Research Article

Output Reachable Set Analysis for Periodic Positive Systems

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Reachability, as a fundamental concept in control theory, has received a lot of attention. Numerous authors have investigated the reachability of positive systems for both discrete and continuous systems [18–23]. The set containing all system outputs that are reachable from the origin under a prescribed set of inputs is called the output reachable set. Many researchers have focused on characterizing the outputs’ reachable sets for dynamical systems, but when the input signal is constrained, transferring the output of the system to an arbitrary desired output from the origin is generally difficult, so the popular technique in the literature is to determine a region as small as possible to bind the output reachable set. A usual strategy is to estimate the output reachable set by a few ellipsoids that can be determined by solving linear matrix inequality (LMI). Reachable set estimation problem has been studied for time-delay systems [24, 25], singular systems [26], periodic systems [27], positive systems [28], switched positive systems [29], etc.

This paper aims to solve the output reachable set estimation problem for periodic positive systems under two possible classes of nonnegative exogenous disturbances based on two norms.

The organization of this paper is the following form. In Section 2, the formulation of the problem is given and the positivity of the considered system is defined. In Section 3, results on the estimation of the output reachable set are given under two classes of exogenous disturbances, and the
optimization techniques are used to obtain the smallest possible hyperpyramids which bound the output reachable set. Some examples are considered to verify the theoretical results.

Notations. The notations used in this work are $\mathbb{Z}_+$ set of nonnegative integers, $\mathbb{Z}_0^+$ set of positive integers, $\mathbb{R}$ set of real numbers, $\mathbb{R}^n$ set of $n$-dimensional real vectors, $\mathbb{R}^{m \times n}$ set of $m \times n$ real matrices, $\mathbb{R}_+^n$ positive orthant of $\mathbb{R}^n$, $\mathbb{R}_-^n$ closed positive orthant of $\mathbb{R}^n$, $A^T$ transpose of $A$, 1 vector $[1, 1, \ldots, 1]^T$, $I$ identity matrix, $x \geq 0$ ($x > 0$) every component of the vector $x$ is nonnegative (positive), $A \geq 0$ ($A > 0$) every entry of matrix $A$ is nonnegative (positive), and $\sigma_i = \{s, s + 1, \ldots, k\}$ is the finite subset of $\mathbb{Z}_+$ with $s \leq k$.

2. Preliminaries

Consider the discrete-time periodic linear system described by

$$
\begin{align*}
    x_{t+1} &= A_t x_t + B_t \theta_t, \quad t \in \mathbb{Z}_+,
    \\
    x_0 &\in \mathbb{R}^n,
    \\
    y_t &= C_t x_t, \quad t \in \mathbb{Z}_+,
\end{align*}
$$

where $x_t \in \mathbb{R}^n$ is the state vector, $\theta_t \in \mathbb{R}^m$ is the exogenous input signal, $y_t \in \mathbb{R}^r$ is the output vector, and $A_t, B_t, C_t$ are real matrices with appropriate dimensions and we assume that there exists $T \in \mathbb{Z}_0^+$ such that, for all $t \in \mathbb{Z}_+$, we have $A_t = A_{t+T}$, $B_t = B_{t+T}$, and $C_t = C_{t+T}$. We note $x := (x_t)_{t \geq 0}$, $y := (y_t)_{t \geq 0}$, and $\theta := (\theta_t)_{t \geq 0}$.

Definition 1. Systems (1)-(2) are said to be positive if, for any initial condition $x_0 \in \mathbb{R}_+^n$ and for any exogenous input $\theta \in \mathbb{R}_+^m$ and $t \in \mathbb{Z}_+$, we have $x_t \in \mathbb{R}_+^n$ and $y_t \in \mathbb{R}_+^r$, for all $t \in \mathbb{Z}_+$.

Remark 1. Systems (1)-(2) are positive if $A_t \geq 0, B_t \geq 0$, and $C_t \geq 0$, for all $t \in \mathbb{Z}_0^{T-1}$.

In the rest of this paper, we assume that systems (1)-(2) are positive.

The results presented in this paper are divided into two cases according to the following norms: $\|\theta\|_{1,1} = \sum_{i=0}^{\infty} \|\theta_i\|_1$ and $\|\theta\|_{\infty,1} = \sup_{r \in \mathbb{N}} \|\theta_r\|_1$, where $\|\theta\|_1 = \sum_{i=0}^{m} |\theta_{i,r}|$, for $\theta = [\theta_{0,1}, \theta_{1,2}, \ldots, \theta_{m,r}]^T \in \mathbb{R}^m$.

Case 1: $\theta \in \Sigma_{1,1}^+ := \{\theta | \|\theta\|_{1,1} \leq 1 \text{ and } \theta_t \geq 0, \text{ for all } t \in \mathbb{Z}_+\}$.

The output reachable set in this case is defined as follows:

$$
\mathcal{R}_{1,1}^+ := \{y | \text{ there is } \theta \in \Sigma_{1,1}^+ \text{ such that } y \text{ and } \theta \text{ satisfy (1) - (2) with } x_0 = 0\}. 
$$

Case 2: $\theta \in \Sigma_{\infty,1}^+ := \{\theta | \|\theta\|_{\infty,1} \leq 1 \text{ and } \theta_t \geq 0, \text{ for all } t \in \mathbb{Z}_+\}$.

The output reachable set in this case is defined as follows:

$$
\mathcal{R}_{\infty,1}^+ := \{y | \text{ there is } \theta \in \Sigma_{\infty,1}^+ \text{ such that } y \text{ and } \theta \text{ satisfy (1) - (2) with } x_0 = 0\}. 
$$

The output reachable set in this paper will be bounded by hyperpyramids of the form

$$
\mathfrak{H}(\xi) = \{\xi \in \mathbb{R}^n | \xi^T \xi \leq 1\}, \text{ where } \xi \in \mathbb{R}_+^n.
$$

3. Main Results

In this section, we will estimate the output reachable set of systems (1)-(2) by hyperpyramids for the two cases mentioned above. For that, we will use optimization techniques to obtain the smallest possible hyperpyramids.

3.1. Estimation of Output Reachable Sets

3.1.1. Case 1. In this case, we consider systems (1)-(2) under zero initial conditions and exogenous input $\theta \in \Sigma_{1,1}^+$.
So, we obtain

$$\varphi_t(y_t) - \varphi_0(y_0) \leq \sum_{s=0}^{t-1} 1^T \xi_s \leq \|\vartheta\|_{1,1} \leq 1.$$  \hfill (8)

Since $\varphi_0(y_0) = \varphi_0(0) = 0$, then we get $\varphi_t(y_t) \leq 1$. The proof is completed. \hfill $\square$

According to this lemma, we get the following bounding of the output reachable set.

**Theorem 1.** Consider systems (1)-(2), and assume that there exist $\xi_t \in \mathbb{R}_e^n$, $l \in \sigma_0$, with $\xi_T = \xi_0$, such that, for all $l \in \sigma_0^{-1},$

$$\left\{ \begin{array}{l} A_l^T C_{l+1}^T \xi_{l+1} - C_l^T \xi_l \leq 0, \\ B_l^T C_{l+1}^T \xi_{l+1} \leq 1. \end{array} \right. \hfill (9)$$

Then, $\mathbb{R}_e^n \subset \cup_{l=0}^{T-1} \mathcal{F}(\xi_l)$ (i.e., the output reachable set is bounded by the union of a set of hyperpyramids).

**Proof.** Let $(\xi_t)_{t \geq 0}$ be the sequence defined by $\xi_t = \xi_{t\cdot T},$ $\forall t \in \mathbb{Z}_+,$ and let, for $t \geq 0$,

$$\varphi_t : \mathbb{R}_e^n \longrightarrow \mathbb{R}_e^n, \quad z \longmapsto \xi_t^T z. \quad \hfill (10)$$

Let $y \in \mathbb{R}_e^n$ and $\vartheta \in \Sigma_1$ be the corresponding input. For any $t \geq 0$, there is $l \in \sigma_0^{-1}$ such that $t \equiv l[T]$. Then, we obtain

$$\varphi_{t+1}(y_{t+1}) - \varphi_t(y_t) = (\xi_{t+1}^T C_{t+1} A_l - \xi_t^T C_l) x_t + \xi_{t+1}^T C_{t+1} B_l \vartheta_t$$

$$= (\xi_{t+1}^T C_{t+1} A_l - \xi_t^T C_l) x_t + \xi_{t+1}^T C_{t+1} B_l \vartheta_t \leq 1^T \vartheta_t. \hfill (11)$$

According to Lemma 1, we get $\varphi_t(y_t) \leq 1$. And, the periodicity of $\xi_t$ implies that $y_t \in \cup_{l=0}^{T-1} \mathcal{F}(\xi_l)$. So, $\mathbb{R}_e^n \subset \cup_{l=0}^{T-1} \mathcal{F}(\xi_l)$. The proof is completed. \hfill $\square$

If $T = 1$, then systems (1)-(2) reduce to the linear positive time-invariant system given by

$$\left\{ \begin{array}{l} x_{t+1} = A x_t + B \vartheta_t, \\ x_0 \in \mathbb{R}^n, \end{array} \right. \quad \hfill (12)$$

$$y_t = C x_t, \quad t \in \mathbb{Z}_+. \hfill (13)$$

From Theorem 1, we can deduce the following corollary to determine an estimation of the output reachable set of systems (12)-(13).

**Corollary 1.** Consider systems (12)-(13), and assume that there exists $\xi \in \mathbb{R}_e^n$ such that the following condition holds:

$$\left\{ \begin{array}{l} A_l^T C_{l+1}^T \xi - C_l^T \xi \leq 0, \\ B_l^T C_{l+1}^T \xi \leq 1. \end{array} \right. \hfill (14)$$

Then, $\mathbb{R}_e^n \subset \mathcal{F}(\xi)$.

**3.1.2. Case 2.** In this case, we consider systems (1)-(2) under zero initial conditions and exogenous input $\vartheta \in \Sigma_1$.

**Lemma 2.** Let $\varphi_t : \mathbb{R}_e^n \longrightarrow \mathbb{R}_e^n$, $t \in \mathbb{Z}_+$, be a set of functions such that

(i) $\varphi_0(0) = 0$.

(ii) There exist $\vartheta_t \in [0,1]$, $t \in \mathbb{Z}_+$, such that if $y \in \mathbb{R}_e^n$ and $\vartheta \in \Sigma_1$ is the corresponding input, then

$$\varphi_{t+1}(y_{t+1}) - \vartheta_t \varphi_t(y_t) \leq (1 - \vartheta_t) I^T \vartheta_t \leq (1 - \vartheta_t) \|\vartheta\|_{1,1} \leq 1 - \vartheta_t. \hfill (15)$$

Then, we obtain $\varphi_t(y_t) \leq 1$, $\forall t \in \mathbb{Z}_+$.

**Proof.** Let $y \in \mathbb{R}_e^n$ and $\vartheta \in \Sigma_1$ be the corresponding input, and we have

$$\varphi_{t+1}(y_{t+1}) - \vartheta_t \varphi_t(y_t) \leq (1 - \vartheta_t) I^T \vartheta_t \leq (1 - \vartheta_t) \|\vartheta\|_{1,1} \leq 1 - \vartheta_t. \hfill (16)$$

Then, $\varphi_{t+1}(y_{t+1}) - 1 \leq \vartheta_t (\varphi_t(y_t) - 1)$.

For any time $t \in \mathbb{Z}_+$, we have

$$\varphi_t(y_t) - 1 \leq \vartheta_{t-1} (\varphi_{t-1}(y_{t-1}) - 1) \leq \vartheta_{t-1} \vartheta_{t-2} (\varphi_{t-2}(y_{t-2}) - 1) \leq \cdots \leq \vartheta_{t-1} \cdots \vartheta_0 (\varphi_0(y_0) - 1). \hfill (17)$$

Since $\varphi_0(y_0) = \varphi_0(0) = 0 \leq 1$, it follows that $\varphi_t(y_t) \leq 1$. The proof is completed. \hfill $\square$

Using this lemma, we can derive the following bounding of the output reachable set.

**Theorem 2.** Consider systems (1)-(2), and assume that there exist $\xi_t \in \mathbb{R}_e^n$, $l \in \sigma_0$, with $\xi_T = \xi_0$ and $\theta_l \in [0,1]$, $l \in \sigma_0^{-1}$, such that, for all $l \in \sigma_0^{-1}$, we have

$$\left\{ \begin{array}{l} A_l^T C_{l+1}^T \xi_{l+1} - \theta_l C_l^T \xi_l \leq 0, \\ B_l^T C_{l+1}^T \xi_{l+1} \leq (1 - \theta_l). \end{array} \right. \hfill (18)$$

Then, $\mathbb{R}_e^n \subset \cup_{l=0}^{T-1} \mathcal{F}(\xi_l)$.

**Proof.** We construct two sequences $(\xi_l)_{l \geq 0}$ and $(\theta_l)_{l \geq 0}$ by posing $\xi_l = \xi_{l\cdot T}$ and $\theta_l = \theta_{l\cdot T}$, $\forall t \in \mathbb{Z}_+$. We consider the family of functions:

$$\varphi_t : \mathbb{R}_e^n \longrightarrow \mathbb{R}_e^n, \quad z \longmapsto z^T \xi_t. \hfill (19)$$

Let $y \in \mathbb{R}_e^n$ and $\vartheta \in \Sigma_1$ be the corresponding input. For all $t \in \mathbb{Z}_+$, there is $l \in \sigma_0$ such that $t \equiv l[T]$. Then, we obtain

$$\varphi_{t+1}(y_{t+1}) - \vartheta_t \varphi_t(y_t) \leq (1 - \vartheta_t) I^T \vartheta_t, \quad \forall t \in \mathbb{Z}_+. \hfill (15)$$

Then, we obtain $\varphi_t(y_t) \leq 1$, $\forall t \in \mathbb{Z}_+$.

**Proof.** Let $y \in \mathbb{R}_e^n$ and $\vartheta \in \Sigma_1$ be the corresponding input, and we have
\[ \varphi_{t+1}(y_{t+1}) - \theta_t \varphi_t(y_t) = \left( \xi_{t+1} C_{t+1} A_t - \theta_t \xi_t^T C_t \right) x_t + \xi_{t+1} C_{t+1} B_t \theta_t \]
\[ = \left( \xi_{t+1} C_{t+1} A_t - \theta_t \xi_t^T C_t \right) x_t + \xi_{t+1} C_{t+1} B_t \theta_t \]
\[ \leq (1 - \theta_t) 1^T \theta_t \]
\[ = (1 - \theta_t) 1^T \theta_t. \]  
(20)

So, Lemma 2 implies that \( \varphi_t(y_t) \leq 1 \), and from the periodicity of \( \xi_t \), we obtain \( y_t \in \cup_{l=0}^{T-1} \mathcal{A}(\xi_t) \). So, \( \mathcal{R}_{x,1}^* \subset \cup_{l=0}^{T-1} \mathcal{A}(\xi_t) \). The proof is completed.

If \( T = 1 \), we can deduce the following corollary for the estimation of the output reachable set of the linear positive time-invariant systems (12)-(13).

**Corollary 2.** Consider systems (12)-(13), and assume that there exists \( \xi \in \mathbb{R}_+^l \) such that the following condition holds:
\[ \begin{cases} A^T C^T \xi - \theta C^T \xi \leq 0, \\ B^T C^T \xi \leq (1 - \theta)1. \end{cases} \]  
(21)

Then, \( \mathcal{R}_{x,1}^* \subset \mathcal{A}(\xi) \).

**Remark 2.** In Theorem 1 (Theorem 2), the norms of the exogenous inputs are no greater than 1. We can deduce the estimation of the output reachable set if the norms of the exogenous inputs are no greater than a scalar \( \bar{\theta} > 0 \). If \( \| \theta \|_{\infty} \leq \bar{\theta} \), then the output reachable set \( \mathcal{R}_{x,1}^* \) can be bounded by the union of a set of hyperpyramids \( \cup_{l=0}^{T-1} \mathcal{A}(\xi_t) \). Indeed, if we assume that \( \| \theta \|_{\infty} \leq \bar{\theta} \), then we obtain \( \| \theta \|_{\infty} \leq 1 \). Systems (1)-(2) can be re-written in the following form:
\[ \begin{cases} x_{t+1} = A_t x_t + B_t \frac{\theta_t}{\bar{\theta}}, \\ y_t = C_t x_t. \end{cases} \]  
(22)

So, according to Theorem 1 (Theorem 2), \( y_t/\theta \) can be bounded by \( \cup_{l=0}^{T-1} \mathcal{A}(\xi_t) \). Then, \( y_t \) can be bounded by \( \cup_{l=0}^{T-1} \mathcal{A}(\xi_t) \).

The hypervolume of the hyperpyramid (5) is equal to \( 1/(r! \prod_{i=1}^{l} (1/\xi_i)) \) (Carter and Champanerkar [30]). The volume of the bounding hyperpyramids considered in the two cases can be minimised by solving the following optimization problem:
\[ \min \left( - \sum_{m=1}^{r} \sum_{l=0}^{T-1} \log \xi_{lm} \right), \]  
(23)
which subject to the conditions in Theorems 1 or 2, with \( \xi_t = [\xi_{t,1}, \xi_{t,2}, \ldots, \xi_{t,T}], \ l \in \sigma_0^{T-1}. \)

To solve optimization problem (23) which subjects to the conditions in Theorem 2, we adopt genetic algorithm (GA). For more details about GA, see [31, 32].

**3.2. Examples.** In this section, we will give examples for the two cases.

**3.2.1. Case 1.** Consider systems (1)-(2) with period \( T = 2 \), and
\[ (A_0, B_0, C_0) = \left( \begin{bmatrix} 0.1 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}, \begin{bmatrix} 0.3 & 0.7 \\ 0.2 & 0.1 \end{bmatrix} \right). \]  
(24)

By solving problem (23), we obtain
\[ \xi_0 = \begin{bmatrix} 1.2321 \\ 4.1090 \end{bmatrix}, \]  
(25)
\[ \xi_1 = \begin{bmatrix} 1.2549 \\ 2.1621 \end{bmatrix}. \]

The bounding hyperpyramids are shown in Figure 1 with exogenous inputs defined by \( \theta_t = (1 - m) \times m', \ m = 0, 0.1, \ldots, 0.9. \)

**3.2.2. Case 2.** Consider systems (1)-(2) with period \( T = 2 \), and
\[ (A_0, B_0, C_0) = \left( \begin{bmatrix} 0.5 & 0.6 \\ 0.4 & 0.3 \end{bmatrix}, \begin{bmatrix} 0.2 & 0.1 \end{bmatrix} \right). \]  
(26)

By solving problem (23), we obtain
\[ \xi_0 = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.3 \end{bmatrix}, \]  
(27)
\[ \xi_1 = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.3 \end{bmatrix}. \]

The optimal values of problem (23) are
\[ \begin{align*}
\theta_0 &= 0.854, \\
\theta_1 &= 0.935, \\
\xi_0 &= \begin{bmatrix} 0.171 \\ 0.317 \end{bmatrix}, \\
\xi_1 &= \begin{bmatrix} 0.531 \\ 0.182 \end{bmatrix}.
\end{align*} \tag{27} \]

The bounding hyperpyramids are shown in Figure 2 with exogenous inputs defined by \( (a) \bar{d}_i = \text{rand}(t) \), \( (b) \bar{d}_i = 0.7 \), and \( (c) \bar{d}_i = [\sin(t)] \).

### 4. Conclusion

In this work, we have studied the estimation of the output reachable set for positive periodic systems. Results (Theorem 1 and Theorem 2) have been found for two exogenous disturbance classes, and optimization techniques have been used to minimize the volume of bounding hyperpyramids. Numerical examples have been used to verify the theoretical results.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### References


