ldots, 1] = r_k(P) P_k \).

\[
\text{Proof: Due to the binomial identity } p^\binom{p-1}{q-1} = q^\binom{p}{q} \text{ for } p, q \geq 1, \text{ we have }
\]

\[
r_{i+1}(P) = (r_i(P); 0) + (0; r_i(P)), \
c_{j+1}(P) = (0; c_j(P)) + (0; c_{j+1}(P)).
\]

Theorem 3.
A matrix $P_k^F = [p_{i,j}]$ is called a flipped matrix of $P_k = [p_{i,j}]$ if it is horizontally flipped sideways of $P_k$. Hence,

$$
p_k^F = \begin{bmatrix}
p_{1,k} & \cdots & p_{1,1} \\
p_{2,k} & \cdots & p_{2,1} \\
\vdots & & \vdots \\
p_{k,k} & \cdots & p_{k,1}
\end{bmatrix}, \quad \text{for } 1 \leq j \leq k.
$$  \hfill (15)

**Theorem 4.** Let $r_i(P_k^F P_k)$ be the $i$th row of $P_k^F P_k$ for $1 \leq i \leq k$. Then,

(1) $r_1(P_k^F P_k) = r_k(P_k)$ and $r_k(P_k^F P_k) = r_k(P_k) P_k$

(2) $r_i(P_k^F P_k) = r_{i-1}(P_k^F P_k) + (r_{i-1}(P_k^F P_k) ; 0) = r_k(P_k) + \sum_{j=1}^{i-1} (r_j(P_k^F P_k) ; 0)$

Proof. Clearly, $r_1(P_k^F P_k) = (0^k ; 1) P_k = r_k(P_k)$ and $r_k(P_k^F P_k) = r_k(P_k^F) P_k = r_k(P_k) P_k$. And, for $1 \leq i < k$, we have

$$r_i(P_k^F P_k) + (r_{i-1}(P_k^F P_k) ; 0) = (r_i(P_k^F)) + (r_{i-1}(P_k^F) ; 0) P_k = r_{i+1}(P_k^F P_k).$$  \hfill (16)

Thus, it follows immediately that

---

We now develop some interrelations of the Bell matrix $B = [b_{i,j}]$, square-type Bell matrix $B^* = [b_{i,j}^*]$, and Pascal matrix $P$.

**Theorem 5.** Let $c_j(B^*)$ be the $j$th column of $B^*$ for $i, j \leq k$. Then,

(1) $r_{k+1}(P) = b_{k+1,k}$, so $P = b_{k+1,k}$

(2) $P_k c_j(B^*) = b_{j+1,1}$, so $r_j(P_k) c_j(B^*) = b_{j+1,1}$

(3) $P_k^F c_j(B^*) = b_{m+1,1}$, so $r_j(P_k^F) c_j(B^*) = b_{m+1,1}$

---

\[\begin{array}{c}
\begin{bmatrix}
b_{i,j} \\
b_{i+1,j} \\
\ldots \\
b_{i+k,j}
\end{bmatrix} = b_{i+k,j+k}, \text{ so } P = b_{i+k,j+k}.
\end{array}\]

\[\begin{array}{c}
\begin{bmatrix}
b_{j+1,1} \\
b_{j+2,1} \\
\ldots \\
b_{j+k,1}
\end{bmatrix} = b_{j+k,1}.
\end{array}\]

\[\begin{array}{c}
\begin{bmatrix}
b_{m+1,1} \\
b_{m+2,1} \\
\ldots \\
b_{m+k,1}
\end{bmatrix} = b_{m+k,1}.
\end{array}\]
Proof

Clearly, $b_{i+1,j+1} = r_2(P) \begin{bmatrix} b_{i,j} \\ b_{i+1,j} \end{bmatrix}$,

$b_{i+2,j+2} = b_{i,j} + 2b_{i+1,j} + b_{i+2,j} = r_3(P) \begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ b_{i+2,j} \end{bmatrix}$.

So if we assume

$r_k(P) \begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ \cdots \\ b_{i+(k-1),j} \end{bmatrix} = r_{k+1}(P) \begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ \cdots \\ b_{i+k,j} \end{bmatrix}$

for some $k$, then

$r_k(P) \begin{bmatrix} b_{i,j} \\ b_{i+1,j} \\ \cdots \\ b_{i+(k-1),j} \end{bmatrix} = b_{i+(k-1),j+(k-1)} + b_{i+k,j+(k-1)} = b_{i+k,j+k}$.

by recurrence (12). Comparing $B^* = \{b^*_{i,j}\}$ with $B = \{b_{i,j}\}$, we have

$\begin{align*}
\begin{bmatrix}
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\cdot & \cdot & \cdots & \cdot \\
\end{bmatrix}
\end{align*}$

Thus, with $c_j(B^*) = \begin{bmatrix} b_{j+1,j} \\ b_{j+2,j} \\ \cdots \end{bmatrix}$, (1) implies

Moreover, (1) gives rise to (3) such that

Theorem 6. $b^*_{i,j} = \begin{cases} b_{i,j+1}^* + r_{j-1}(P)c_1(B^*), & \text{if } j > 1, \\ b_{i,1}^*, & \text{if } j = 1. \end{cases}$

Proof. Clearly, $b^*_{i,j} = b_{i,j} = r_{j-1}(P)c_1(B^*)$ by Theorem 5 (2).

And, observe $b^*_{3,3} (1 < j \leq 3) from $B_3^* = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 10 \\ 2 & 7 & 27 \end{bmatrix}$ such that

$\begin{align*}
b_{3,2}^* &= 7 = 2 + (1, 1)(2, 3)^T = b_{3,3}^* + r_1(P^TP_3)(b_{3,3}^*, b_{2,3}^*)^T \\
b_{3,3}^* &= 27 = 7 + (2, 1)(5, 10)^T = b_{3,2}^* + r_2(P^TP_3)(b_{3,3}^*, b_{2,3}^*)^T.
\end{align*}$

From $B_4^* = \begin{bmatrix} b_{4,1}^* \\ b_{4,2}^* \\ b_{4,3}^* \\ b_{4,4}^* \end{bmatrix} = \begin{bmatrix} 15 \\ 37 \\ 114 \\ 409 \end{bmatrix}$, $b_{4,j}^* (1 < j \leq 4)$ satisfy $b_{4,2}^* = 20 = 5 + (1, 2, 1)(2, 3, 7)^T = b_{3,1}^* + r_1(P^TP_3)(b_{3,1}^*, b_{2,1}^*, b_{1,2}^*, b_{2,2}^*, b_{3,2}^*, b_{2,3}^*)^T$, $b_{4,3}^* = 87 = 20 + (2, 3, 1)(5, 10, 27)^T = b_{3,1}^* + r_2(P^TP_3)(b_{3,1}^*, b_{2,1}^*, b_{1,2}^*, b_{2,2}^*, b_{3,2}^*, b_{2,3}^*)^T$, and $b_{4,4}^* = 409 = 87 + (4, 4, 1)(15, 37, 114)^T = b_{3,1}^* + r_3(P^TP_3)(b_{3,1}^*, b_{2,1}^*, b_{1,2}^*, b_{2,2}^*, b_{3,2}^*, b_{2,3}^*)^T$. 

$\Box$
Proof. Let $U_k = [u_{i,j}]$ \((1 \leq i, j \leq k)\) be with
\[
u_{1,j} = b_{1,j}^*, \\
u_{2,j} = b_{2,j}^* - u_{1,j},
\]
for all \(j \geq 1\). \hfill (28)
Then, \(B_2^* = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\) yields \(U_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\). And, \(B_2^* U_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) is a lower triangular matrix; denote it by \(L_2\).

Now, for some \(i\), we assume \(b_{1,i}^* = b_{1,i-1} + r_{i-1} (p_{i-1}^T p_{i-1}) c_j (B^*) \) for all \(1 \leq j < i\). Then, \(b_{i+1,j}^* = b_{i+1,j-1} - b_{i,j}^*\) equals
\[
b_{i+1,j}^* = (b_{i,j}^* + r_{i-1} (p_{i-1}^T p_{i-1}) c_{j+1} (B^*)) - (b_{i,j-1}^* + r_{i-1} (p_{i-1}^T p_{i-1}) c_j (B^*))
= (b_{i,j}^* - b_{i,j-1}^*) + (r_{i-1} (p_{i-1}^T p_{i-1}) c_{j+1} (B^*)) - r_{i-1} (p_{i-1}^T p_{i-1}) c_j (B^*)
\] (25)

But, since
\[
r_j (p_{i-1}^T p_{i-1}) c_{j+1} (B^*) = r_j (p_{i-1}^T p_{i-1}) c_{j+1} (B^*) = r_j (p_{i-1}^T p_{i-1}) = b_{i+1,j}
\] (26)
by Theorem 5, we have
\[
b_{i+1,j}^* = (b_{i,j}^* - b_{i,j-1}^*) + (b_{i+1,j} - b_{i+1,j-1})
= b_{i+1,j}^* + b_{i+1,j-1} = b_{i+1,j}^* + r_{j-1} (p_{j-1}^T p_{j-1}) = b_{i+1,j}^* + r_{j-1} (p_{j-1}^T p_{j-1}) c_j (B^*)
\] (27)

4. LU Factorization of the Square-Type Bell Matrix

We are ready to have an LU factorization of \(B_k^* = [b_{i,j}^*]\) with diagonal entries.

Theorem 7. \(B_k^* = L_k U_k\) \((1 \leq k \leq 5)\) where the lower triangular matrix \(L_k = \bar{S}_k P_k\) and the upper triangular matrix \(U_k\) has diagonal entries \([1, 1, 2, 1, 3, 1]\).

Proof. Let \(U_k = [u_{i,j}]\) \((1 \leq i, j \leq k)\) be with
\[
u_{1,j} = b_{1,j}^*, \\
u_{2,j} = b_{2,j}^* - u_{1,j},
\]
for all \(j \geq 1\). \hfill (28)
Then, \(U_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\) yields \(U_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\). And, \(B_2^* U_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) is a lower triangular matrix; denote it by \(L_2\).

From \(B_3^* = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 7 & 27 \end{bmatrix}\), clearly \(u_{1,3} = b_{1,3}^*\) and
\[
u_{2,3} = b_{2,3}^* - u_{1,3} = 5\) by (28). Let \(u_{3,j} = [b_{3,j}^* - r_{j-1} (p_{j-1}^T p_{j-1}) c_j (B^*)] - 2u_{2,j}\), for all \(j \geq 1\). \hfill (29)

Then, the identity \(b_{k+2,1}^* = r_{k+1} (p_1) c_1 (B^*)\) in Theorem 5 (2) implies \(b_{1,1}^* = r_{k} (p_1) (b_{1,1}^*, b_{2,1}^*)^T\), so \(u_{1,1} = 0\) because \(u_{2,1} = 0\).
Similarly, \(u_{3,2} = [b_{3,2}^* - r_{j-1} (p_{j-1}^T p_{j-1}) (b_{1,2}^*, b_{2,2}^*)^T - 2u_{2,2}] = 2 - 2 \cdot 1 = 0\). \(u_{3,3} = [b_{3,3}^* - r_{j-1} (p_{j-1}^T p_{j-1}) (b_{1,3}^*, b_{2,3}^*)^T - 2u_{2,3}] = 12 - 2 \cdot 5 = 2\), so
\[
U_3 = \begin{bmatrix} U_2 \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 5 \\ 2 \end{bmatrix},
\] (30)

\(B_3^* U_3^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 \end{bmatrix}\) is \(\bar{S}_3 P_3\).
is a lower triangular matrix and denote it by $L_3$.

From $B_4^* = \begin{bmatrix} 15 \\ 37 \\ 114 \\ 5,20,87,409 \end{bmatrix}$, (28), and (29), we have
\[
\begin{align*}
 u_{1,4} &= b_{4,1}^* - u_{1,4} = 22, & u_{2,4} &= b_{4,1}^* - u_{2,4} = 22, & u_{3,4} &= [b_{3,4}^* - r_2(P) (b_{3,4}^*)^T] - 2u_{2,4} = 18. \\
\text{Now, let} & \\
 u_{4,j} &= [b_{4,j}^* - r_3(P) c_j(B^*)] - (5,5)(u_{2,j}, u_{3,j})^T, \text{ for all } j \geq 1.
\end{align*}
\]
\[(31)\]

Since $b_{4,1}^* = r_3(P) c_1(B^*)$ and $u_{2,1} = u_{3,1} = 0$, we have $u_{4,1} = 0$, $u_{4,2} = [b_{4,2}^* - r_3(P) c_2(B^*)] - (5,5)(u_{2,2}, u_{3,2})^T = 5 - (5,5)(1,0)^T = 0$, $u_{4,3} = [b_{4,3}^* - r_3(P) c_3(B^*)] - (5,5)(u_{2,3}, u_{3,3})^T = 35 - (5,5)(5,2)^T = 0$, and $u_{4,4} = [b_{4,4}^* - r_3(P) c_4(B^*)] - (5,5)(u_{2,4}, u_{3,4})^T = 6$. Therefore,
\[
U_4 = \begin{bmatrix} 15 \\ 22 \\ 18 \\ 6 \end{bmatrix}
\]
\[
B_4^* U_4^{-1} = \begin{bmatrix} L_3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{S}_{_5} P_4.
\]
\[(32)\]

Denote it by $L_4$.

From
\[
B_5^* = \begin{bmatrix} 152 \\ 523 \\ 2066 \\ 9089 \end{bmatrix}
\]
\[(33)\]

(28), (29), and (31), give $u_{4,5} = b_{4,5}^* - u_{1,5} = 99$, $u_{4,5} = [b_{5,5}^* - r_3(P) c_5(B^*)] - 2u_{2,5} = 122$, and $u_{4,5} = [b_{6,5}^* - r_3(P) c_5(B^*)] - (5,5)(u_{2,5}, u_{3,5})^T = 84$. Now, let
\[
W_{5,j} = [b_{5,j}^* - r_4(P) c_j(B^*)] - 3[b_{4,j}^* - r_3(P) c_j(B^*)],
\]
\[(34)\]

and let $u_{6,j} = W_{5,j} - (7,6)(u_{5,j}, u_{4,j})^T$ for $j \geq 1$.

Since $u_{3,1} = u_{4,1} = 0$, $b_{3,1}^* = r_3(P) c_1(B^*)$, and $b_{5,1}^* = r_4(P) c_1(B^*)$, we have $u_{5,1} = 0$. Also, $u_{5,2} = u_{4,2} = u_{4,3} = 0$ and $u_{3,3} = 2$ imply $u_{5,2} = W_{5,2} = (7,6)(u_{1,2}, u_{1,3})^T = 15 - 3 \cdot 5 = 0$, $u_{5,3} = W_{5,3} - (7,6)(u_{1,3}, u_{1,4})^T = 0 = W_{5,4} - (7,6)(18,6)^T = u_{5,4}$, and $u_{5,5} = W_{5,5} - (7,6)(122,84)^T = 4!$. Hence,
\[
B_5^* U_5^{-1} = \begin{bmatrix} 52 \\ 122 \\ 84 \\ 0 \end{bmatrix} = \mathbf{S}_{_5} P_5.
\]
\[(35)\]

Denote it by $L_5$.

Note $u_{5,j} = (j - 1)!$ for $1 \leq j \leq 5$. From (34), we let
\[
W_{5,j} = \left[ b_{5,j}^* - r_{j-1}(P) c_j(B^*) \right] - 3 \left[ b_{4,j-1}^* - r_{j-2}(P) c_j(B^*) \right],
\]
\[(36)\]

**Theorem 8.** Assume the matrix $U_k$ in Theorem 7 further satisfies $u_{6,j} = W_{6,j} - \Gamma_4(u_{5,j}, \ldots, u_{6,j})^T$ and $u_{6,j} = W_{5,j} - \Gamma_5(u_{5,j}, \ldots, u_{6,j})^T$ with row matrices $\Gamma_4 = (7,33,34,11)$ and $\Gamma_5 = (47,174,202,93,17)$. Then, for $1 \leq k \leq 7$, $U_k$ is an upper triangular matrix having $u_{6,k} = (k - 1)!$ and $L_k = B_k^* U_k^{-1}$ is a lower triangular matrix such that $L_k = \mathbf{S}_k P_k$.

**Proof.**

\[
B_6^* = \begin{bmatrix} 203 \\ 674 \\ 2589 \\ 52922 \\ 11155 \\ 43833 \\ 27774 \end{bmatrix}
\]
\[(37)\]

(28), ..., (34), show $u_{1,6} = b_{1,6}^*$, $u_{2,6} = 471$, $u_{3,6} = 770$, $u_{4,6} = 810$, and $u_{6,6} = 480$. Let
\[
U_{6,j} = W_{6,j} - (t_1, t_2, t_3, t_4)(u_{5,j}, u_{4,j}, u_{3,j}, u_{2,j})^T,
\]
\[(38)\]

for some $t_1 \in \mathbb{Z}$.

Note from Theorem 5 that $u_{2,1} = u_{3,1} = u_{4,1} = u_{5,1} = 0$, $b_{5,1}^* = r_3(P) c_1(B^*)$, and $b_{6,1}^* = r_4(P) c_1(B^*)$. Thus, $u_{6,1} = 0$ by (38). And, we also observe $u_{6,2} = W_{6,2} = (t_1, t_2, t_3, t_4)(u_{5,2}, u_{5,3}, u_{5,4}, u_{5,5})^T = 52 - 3 \cdot 15 - (t_1, t_2, t_3, t_4) (1; 0)^T$, so $u_{6,2} = 0$ if $t_1 = 7$. Similarly, since $W_{6,3} = 458 - 3 \cdot 119$, $W_{6,4} = 3292 - 3 \cdot 780$, and $W_{6,5} = 22686 - 3 \cdot 4949$ from (36), in order to be $u_{6,3}, u_{6,4}$, and $u_{6,5}$ all zeros, the identities
\[ u_{6,3} = W_{6,3} - (t_1, \ldots, t_4)(u_{2,3}, \ldots, u_{5,3})^T = 101 - (7, t_2, t_3, t_4)(5, 2; \overline{1})^T, \]
\[ u_{6,4} = W_{6,4} - (t_1, \ldots, t_4)(u_{2,4}, \ldots, u_{5,4})^T = 952 - (7, t_2, t_3, t_4)(22, 18, 6, 0)^T, \]
\[ u_{6,5} = W_{6,5} - (t_1, \ldots, t_4)(u_{2,5}, \ldots, u_{5,5})^T = 7839 - (7, t_2, t_3, t_4)(99, 122, 84, 24)^T, \]

yield \( t_2 = 33, \ t_3 = 34, \) and \( t_4 = 11. \) Thus, with \( W_{6,6} = 156972 - 3 \cdot 31775 \) in (36), we have \( u_{6,6} = W_{6,6} - (7, 33, 34, 11)(u_{2,6}, \ldots, u_{5,6})^T = 5! \).

Hence,

\[
U_6 = \begin{bmatrix}
203 \\
471 \\
U_5 \\
770 \\
810 \\
480 \\
\overline{0}^5 \\
5! \\
\end{bmatrix},
\]

\[
L_6 = B_6^* U_6^{-1} = \begin{bmatrix}
L_5 & 0 \\
52, 151, 160, 75, 15 & 1 \\
\end{bmatrix} = \overline{S}_6 P_6.
\]

Similarly, from

\[
B_7^* = \begin{bmatrix}
B_6^* \\
877 \\
3263 \\
13744 \\
64077 \\
325869 \\
1788850 \\
203, 1080, 6097, 36401, 229114, 1515903, 10515147 \\
\end{bmatrix}
\]

we get \( u_{7,1} = 0 \) for some \( t_j \in Z. \)

Since \( u_{i,1} = 0 \) \( (1 \leq i \leq 6), \) \( b_{7,1}^* = r_6(P)c_1(B^*), \) and \( b_{6,1}^* = r_5(P)c_1(B^*), \) we also have \( u_{7,1} = 0. \) In order to get \( u_{7,j} = 0 \) \( (2 \leq j \leq 6), \) the integers \( t_i (1 \leq i \leq 6) \) are determined as follows: Note that \( W_{7,2} = 203 - 3 \cdot 52 \) from (36), so \( 0 = u_{7,2} = W_{7,2} - (t_1, \ldots, t_4)(1; \overline{0})^T \) implies \( t_1 = 47. \)

Analogously, with \( W_{7,3} = 1957 - 3 \cdot 458, \) \( W_{7,4} = 15254 - 3 \cdot 3292, \) \( W_{7,5} = 113139 - 3 \cdot 22686, \) and \( W_{7,6} = 837333 - 3 \cdot 156972 \) from (36), we also have

\[
\begin{align*}
u_{7,3} &= W_{7,3} - (47, 174, t_3, t_4, t_5)(5, 2; \overline{1})^T = 0, \\
u_{7,4} &= W_{7,4} - (47, 174, 202, t_4, t_5)(22, 18, 6; \overline{0})^T = 0, \\
u_{7,5} &= W_{7,5} - (47, 174, 202, 93, t_5)(99, 122, 84, 24, 0)^T = 0, \\
u_{7,6} &= W_{7,6} - (47, 174, 202, 93, 17)(471, 770, 810, 480, 120)^T = 0.
\end{align*}
\]

Thus, with \( (t_1, t_2, t_3, t_4, t_5) = (47, 174, 202, 93, 17) \) and \( W_{7,7} = 6301550 - 3 \cdot 1110280 \) from (36), we have \( u_{7,7} = W_{7,7} - (t_1, t_2, t_3, t_4, t_5)(u_{2,7}, u_{3,7}, \ldots, u_{6,7})^T = 6!. \)

Therefore,
\[ U_7 = \begin{bmatrix} 877 \\ 2386 \\ U_6 \\ 4832 \\ 6840 \\ 6240 \\ 3240 \\ 0^6 \\ 6! \end{bmatrix} \]

\[ L_7 = \begin{bmatrix} L_6 \\ 0 \\ 0 \end{bmatrix} = S_7 P_7. \]

\[ \begin{bmatrix} 203, 674, 856, 520, 155, 21, 1 \end{bmatrix} \]

**Theorem 9.** The above upper triangular matrix \( U_k = [u_{i,j}] \) (\( j \geq i \)) satisfies

\[ \text{Write a lower triangular matrix} \]

\[ u_{1,j} = m_{1,j} b^*_{1,j} \text{ with } m_{1,1} = 1, \]
\[ u_{2,j} = (m_{2,1}, m_{2,2}) c_j (B^*_2), \text{ with } \{m_{2,1}\} = (-1, 1), \]
\[ u_{3,j} = (m_{3,1}, m_{3,2}, m_{3,3}) c_j (B^*_3) \text{ with } \{m_{3,1}\} = (1, -3, 1), \]
\[ u_{4,j} = (m_{4,1}, \ldots, m_{4,4}) c_j (B^*_4) \text{ with } \{m_{4,1}\} = (-1, -8, -6, 1), \]
\[ u_{5,j} = (m_{5,1}, \ldots, m_{5,5}) c_j (B^*_5) \text{ with } \{m_{5,1}\} = (1, -24, 29, -10, 1), \]
\[ u_{6,j} = (m_{6,1}, \ldots, m_{7,6}) c_j (B^*_6) \text{ with } \{m_{6,1}\} = (-1, -89, -145, 75, -15, 1), \]
\[ u_{7,j} = (m_{7,1}, \ldots, m_{7,7}) c_j (B^*_7) \text{ with } \{m_{7,1}\} = (1, -415, 814, -545, 160, -21, 1). \]

**Proof.** \( u_{1,j} = b^*_{1,j} \) and \( u_{2,j} = b^*_{2,j} - u_{1,j} = -b^*_{1,j} + b^*_{2,j} = (-1, 1) c_j (B^*_2) \) by Theorems 7 and 8. Similarly, we have

\[ u_{3,j} = \left[ b^*_{3,j} - r_2 (P) \left( b^*_{1,j}, b^*_{2,j} \right)^T \right] - 2u_{2,j} \]
\[ = \left[ b^*_{3,j} - (1, 1) (b^*_{1,j}, b^*_{2,j})^T \right] - 2 (1, 1) (b^*_{1,j}, b^*_{2,j})^T \]
\[ = (1, -3, 1) (b^*_{1,j}, b^*_{2,j}, b^*_{3,j})^T = (1, -3, 1) c_j (B^*_3) \]
\[ u_{4,j} = \left[ b^*_{4,j} - r_3 (P) \left( b^*_{1,j}, b^*_{2,j}, b^*_{3,j} \right)^T \right] - 5 \left( u_{2,j}, u_{3,j} \right)^T = (1, -8, -6, 1) c_j (B^*_4). \]

Moreover, with \( c_j (B^*) = (b^*_{1,j}, b^*_{2,j}, \ldots)^T \), we also have

\[ u_{5,j} = \left[ b^*_{5,j} - r_4 (P) c_j (B^*_4) \right] - 3 \left[ b^*_{4,j} - r_3 (P) c_j (B^*_3) \right] - (7, 6) \left( u_{3,j}, u_{4,j} \right)^T \]
\[ = (1, -24, 29, -10, 1) c_j (B^*_5), \]

and, similarly, the rest \( u_{6,j} \) and \( u_{7,j} \) follow immediately. \( \square \) With all \( m_{1,j} \)'s in Theorem 9, write a lower triangular matrix.
satisfies

\[ M_7 = \begin{bmatrix} 1 \\ -1 & 1 \\ 1 & -3 & 1 \\ -1 & 8 & -6 & 1 \\ 1 & -24 & 29 & -10 & 1 \\ -1 & 89 & -145 & 75 & -15 & 1 \\ 1 & -415 & 814 & -545 & 160 & -21 & 1 \end{bmatrix}. \]

(48)

Then,

\[ U_7 = M_7B_7^* = \begin{bmatrix} 1 & 2 & 5 & 15 & 52 & 203 & 877 \\ 1 & 5 & 22 & 99 & 471 & 2386 \\ 2 & 18 & 122 & 770 & 4832 \\ 6 & 84 & 810 & 6840 \\ 24 & 480 & 6240 \\ 120 & 3240 \\ 720 \end{bmatrix}, \]

(49)

gives an LU factorization \( B_7^* = M_7^{-1}U_7 \), where

\[ L_7 = M_7^{-1} = \begin{bmatrix} 1 \\ 1 & 1 \\ 2 & 3 & 1 \\ 5 & 10 & 6 & 1 \\ 15 & 37 & 31 & 10 & 1 \\ 52 & 151 & 160 & 75 & 15 & 1 \\ 203 & 674 & 856 & 520 & 155 & 21 & 1 \end{bmatrix} = \mathcal{S}_7 P_7. \]

(50)

Observe that the matrix \( M_7 = [m_{i,j}] \) is the exponential Riordan array (without signs) (refer to [7]) satisfying a recurrence rule

\[ m_{i,j} = (m_{i-1,t+1}, m_{i,t+1}, m_{i+1,t+1}) (i-1, i, 1)^T. \]

(51)

\( M \) is also known as the coefficients of the Charlier polynomial [8], and we may refer Table 3 in [9] for \( M^{-1} = L \).

**Theorem 10.** Let \( M = [m_{i,j}] \) be a matrix satisfying recurrence (51) with \( m_{i,1} = (-1)^{i-1} \) and \( M_3 = \begin{bmatrix} 1 & 1 \\ 1 & -3 & 1 \end{bmatrix} \). Then, the diagonal entries of the upper triangular matrix \( U_k = M_kB_k^* \) are \((k-1)!\) for all \( k > 1 \).

**Proof.** Let \( U = [u_{i,j}] \). Then, \( U_k = M_kB_k^* \) has diagonal entries \( u_{i,j} (1 \leq i \leq 7) \) as \( 1, 1, 2!, \ldots, 6! \) due to Theorem 9. We note the following identities.

Since \( U = MB^* \) is an upper triangular matrix, we have

\[ r_i(M)c_j(B^*) = u_{i,j} = 0, \quad \text{for all } 1 \leq j < i. \]

(52)

Recurrence (51) gives

\[ m_{i,k} = m_{i-1,t-1} - (i-2)m_{i-2,t} - (i-1)m_{i-1,t}, \]

(53)

over \( M \). Hence, with (52), the \( i \) th row \( r_i(M) = (m_{i,1}, \ldots, m_{i,j}) \) satisfies

\[ r_i(M) = (0, m_{i-1,1}, \ldots, m_{i-1,j-1}) - (i-2)(m_{i-2,1}, \ldots, m_{i-2,j-2}, 0, 0) - (i-1)(m_{i-1,1}, \ldots, m_{i-1,j-1}, 0) \]

\[ = (0; r_{i-1}(M)) - (i-2)(r_{i-2}(M); \mathbf{0}) - (i-1)(r_{i-1}(M); 0). \]

(54)

On the other hand, by (22), the \( i \) th column \( c_j(B^*) \) of \( B^* \) satisfies

\[ c_j(B^*) = \begin{bmatrix} b_{1,j}^* \\ b_{2,j}^* \\ \vdots \end{bmatrix} \begin{bmatrix} b_{1,j-1} + b_{2,j-1} \\ b_{2,j-1} + b_{3,j-1} \\ \vdots \end{bmatrix} = c_{j-1}(B^*) + \begin{bmatrix} b_{2,j-1} \\ b_{3,j-1} \\ \vdots \end{bmatrix}. \]

(55)

Thus, (52) and (55) together show

\[ r_j(M) \begin{bmatrix} b_{2,j-1}^* \\ b_{3,j-1}^* \\ \vdots \end{bmatrix} = r_j(M)c_j(B^*) - r_j(M)c_{j-1}(B^*) \]

(56)

and similarly,
in $\Gamma$.

Moreover, by (28), (55), and (56), $u_{i,j} = r_i(M)c_i(B^*)$

$$= (0; r_{i-1}(M)c_{i-1}(B^*) - (i-1)r_{i-2}(M)c_{i-1}(B^*))$$

$$- (i-1)r_{i-1}(M)c_{i-1}(B^*) + r_i(M)\left[\begin{array}{c}
  b^*_{2,i-1} \\
  b^*_{3,i-1} \\
  \vdots \\
  b^*_{t+1,i-1}
\end{array}\right]$$

$$= (u_{i-1,j} - u_{i-1,j-1}) - (i-2)u_{i-2,j-1} - (i-1)u_{i-1,j-1} + u_{i,j}$$

(57)

$$u_{i+1,j+1} = r_{i+1}(M)c_{i+1}(B^*)$$

$$= (0; r_j(M)c_j(B^*) - (i-1)(r_{i-1}(M); \bar{\overrightarrow{c}}^2)c_i(B^*) - i(r_j(M); 0)c_i(B^*) + r_{i+1}(M)\left[\begin{array}{c}
  b^*_{2,i} \\
  b^*_{3,i} \\
  \vdots \\
  b^*_{t,i}
\end{array}\right]$$

$$= (0; r_j(M)c_j(B^*) - (i-1)r_{i-1}(M)c_j(B^*) - ir_j(M)c_j(B^*) + r_{i+1}(M)\left[\begin{array}{c}
  b^*_{2,i} \\
  b^*_{3,i} \\
  \vdots \\
  b^*_{t,i}
\end{array}\right]$$

$$= i! - (i-1)! + i! = i!.$$

(58)

**Theorem 11.** The lower triangular matrix $L_k = [l_{i,j}] = \overline{S}_kP_k$ in Theorem 8 satisfies $l_{i+1,j+1} = (l_{i,j}, l_{i,j+1}, l_{i,j+2})^{T}$ in (1, $j+1, j+1)^T$.

**Proof.** Since $L_k = \overline{S}_kP_k$, for any $1 \leq i, j \leq k$, Theorem 3 shows

$$l_{i+1,j+1} = r_{i+1}(\overline{S})c_{i+1} (P) = \left(0; r_j(\overline{S}) + r_j(\overline{S})\bar{\overrightarrow{d}}[0, 1, \ldots, i-1]\right)\left(0; c_j (P) \right) + \left(0; c_{j+1} (P) \right)$$

$$+ \left(0; r_j(\overline{S})\bar{\overrightarrow{d}}[0, 1, \ldots, i-1]\right)\left(0; c_j (P) \right) + \left(0; c_{j+1} (P) \right)$$

$$= l_{i,j} + l_{i,j+1} + r_j(\overline{S})(\bar{\overrightarrow{d}}[0, 1, \ldots, i-1])\left(0; c_j (P) \right) + r_j(\overline{S})(\bar{\overrightarrow{d}}[0, 1, \ldots, i-1])\left(0; c_{j+1} (P) \right)$$

$$= l_{i,j} + l_{i,j+1} + jr_j(\overline{S})c_{j+1} (P) + (j+1)r_j(\overline{S})c_{j+2} (P)$$

$$= l_{i,j} + l_{i,j+1} + j[l_{i,j+1} + (j+1)l_{i,j+2}] = (l_{i,j}, l_{i,j+1}, l_{i,j+2})^{T}(1, j+1, j+1)^T.$$

(60)
The $L_k$ and $U_k$ are obtained by the recurrence in Theorems 10 and 11. In fact, from $M'_7$ and $L_7$, we get

\[
L_9 = \begin{bmatrix}
\tilde{S}_7P_7 & 0 \\
877 & 3263 & 4802 & 3556 & 1400 & 287 & 28 & 1 \\
4140 & 17007 & 28337 & 24626 & 11991 & 3290 & 490 & 36 & 1
\end{bmatrix},
\]

\[
M_9 = \begin{bmatrix}
M_7 & 0 \\
-1 & 2372 & -5243 & 4179 & -1575 & 301 & -28 & 1 \\
1 & -16072 & 38618 & -34860 & 15659 & -3836 & 518 & -36 & 1
\end{bmatrix},
\]

so

\[
M_9B'_9 = \begin{bmatrix}
U_7 & \ddots \\
0 & 7! & 221760 \\
0 & 8!
\end{bmatrix} = U_9,
\]

in which all diagonal entries are $(k-1)!$.

**Theorem 12.** $\det B'_k = \prod_{i=1}^{k-1} i! = (k-1)! \det B'_{k-1}$.

Clearly, $\det B'_k = \det L_k \det U_k = \det U_k = \prod_{i=1}^{k-1} i! = (k-1)! \det B'_{k-1}$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**References**


