Research Article

Biderivations and Commuting Linear Maps on Topologically Simple $L^*$-Algebras

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Received 22 March 2021; Revised 31 May 2021; Accepted 6 July 2021; Published 17 July 2021

1. Introduction

Let $(L, [., .])$ be a Lie algebra over a field $F$ of a characteristic different from two. A linear map $D: L \longrightarrow L$ is called derivation on $(L, [., .])$ if it satisfies the following identity:

$$D([x, y]) = [D(x), y] + [x, D(y)],$$

for all $x, y \in L$.

A bilinear map $\delta: L \times L \longrightarrow L$ is called biderivation on $(L, [., .])$ if it satisfies the following identities:

$$\delta([x, y], z) = [x, \delta(y, z)] + [\delta(x, z), y],$$
$$\delta(x, [y, z]) = [\delta(x, y), z] + [y, \delta(x, z)].$$

For all $x, y, z \in L$, which means that it is a derivation with respect to both components. In addition, if $\delta(x, y) = -\delta(y, x)$, $\delta$ will be called skew-symmetric biderivation. Let $\lambda \in F$ and $f$ be the bilinear map $f: L \times L \longrightarrow L$ sending $(x, y)$ to $\lambda[x, y]$; it is straightforward to prove that $f$ is a biderivation of $(L, [., .])$; and the biderivations of this type are called inner biderivations of $(L, [., .])$. A linear map $\phi: L \longrightarrow L$ is called a commuting linear map on $(L, [., .])$ if it satisfies the following identity: $[\phi(x), x] = 0$, for all $x \in L$. It is easy to show that

$$[\phi(x), y] = [x, \phi(y)], \quad \forall x, y \in L,$$

which implies that the bilinear map $\delta$ defined by

$$\delta(x, y) = [\phi(x), y] = [x, \phi(y)],$$

is a skew-symmetric biderivation on $(L, [., .])$.

Commuting maps and biderivations arose first in the associative ring theory [1, 2]. Since then, many authors have made considerable efforts to make their study very successful (see, for example, [3–10]). The way used in [8] requires the finiteness of the dimension of the simple Lie algebra. However, the purpose of this paper is to extend the results given in [8] concerning the commuting linear maps to topologically simple $L^*$-algebras, which are of arbitrary dimension. To overcome the problem of the nonfiniteness of the dimension, we use some techniques related to these algebras. The $L^*$-algebras are introduced by Schue in [11].

We recall that an $L^*$-algebra over $C$ (the complex field) is a Lie algebra $L$, which is also a complex Hilbert space with the inner product $(., .)$ endowed with a (conjugate-linear) algebra involution * such that $([x, y], z) = (y, [x^*, z])$, for all $x, y, z \in L$.

Let $L$ be an $L^*$-algebra; for subsets $M$ and $N$ of $L$, we recall that $[M, N]$ denotes the closed subspace spanned by $\{m, n\}; m \in M, n \in N$. $L$ is said to be semisimple as an $L^*$-algebra if and only if $L = [L, L]$. From [11], a finite-dimensional Lie algebra $L$ is semisimple as an $L^*$-algebra if and only if it is semisimple in the usual sense. The
Let $\mathcal{L}^*$-algebra $\mathcal{L}$ be called topologically simple if and only if there are no nontrivial closed ideals. In [11], the author shows that the $\mathcal{L}^*$-algebras are reductive, the semisimple ones are the Hilbert space direct sum of its closed topologically simple ideals, and the author also gives the classification of the topologically simple $\mathcal{L}^*$-algebras in the separable case. The classification of the topologically simple $\mathcal{L}^*$-algebras in the arbitrary dimensional case can be found in [12]. In [13], Schue shows that every semisimple $\mathcal{L}^*$-algebra has a Cartan decomposition relative to a Cartan subalgebra. We recall that a Cartan subalgebra of a semisimple $\mathcal{L}^*$-algebra is defined as a maximal self-adjoint abelian subalgebra.

The paper is organized as follows. In the second section, we give some definitions and basic results related to $\mathcal{L}^*$-algebras. In Section 3, we introduce the notion of semi-inner biderivation in order to show that every semi-inner biderivation on an arbitrary dimensional topologically simple $\mathcal{L}^*$-algebra is inner; using this result, we determine all continuous commuting linear maps on an arbitrary dimensional topologically simple $\mathcal{L}^*$-algebras.

2. Preliminaries

In this section, we summarize some basic results related to $\mathcal{L}^*$-algebras, collected from [14, 15]. First of all, we point out that the notation concerning Lie algebras follows principally from [8, 14]. Let $\mathcal{L}$ be the complex numbers field, $\mathcal{L}$ a semisimple $\mathcal{L}^*$-algebra, $H$ a fixed Cartan subalgebra of $\mathcal{L}$, and $\overline{\cdot}$ denotes the conjugation operator on $\mathcal{L}$, a root of $\mathcal{L}$ relative to $H$ is a linear form commuting with the involution:

$$\alpha: (H, \ast) \longrightarrow (\mathbb{C}, \ast).$$

That is, $\alpha(h^*) = \overline{\alpha(h)}$ for any $h \in H$, such that there exists $v_\alpha \in \mathcal{L}$, $v_\alpha \neq 0$ satisfying $[h, v_\alpha] = \alpha(h)v_\alpha$ for any $h \in H$. The subspace

$$\mathcal{Y}_\alpha = \{v_\alpha \in \mathcal{L} : [h, v_\alpha] = \alpha(h)v_\alpha \text{ for all } h \in H\},$$

is called the root space associated to $\alpha$; it follows from this that if $\alpha$ is a root, then $-\alpha$ is also one and $(\mathcal{Y}_\alpha)^* = \mathcal{Y}_{-\alpha}$. The root space associated to the zero root is equal to the Cartan subalgebra $H$, using the Jacobi identity; one proves that if $\alpha + \beta$ is a root, then $[\mathcal{Y}_\alpha, \mathcal{Y}_\beta] \subseteq \mathcal{Y}_{\alpha + \beta}$, and if $\alpha + \beta$ is not a root, then $[\mathcal{Y}_\alpha, \mathcal{Y}_\beta] = 0$. Let $\Phi$ denote the set of nonzero roots of $\mathcal{L}$ relative to $H$, then we have the following Cartan decomposition $\mathcal{L} = H \oplus (\sum_{\alpha \in \Phi} \mathcal{Y}_\alpha)$, where $\oplus$ is the usual Hilbert space direct sum.

Let $\alpha$ be a root of $\mathcal{L}$ relative to $H$, then $\alpha$ is a linear functional on $H$; this implies that there exists a unique vector $h_\alpha \in H$ such that $\alpha(h) = \langle h, h_\alpha \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the inner product of $\mathcal{L}$. Consequently, $h_\alpha$ is self-adjoint, which means that $h_\alpha^* = h_\alpha$ and $h_\alpha = [v_\alpha, v_\alpha^*]$ for any $v_\alpha \in \mathcal{Y}_\alpha$ with $\|v_\alpha\| = 1$. Then, we have the following result.

Lemma 1 (see [15]). The set $\{h_\alpha : \alpha \in \Phi\}$ is total in $H$, i.e., for any $h \in H$, $(h, h_\alpha) = 0$ for all $h_\alpha$, implies $h = 0$.

Let $\mathcal{H}$ be a Hilbert space, the orthogonal dimension of $\mathcal{H}$ is denoted by $\dim \mathcal{H}$, i.e., the cardinality of an orthonormal basis for $\mathcal{H}$. We will define the cardinality of an arbitrary set $E$ by $|E|$.

Now, we will define the root system relative to a Cartan subalgebra of the semisimple $\mathcal{L}^*$-algebra $\mathcal{L}$.

Definition 1. Let $\mathcal{L}$ be a semisimple $\mathcal{L}^*$-algebra, $H$ a Cartan subalgebra of $\mathcal{L}$, and $\Phi$ the set of nonzero roots of $\mathcal{L}$ relative to $H$. A subset $\Phi_0$ of $\Phi$ will be called a root system of $\Phi$ if the following conditions are satisfied:

(i) If $\alpha \in \Phi_0$, then $-\alpha \in \Phi_0$

(ii) If $\alpha, \beta \in \Phi_0$, such that $\alpha + \beta \in \Phi$, then $\alpha + \beta \in \Phi_0$

We need some further notations; $\mathbb{R}$ and $\mathbb{Q}$ will refer to the real and the rational fields, respectively. For a subset $\mathcal{S}$ of $\Phi$, the set of all $\mathbb{C}$-linear combinations of elements of $\mathcal{S}$ will be denoted by $\mathcal{S} \mathbb{C}$ and the set of all $\mathbb{Q}$-linear combinations of elements of $\mathcal{S}$ by $\mathcal{S} \mathbb{Q}$. If we write $(\mathcal{S}) \mathbb{Q} = \mathcal{S} \mathbb{Q} \cap \Phi$, then $(\mathcal{S}) \mathbb{Q}$ is obviously a root system. The following results will be useful in our main proofs.

Lemma 2. Let $\mathcal{L}$ be a topologically simple $\mathcal{L}^*$-algebra and $\Phi$ the set of its nonzero roots relative to some Cartan subalgebra $H$. For any subset $\mathcal{S}$ of $\Phi$, there exists a topologically simple $\mathcal{L}^*$-subalgebra $\mathcal{L}_\mathcal{S}$ of $\mathcal{L}$, with Cartan subalgebra $H_\mathcal{S}$, such that

(i) $\mathcal{S} \subseteq \Phi_\mathcal{S}$, where $\Phi_\mathcal{S}$ is the set of roots of $\mathcal{L}_\mathcal{S}$ (relative to $H_\mathcal{S}$) and $\Phi_\mathcal{S} = (\mathcal{S}) \mathbb{Q} \cap \Phi$

(ii) $\mathcal{L}_\mathcal{S}$ is finite-dimensional if $\mathcal{S}$ is finite

(iii) $\mathcal{L}_\mathcal{S}$ is infinite-dimensional and $\dim \mathcal{L}_\mathcal{S} = |\mathcal{S}|$ if $\mathcal{S}$ is infinite.

Proof. See Proposition 3 in [14].

Definition 2. Let $\mathcal{L}$ be a topologically simple $\mathcal{L}^*$-algebra and $\Phi$ the set of nonzero roots relative to a Cartan subalgebra $H$. Let $\alpha, \beta \in \Phi$; we say that $\alpha$ is connected to $\beta$ if there are some $y_1, y_2, \ldots, y_k \in \Phi$ such that $\alpha + \beta = y_1 + y_2 + \ldots, y_k + \beta \in \Phi \cup \{0\}$.

Obviously, the connected relation is an equivalence relation on $\Phi$.

Lemma 3. Any two roots of a topologically simple $\mathcal{L}^*$-algebra are connected.

Proof. Let $\mathcal{S} = \{\beta_1, \beta_2\} \subseteq \Phi$, then by Lemma 2 there exists a finite-dimensional simple $\mathcal{L}^*$-algebra $\mathcal{L}_\mathcal{S}$ of $\mathcal{L}$, with Cartan subalgebra $H_\mathcal{S}$, such that $\beta_1$ and $\beta_2$ are roots of $\mathcal{L}_\mathcal{S}$. Using Lemma 1.3 in [8], we obtain that $\beta_1$ and $\beta_2$ are connected in $\mathcal{L}_\mathcal{S}$. Since $\Phi_\mathcal{S} \subseteq \Phi$, then $\beta_1$ and $\beta_2$ are connected in $\mathcal{L}$. 

3. Semi-Inner Biderivations on $\mathcal{L}^*$-Algebras and Commuting Linear Maps on Topologically Simple $\mathcal{L}^*$-Algebras

In this section, we introduce the notion of semi-inner biderivation in order to show that every continuous commuting linear map on a topologically simple $\mathcal{L}^*$-algebra $\mathcal{L}$
is a scalar multiple of the identity mapping. The aim of the first main theorem of this section is to prove that every semi-inner biderivation of a topologically simple $\mathcal{L}^*$-algebra is inner. To get this result, we have to show two lemmas.

**Definition 3.** Let $\mathcal{L}$ be an $\mathcal{L}^*$-algebra and $f: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ a biderivation of $\mathcal{L}$; $f$ is said to be a semi-inner biderivation of $\mathcal{L}$ if there exists two continuous linear maps $\phi: \mathcal{L} \to \mathcal{L}$ and $\psi: \mathcal{L} \to \mathcal{L}$ such that
\[
f(x, y) = [\phi(x), y] = [x, \psi(y)], \quad \forall x, y \in \mathcal{L},
\]
where $f$ will be denoted by $f_{\phi, \psi}$.

**Remark 1.** By using Lemma 2.1 in [8], any biderivation of a finite-dimensional simple complex Lie algebra is semi-inner (a linear map between two finite-dimensional vector spaces is continuous).

From now on, $\mathcal{L}$ will represent an infinite-dimensional simple topologically simple $\mathcal{L}^*$-algebra of arbitrary dimension and $\mathcal{L} = H \oplus \left( \sum_{\alpha \in \Phi, |\alpha| \neq 1} \mathcal{Y}^{\alpha} \right)$, where its Cartan decomposition is relative to a Cartan subalgebra $H (\Phi$ is the set of nonzero roots relative to $H)$.

**Lemma 4.** Let $f_{\phi, \psi}: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ be a semi-inner biderivation of $\mathcal{L}$, then for any $h \in H$, we have $\phi(h), \psi(h) \in H$.

**Proof.** For any $\alpha \in \Phi$, we select $x_\alpha \in \mathcal{Y}^{\alpha}$ such that $\|x_\alpha\|_1 = 1$, $x_{-\alpha} = x_\alpha \in \mathcal{Y}^{-\alpha}$, and $h \in H$. Then, we have
\[
\begin{align*}
[x_\alpha, x_{-\alpha}] &= h_\alpha, \\
h_{-\alpha} x_\alpha &= \alpha(h_\alpha) x_{-\alpha}, \\
h_\alpha x_{-\alpha} &= -\alpha(h_\alpha) x_\alpha.
\end{align*}
\]
Let $\alpha \in \Phi$, we denote by $\bar{\Phi}_\alpha$ the set $\Phi \setminus \{\alpha, -\alpha\}$. Let
\[
\begin{align*}
\phi(h_\alpha) &= a_1 h_1 + a_2 x_\alpha + a_3 x_{-\alpha} + \sum_{\beta \in \bar{\Phi}_\alpha} k_\beta x_\beta, \quad (9) \\
\phi(x_\alpha) &= b_1 h_2 + b_2 x_\alpha + b_3 x_{-\alpha} + \sum_{\beta \in \bar{\Phi}_\alpha} t_\beta x_\beta, \quad (10) \\
\phi(x_{-\alpha}) &= c_1 h_3 + c_2 x_\alpha + c_3 x_{-\alpha} + \sum_{\beta \in \bar{\Phi}_\alpha} s_\beta x_\beta, \quad (11) \\
\psi(h_\alpha) &= s_1 h_4 + s_2 x_\alpha + s_3 x_{-\alpha} + \sum_{\beta \in \bar{\Phi}_\alpha} m_\beta x_\beta, \quad (12) \\
\psi(x_\alpha) &= p_1 h_5 + p_2 x_\alpha + p_3 x_{-\alpha} + \sum_{\beta \in \bar{\Phi}_\alpha} n_\beta x_\beta, \quad (13) \\
\psi(x_{-\alpha}) &= q_1 h_6 + q_2 x_\alpha + q_3 x_{-\alpha} + \sum_{\beta \in \bar{\Phi}_\alpha} r_\beta x_\beta. \quad (14)
\end{align*}
\]
By equations (10) and (14), we have
\[
\begin{align*}
f(h_\alpha, x_\alpha) &= \left[\phi(h_\alpha), x_\alpha\right] = a_1 a(h_\alpha) x_{-\alpha} - a_3 h_\alpha \\
&\quad + \sum_{\beta \in \bar{\Phi}_\alpha} k_\beta [x_\beta, x_\alpha],
\end{align*}
\]
\[
\begin{align*}
f(h_\alpha, x_{-\alpha}) &= \left[\phi(h_\alpha), x_{-\alpha}\right] = a_1 a(h_\alpha) p_2 x_{-\alpha} - a_3 h_\alpha \\
&\quad + \sum_{\beta \in \bar{\Phi}_\alpha} m_\beta [x_\beta, x_{-\alpha}]
\end{align*}
\]
If we compare the two equations above and $[x_\beta, x_\alpha] \in L_{\alpha, \beta} \neq H$ since $\beta \in \bar{\Phi}_\alpha$, we obtain $a_j = 0$. In the same way, by considering $f(h_\alpha, x_{-\alpha})$ with equations (10) and (14), $a_3 = 0$. Similarly, considering the images $f(x_\alpha, h_\alpha)$ and $f(x_{-\alpha}, h_\alpha)$, we obtain $s_2 = s_3 = 0$, by equations (11)-(13).

If we put $a_1 h_1 = \tilde{h}^\alpha \in \eta$ and $s_1 h_3 = \tilde{h}^\alpha \in \eta$, then equations (9) and (12) can be written as follows:
\[
\begin{align*}
\phi(h_\alpha) &= \tilde{h}^\alpha + \sum_{\beta \in \bar{\Phi}_\alpha} k_\beta x_\beta, \quad (16) \\
\psi(h_\alpha) &= \tilde{h}^\alpha + \sum_{\beta \in \bar{\Phi}_\alpha} m_\beta x_\beta. \quad (17)
\end{align*}
\]
Now, for any root $\gamma \in \bar{\Phi}_\alpha$, the set $\Phi \setminus \{\alpha, -\alpha, \gamma, -\gamma\}$ is denoted by $\bar{\Phi}_{\alpha, \gamma}$. By equations (16) and (17), we can write
\[
\begin{align*}
\phi(h_\gamma) &= \tilde{h}^\gamma + k_\gamma x_\gamma + k_{-\gamma} x_{-\gamma} + \sum_{\beta \in \bar{\Phi}_{\alpha, \gamma}} k_\beta x_\beta, \\
\psi(h_\gamma) &= \tilde{h}^\gamma + m_\gamma x_\gamma + m_{-\gamma} x_{-\gamma} + \sum_{\beta \in \bar{\Phi}_{\alpha, \gamma}} m_\beta x_\beta.
\end{align*}
\]
The two equations above imply that
\[
\begin{align*}
f(h_\gamma, x_\gamma) &= \left[\phi(h_\gamma), x_\gamma\right] = -\gamma(h_\gamma) k_\gamma x_\gamma + \gamma(h_\gamma) k_{-\gamma} x_{-\gamma} \\
&\quad - \sum_{\beta \in \bar{\Phi}_{\alpha, \gamma}} k_\beta [x_\beta, x_\gamma],
\end{align*}
\]
\[
\begin{align*}
f(h_\gamma, x_{-\gamma}) &= \left[\phi(h_\gamma), x_{-\gamma}\right] = \gamma(h_\gamma) m_\gamma x_{-\gamma} - \gamma(h_\gamma) m_{-\gamma} x_\gamma \\
&\quad + \sum_{\beta \in \bar{\Phi}_{\alpha, \gamma}} m_\beta [x_\beta, x_{-\gamma}]
\end{align*}
\]
By comparing, one has $k_\gamma = k_{-\gamma} = \mu_\gamma = \mu_{-\gamma} = 0$. Due to the arbitrariness of $\gamma$, we obtain $\phi(h_\alpha) = \tilde{h}^\alpha \in H$ and $\psi(h_\alpha) = \tilde{h}^\alpha \in H$. By Lemma 1, the set $\{h_\alpha, \alpha \in \Phi\}$ is total in $H$ and $\phi$ and $\psi$ are continuous, and we have $\phi(H) \subset H$ and $\psi(H) \subset H$.

**Lemma 5.** Let $f_{\phi, \psi}: \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ be a semi-inner biderivation of $\mathcal{L}$, then there is a complex number $\lambda$ such that
\[ \phi(x) = \psi(x) = \lambda x, \quad \forall x \in \mathcal{L}_a, \alpha \in \Phi. \]  
(20)

**Proof.** Let \( h \in H \) and \( \alpha, \gamma \in \Phi \) such that \( \alpha \neq \gamma \), then we have
\[ f(x_{\alpha}, h) = [\phi(x_{\alpha}), h] = [-h, \phi(x_{\alpha})], \]  
(21)
\[ f(x_{\alpha}, h) = [x_{\alpha}, \psi(h)] = [-\psi(h), x_{\alpha}] = -\alpha(\psi(h))x_{\alpha}. \]  
(22)

Let
\[ \phi(x_{\alpha}) = \tilde{h}^\alpha + \sum_{\beta \in \Phi} t_{\beta}x_{\beta}, \]  
(23)
where \( t_{\beta} \in \mathbb{C} \) and \( \tilde{h}^\alpha \in H \). Using equations (21) and (23), we obtain
\[ f(x_{\alpha}, h) = -\sum_{\beta \in \Phi} t_{\beta}\beta(h)x_{\beta}. \]  
(24)

Combining equations (22) and (24), we obtain \( t_{\alpha}(\alpha(h)) = a(\psi(h)) \) and \( t_{\beta}(\beta(h)) = 0 \) for every \( \beta \in \Phi \).\( \Box \)

**Remark 2.** In the above proof, there is another method to show that \( t_{\gamma} = 0 \) if \( \gamma \neq \alpha \). Indeed, using equation (23), we have
\[ f(x_{\alpha}, h_{\gamma}) = -[\phi(x_{\alpha}), h_{\gamma}] = -\phi(x_{\alpha}), \]  
(25)
\[ f(x_{\alpha}, h_{\gamma}) = [x_{\alpha}, \psi(h_{\gamma})] = k_{\alpha}[x_{\alpha}, h_{\gamma}] - \alpha(h_{\gamma})x_{\alpha}. \]  
(26)

By combining equations (27) with (28), we first see that \( \beta(\tilde{h}^\alpha) = \alpha(\tilde{h}^\alpha) = 0 \) if \( \alpha \neq \beta \). However, \( -\alpha(\tilde{h}^\alpha) = \alpha(-\tilde{h}^\alpha) = 0 \) by taking \( \beta = -\alpha \). This means that \( \beta(\tilde{h}^\alpha) = 0 \) for all \( \beta \in \Phi \), i.e., \( \tilde{h}^\alpha \in \bigcap_{\beta \in \Phi} \ker \beta \), which gives \( \tilde{h}^\alpha = 0 \). Similarly, by taking \( \alpha = -\beta \), we have \( \tilde{h}^\beta = 0 \). Therefore, by equations (25) and (26), one can obtain
\[ \phi(x_{\alpha}) = t_{\alpha}x_{\alpha}, \]  
(29)
\[ \psi(x_{\alpha}) = k_{\alpha}x_{\alpha}. \]  
(30)

Using equation (29) and \( f(x_{\alpha}, x_{\gamma}) = [\phi(x_{\alpha}), x_{\gamma}] = [x_{\alpha}, \psi(x_{\gamma})] \), it follows that
\[ t_{\alpha} = k_{-\alpha}, \quad \forall \alpha \in \Phi. \]  
(31)

Then, comparing equations (27) with (28), we obtain
\[ t_{\gamma} = k_{\gamma}, \quad \text{for} \, \alpha + \beta \in \Phi. \]  
(32)

Let \( S = [\alpha] \subseteq \Phi \), then by Lemma 2 there exists a finite-dimensional simple \( \mathcal{L}^* \)-algebra \( S \) of \( \mathcal{L} \), with Cartan subalgebra \( H_S \), such that \( \alpha \) is a root of \( \mathcal{L}_S \). Now, let us prove that \( f_{\phi_\psi}(\mathcal{L}_S \times \mathcal{L}_\gamma) \subseteq \mathcal{L}_S \) indeed; let \( x, y \in \mathcal{L}_S \), then
\[ x = h_{\gamma} + \sum_{\beta \in \Phi_\gamma} x_{\beta} \quad \text{and} \quad y = h_{\gamma} + \sum_{y \in \Phi_\gamma} y_{\gamma}. \]  
(33)
\[ f_{\phi_\psi}(x, y) = [\phi(x), y] = \sum_{\beta \in \Phi_\gamma} [\phi(h_{\gamma}), y_{\gamma}] + \sum_{\beta \in \Phi_\gamma} [y_{\beta}x_{\beta}, h_{\gamma}] + \sum_{\beta \in \Phi_\gamma} \sum_{y \in \Phi_\gamma} [t_{\beta}x_{\beta}, y_{\gamma}]. \]  
(34)

Then, \( f_{\phi_\psi}(\mathcal{L}_S \times \mathcal{L}_\gamma) \) is a biderivation of \( \mathcal{L}_S \) by Theorem 2.4 in [8], there exists \( \mu \in \mathbb{C} \) such that \( f_{\phi_\psi}(x, y) = \phi(x, y) \) for any \( x, y \in \mathcal{L}_S \). Then, \( \mu[x_{\beta}, y] = t_{\beta}[x_{\beta}, y] = -k_{\beta}[y, x_{\beta}] \) for any \( \beta \in \Phi_\gamma \) and \( y \in \mathcal{L}_\gamma \); this implies that
\[ t_{\alpha} = t_{-\alpha} = k_{\alpha} = k_{-\alpha} = \mu, \quad \text{for arbitrary} \, \alpha \in \Phi. \]  
(35)

If we pose \( t_{\alpha} = \lambda \) in equation (34), we get our result. \( \Box \)

**Remark 3.** In the case where \( \mathcal{L} \) is separable, Theorem 2 in [16] will facilitate some difficulties encountered in the above proof (the above proof will look like the proof of Lemma 2.2 in [8]).

Thanks to the above Lemmas, we can state our first main theorem.

**Theorem 1.** Let \( \mathcal{L} \) be a topologically simple \( \mathcal{L}^* \)-algebra. Then, \( f \) is a semi-inner biderivation of \( \mathcal{L} \) if and only if it is inner, i.e., there is a complex number \( \lambda \) such that \( f(x, y) = \lambda[x, y] \).

**Proof.** Let \( \mathcal{L} \) be a topologically simple \( \mathcal{L}^* \)-algebra, any inner biderivation \( f \) of \( \mathcal{L} \) such that \( f(x, y) = \lambda[x, y] \) is a semi-inner biderivation \( f_{\phi_\psi} \) with
\[ \phi = \psi = \lambda \text{id}_{\mathcal{L}^*}. \]  
(36)

Now, let us prove the “only if direction.”
The first case: if $\mathcal{L}$ is finite-dimensional, the result follows from Theorem 2.4 in [8]. The second case: if $\mathcal{L}$ is infinite-dimensional, then, for $f_{\phi, \psi}$ a semi-inner biderivation of $\mathcal{L}$, by using Lemma 5, there is $\lambda \in \mathbb{C}$ such that $\phi(x) = \psi(x) = \lambda x$, $\forall x \in L$, $\alpha \in \Phi$. For any $h \in H$ and $\alpha \in \Phi$, we have $f(h, x) = [\phi(h), x] = [h, \psi(x)]$. This implies that $\alpha(\phi(h)x) = \lambda(h)x$, which means that

$$a(\lambda h - \phi(h)) = 0, \quad \forall \alpha \in \Phi, \quad (37)$$

this implies $h = \lambda h$ for all $h \in H$. For any $x, y \in L$, with $x = h + \sum_{\alpha \in \Phi} l_{\alpha}x_{\alpha}$, where $l_{\alpha} \in \mathbb{C}$, with $l_{\alpha} = 0$ except for a countable number. Therefore,

$$f(x, y) = [\phi(x), y] = \sum_{\alpha \in \Phi} l_{\alpha} \phi(x_{\alpha}) + \phi(h), y = \sum_{\alpha \in \Phi} l_{\alpha} \lambda x_{\alpha} + \lambda h, y = [\lambda x, y]. \quad (38)$$

In recent years, many authors have studied commuting linear maps of certain algebra structures; for some of these achievements, refer to [3, 4, 6, 8, 9, 17, 18]. It should be noted that this subject is not new since it was studied in 1957, exactly in Posner’s works [19]. As we said in the introduction, if $f$ is a commuting linear map on $\mathcal{L}$, then $[\phi(x), y] = [x, \phi(y)]$ for any $x, y \in \mathcal{L}$, and $f(x, y) = [\phi(x), y] = [x, \phi(y)]$ is a biderivation of $\mathcal{L}$. Using Theorem 1, the present theorem aims to determine all continuous commuting linear maps on topologically simple $L^\ast$-algebras. \hfill $\square$

**Theorem 2.** Let $\mathcal{L}$ be a topologically simple $\mathcal{L}^\ast$-algebra. Then, every continuous linear map $\phi$ on $\mathcal{L}$ is a commuting linear map if and only if it is a scalar multiplication map on $\mathcal{L}$.

**Proof.** Let $\phi$ be a continuous commuting linear map on a topologically simple $\mathcal{L}^\ast$-algebra $\mathcal{L}$. Then, $f$ defined by $f(x, y) = [\phi(x), y] = [x, \phi(y)]$ is a semi-inner biderivation of $\mathcal{L}$. By Theorem 1, we have $f(x, y) = [\phi(x), y] = \lambda[x, y]$ for some $\lambda \in \mathbb{C}$. Since $\mathcal{L}$ is topologically simple $\mathcal{L}^\ast$-algebra and $y$ is arbitrary, therefore, we have $\phi(x) = \lambda x$.

The following Lemma is one of the interesting results given in (see p.7 in [3]). \hfill $\square$

**Lemma 6.** Let $\mathcal{L}$ be a simple Lie algebra over an algebraically closed field $F$ of characteristic different from 2 such that $\text{card}(\mathcal{F}) > \dim(\mathcal{L})$ (cardinal of $\mathcal{F}$ is the cardinality of $\mathbb{F}$ and $\dim(\mathcal{L})$ is the dimension of $\mathcal{L}$). For any skew-symmetric biderivation $f$ of $\mathcal{L}$, there exists $\lambda \in F$ such that

$$f(x, y) = \lambda[x, y], \quad \text{for all } x, y \in \mathcal{L}. \quad (39)$$

**Remark 4.** Let $\mathcal{L}$ be a simple complex Lie algebra of countable dimension and $\phi$ a commuting linear map on $\mathcal{L}$. Then, $\phi$ is of the form $\phi(x) = \lambda x$ for all $x \in \mathcal{L}$ where $\lambda \in \mathbb{C}$. Indeed, let $\phi$ be a commuting linear map on $\mathcal{L}$, then $f(x, y) = [\phi(x), y] = [x, \phi(y)]$ for all $x, y \in \mathcal{L}$ which is a skew-symmetric biderivation of $\mathcal{L}$. By Lemma 6, there exists $\lambda \in \mathbb{C}$, such that $f$ is of the form $f(x, y) = \lambda[x, y]$ for all $x, y \in \mathcal{L}$. Then, $\phi$ is of the form $\phi(x) = \lambda x$ for all $x \in \mathcal{L}$.

We mention here that this proof does not seem to work in our case when the underlying Hilbert space of the $\mathcal{L}^\ast$-algebra is infinite-dimensional. However, Corollary 2.4 in [3] may simplify some of the proofs in our paper.

4. **Conclusions**

This paper aimed to show that every continuous commuting linear map on a topologically simple $\mathcal{L}^\ast$-algebra $\mathcal{L}$ is a scalar multiplication map on $\mathcal{L}$.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest.

**Acknowledgments**

The author would like to thank Professor M. Ait Ben Haddou and Professor M. Raouyane.

**References**


