Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2021, Article ID 8895949, 3 pages
https://doi.org/10.1155/2021/8895949

Research Article

On a Characterization of Finite-Dimensional Vector Spaces

Marat V. Markin

Department of Mathematics, California State University, Fresno, 5245 N. Backer Avenue, M/S PB 108, Fresno 93740-8001, CA, USA

Correspondence should be addressed to Marat V. Markin; mmarkin@csufresno.edu

Received 12 September 2020; Revised 8 January 2021; Accepted 13 January 2021; Published 12 February 2021

Academic Editor: Sergejs Solovjovs

We provide a characterization of the finite dimensionality of vector spaces in terms of the right-sided invertibility of linear operators on them.

1. Introduction

In [1], found is a characterization of one-dimensional (real or complex) normed algebras in terms of the bounded linear operators on them, echoing the celebrated *Gelfand–Mazur theorem* characterizing complex one-dimensional Banach algebras (see, e.g., [2–6]).

Here, continuing along this path, we provide a simple characterization of the finite dimensionality of vector spaces in terms of the right-sided invertibility of linear operators on them.

2. Preliminaries

As is well-known (see, e.g., [7, 8]), a square matrix A with complex entries is invertible iff it is one-sided invertible, i.e., there exists a square matrix C of the same order as A such that

$$\begin{aligned} \mathrm{AC} &= I\,(\text{right inverse}) \\ \text{or } \mathrm{CA} &= I\,(\text{left inverse}), \end{aligned} \tag{1}$$

where I is the *identity matrix* of an appropriate size, in which case C is the (two-sided) inverse of A, i.e.,

$$\mathrm{AC} = \mathrm{CA} = I. \tag{2}$$

Generally, for a linear operator on a (real or complex) vector space, the existence of a *left inverse* implies is *invertible*, i.e., *injective*. Indeed, let $A\colon X \longrightarrow X$ be a linear operator on a (real or complex) vector space X and a linear operator $C\colon X \longrightarrow X$ be its *left inverse*, i.e.,

$$\mathrm{CA} = I, \tag{3}$$

where I is the *identity operator* on X. Equality (3), obviously, implies that

$$\ker A = \{0\}, \tag{4}$$

and hence, there exists an inverse $A^{-1}\colon R(A) \longrightarrow X$ for operator A, where $R(A)$ is its range (see, e.g., [9]). Equality (3) also implies that the inverse operator A^{-1} is the restriction of C to $R(A)$.

Furthermore, as is easily seen, for a linear operator on a (real or complex) vector space, the existence of a *right inverse*, i.e., a linear operator $C\colon X \longrightarrow X$ such that

$$\mathrm{AC} = I, \tag{5}$$

immediately implies being *surjective*, which, provided the underlying vector space is *finite dimensional*, by the *rank-nullity theorem* (see, e.g., [9, 10]), is equivalent to being *injective*, i.e., being *invertible*.

With the underlying space being *infinite-dimensional*, the arithmetic of infinite cardinals does not allow to directly infer by the *rank-nullity theorem* that the *surjectivity* of a linear operator on the space is equivalent to its *injectivity*. In this case, the right-sided invertibility for linear operators need not imply invertibility. For instance, on the (real or complex) *infinite-dimensional* vector space l_∞ of bounded sequences, the left shift linear operator

$$l_\infty \ni x := (x_1, x_2, x_3, \ldots) \longmapsto Lx := (x_2, x_3, x_4, \ldots) \in l_\infty \tag{6}$$

is *noninvertible* since

$$\ker L = \{(x_1, 0, 0, \ldots)\} \neq \{0\} \tag{7}$$

(see, e.g., [9, 10]), but the right shift linear operator

$$l_\infty \ni x := (x_1, x_2, x_3, \ldots) \mapsto Rx := (0, x_1, x_2, \ldots) \in l_\infty \tag{8}$$

is its *right inverse*, i.e.,

$$\mathrm{LR} = I, \tag{9}$$

where I is the *identity operator* on l_∞.

Not only does the above example give rise to the natural question of whether, when the right-sided invertibility for linear operators on a (real or complex) vector space implies their invertibility, i.e., *injectivity*, the underlying space is necessarily *finite dimensional* but also serve as an inspiration for proving the *"if"* part of the subsequent characterization.

3. Characterization

Theorem 1 (characterization of finite-dimensional vector spaces). *A (real or complex) vector space X is finite-dimensional iff, for linear operators on X, right-sided invertibility implies invertibility.*

Proof.

"Only if" part. Suppose that the vector space X is *finite-dimensional* with $\dim X = n\,(n \in \mathbb{N})$ and let $B := \{x_1, \ldots, x_n\}$ be an ordered basis for X.

For an arbitrary linear operator $A\colon X \longrightarrow X$ on X, which has a *right inverse*, i.e., a linear operator $C\colon X \longrightarrow X$ such that

$$\mathrm{AC} = I, \tag{10}$$

where I is the *identity operator* on X. Let $[A]_B$ and $[C]_B$ be the *matrix representations* of the operators A and C relative to the basis B, respectively (see, e.g., [7, 8]), then

$$[A]_B [C]_B = I_n, \tag{11}$$

where I_n is the *identity matrix* of size n (see, e.g., [7, 8]).

By the *multiplicativity of determinant* (see, e.g., [7, 8]), equality (11) implies that

$$\det([A]_B)\det([C]_B) = \det([A]_B [C]_B) = \det(I_n) = 1. \tag{12}$$

Whence, we conclude that

$$\det([A]_B) \neq 0, \tag{13}$$

which, by the *determinant characterization of invertibility*, it implies that matrix $[A]_B$ is invertible, and hence, so is the operator A (see, e.g., [7, 8]).

"If" part. Let us prove this part *by contrapositive*, assuming that the vector space X is *infinite-dimensional*. Suppose that $B := \{x_i\}_{i \in I}$ is a (Hamel) basis for X (see, e.g., [9, 10]), where I is an infinite indexing set and that $J := \{i(n)\}_{n \in \mathbb{N}}$ is a *countably infinite* subset of I.

Let us define a linear operator $A\colon X \longrightarrow X$ as follows:

$$\begin{aligned} Ax_{i(1)} &:= 0, \\ Ax_{i(n)} &:= x_{i(n-1)}, \quad n \geq 2, \\ Ax_i &:= x_i, \quad i \in I \backslash J, \\ X \ni x = \sum_{i \in I} c_i x_i &\mapsto Ax := \sum_{i \in I} c_i A x_i, \end{aligned} \tag{14}$$

where

$$\sum_{i \in I} c_i x_i \tag{15}$$

is the *basis representation* of a vector $x \in X$ relative to B, in which all but a finite number of the coefficients c_i, $i \in I$, called the *coordinates* of x relative to B, are zero (see, e.g., [9, 10]).

As is easily seen, A is a linear operator on X, which is *noninvertible*, i.e., *noninjective*, since

$$\ker A = \operatorname{span}\left(\left\{x_{i(1)}\right\}\right) \neq \{0\}. \tag{16}$$

The linear operator $C\colon X \longrightarrow X$ on X defined as follows:

$$\begin{aligned} Cx_{i(n)} &:= x_{i(n+1)}, \quad n \in \mathbb{N}, \\ Cx_i &:= x_i, \quad i \in I \backslash J, \\ X \ni x = \sum_{i \in I} c_i x_i &\longmapsto Cx := \sum_{i \in I} c_i C x_i, \end{aligned} \tag{17}$$

which is a *right inverse* for A since

$$\begin{aligned} \mathrm{AC}x_{i(n)} &= Ax_{i(n+1)} = x_{i(n)}, \quad n \in \mathbb{N}, \\ \mathrm{AC}x_i &= Ax_i = x_i, \quad i \in I \backslash J. \end{aligned} \tag{18}$$

Thus, on a (real or complex) infinite-dimensional vector space, there exists a noninvertible linear operator with a right inverse, which completes the proof of the *"if"* part, and hence, of the entire statement.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] M. V. Markin, "A Gelfand-Mazur type theorem for normed algebras," *Anal Topology*, vol. 11, no. 1, pp. 63-64, 2005.

[2] G. Bachman and L. Narici, *Functional Analysis*, Dover Publications, Inc., Mineola, NY, USA, 2000.

[3] I. M. Gelfand, "On normed rings," *Dokl. Akad. Nauk SSSR*, vol. 23, pp. 430–432, 1939.

[4] I. M. Gelfand, “Normierte Ringe,” *Matem. Sbornik*, vol. 9, pp. 3–24, 1941.
[5] M. A. Naimark, *Normed Algebras*, Wolters-Noordhoff Publishing, Groningen, Netherland, 1972.
[6] C. E. Rickart, “An elementary proof of a fundamental theorem in the theory of Banach algebras,” *The Michigan Mathematical Journal*, vol. 5, no. 1, pp. 75–78, 1958.
[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, NY, USA, 1986.
[8] M. O’Nan, *Linear Algebra*, Harcourt Brace Jovanovich, New York, NY, USA, 1976.
[9] M. V. Markin, *Elementary Operator Theory*, De Gruyter Graduate, Walter de Gruyter GmbH, Berlin, Germany, 2020.
[10] M. V. Markin, *Elementary Functional Analysis*, De Gruyter Graduate, Walter de Gruyter GmbH, Berlin, Germany, 2018.