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Research Article

Affine Magic Cubes of Order 4: Concepts and Applications

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We convert a classical magic cube of order 4, which is an arrangement of $\{1, 2, \ldots, 64\}$, to a cube of order 4 whose entries belong to F_2^6. By using finite-dimensional vector spaces over the field F_2, we introduce the notion of affine magic cubes and study their properties. The obtained results can be applied to describe some features of various types of magic cubes of order 4.

1. Introduction

A magic cube of order n is an arrangement of the numbers $\{1, 2, \ldots, n^3\}$ in an $n \times n \times n$ cube in which the entries in each row, column, pillar, and triagonal sum to the same number. In 1650, Pierre de Fermat found a cube of order 4 which is almost magic but its four triagonals failed to have the same sum. This type of cube is called semi-magic. Joseph Sauveur later in 1710, published a cube of order 5 with all correct triagonal sums, however, it is still not magic because most orthogonal lines sum incorrectly. The first published magic cube we have been able to locate was published in Germany in 1898 by Hermann Schubert [1]. The appearance of magic cubes has subsequently been found by many mathematicians, for example, other well-known magic cubes are found by Hermann Weidemann and John Hendricks.

Appearing as a 3-dimensional shape, a magic cube is more complicated to study than a magic square. This leads to a study on magic cubes by transferring them to magic squares first, e.g., Allan Adler and Shuo-Yen Robert Li in [2] demonstrated a method for transforming a magic cube of order m into a magic square of order $2m$.

In our work, we transform a magic cube of order 4 into a magic square of order 8 and study the relations between magic cubes and linear algebra. Motivated by John Henrich in [3] where he introduced an affine magic square of order 4, we introduce the notion of affine magic cubes of order 4 and study their properties. Moreover, we show that the famous magic cubes by Hermann Schubert, Hermann Weidemann, and John Hendricks are affine. Their different features are also discussed.

2. Preliminary

In this section, the definitions of affine subspaces and affine transformations are given. We also introduce the definitions of magic cubes and their special kinds and provide some examples.

2.1. Affine Subspaces and Affine Functions

Definition 1 (see [4]). Let V be a finite-dimensional vector space over a field F. For a vector v in V and a subspace W of V, the set $v + W := \{v + w \mid w \in W\}$ is called an *affine subspace* of V.

Definition 2 (see [4]). Let V be a finite-dimensional vector space over a field F. We say that a function $f: V \longrightarrow F$ is an *affine function* if there are a linear functional $g: V \longrightarrow F$ and a constant $\beta \in F$ such that $f(x) = g(x) + \beta$ for all $x \in V$. Here, the linear functional g and the constant β with respect to f are unique.

From the above definitions, an affine subspace of V needs not to be a subspace of V, and an affine function does not need to be a linear functional.

Example 1. Let F be the field $F_2 = \{0, 1\}$ and let V be the vector space F_2^3 over F_2, i.e., $V = \{000, 001, 010, 011, 100, 101, 110, 111\}$. The set $\{010, 101\}$ is not a subspace of V. However, it is an affine subspace of V because $\{010, 101\} = 010 + \{000, 111\}$ is the translation of the subspace $\{000, 111\}$. The constant function $f(x) = 1$ is not a linear functional but f is an affine function when the corresponding g is the zero linear functional and $\beta = 1$.

Example 2. Let F be a field and $\alpha_1, \alpha_2, \ldots, \alpha_n$ scalars in F. Let $g\colon F^n \longrightarrow F$ be the function defined by

$$g(x_1, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n. \tag{1}$$

Clearly, g is a linear function. For any $\beta \neq 0$, the function $f(x_1, \ldots, x_n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + \beta = g(x_1, \ldots, x_n) + \beta$ is not a linear functional, but it is an affine function.

2.2. Magic Cubes

Definition 3. For $n \in \mathbb{N}$, a *cube of order n* is an $n \times n \times n$ cube of n^3 numbers. In a cube of order n, a *row* is a set of n numbers on a horizontal line from left to right or right to left. A *column* is a set of n numbers on a line from back to front or front to back. A *pillar* is a set of n numbers on a vertical line from top to bottom or bottom to top. A *face diagonal* is a diagonal on one of the faces of a cube. The *main triagonal* is a set of n numbers on a line from a corner of a cube, though the center of the cube, toward the opposite corner. A *bent triagonal* is a set of n numbers on a line from a corner of a cube, through the center of the cube, and then toward the other corners different from the opposite corner. A *broken triagonal* is a set of numbers that follow a line parallel to the main triagonal of a cube and continues on the corresponding point of an opposite face whenever it reaches the face of the cube.

Note that a broken triagonal may consist of 2 or 3 segments. For $n \geq 2$, a cube of the order n has n^2 rows, n^2 columns, n^2 pillars, and 4 main triagonals.

Definition 4. A *magic cube of order n* is an $n \times n \times n$ cube of n^3 numbers such that the sums of the numbers on each row, each column, each pillar, and each of the main triagonals are equal to the same number called a *magic sum*.

In this paper, all entries in a cube are thought of as natural numbers $1, 2, \ldots, n^3$ where each number is used only once. Then, the magic sum of a magic cube of order n becomes $n(n^3 + 1)/2$.

Example 3. In Figure 1, there are 4 rows $\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}$, 4 columns $\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}$, 4 pillars $\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}$, and 4 main triagonals $\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$.

Example 4. From Figure 2, this cube is a magic cube of order 3, and there are 9 rows, 9 columns, 9 pillars, and 4 main triagonals. The sum of the numbers in any row, column, and pillar is 42 (for example, row: $2 + 13 + 27$, column: $2 + 22 + 18$, and pillar: $2 + 16 + 24$). The sum along each of the main triagonals is also 42 $(2 + 14 + 26, 1 + 14 + 27, 18 + 14 + 10, 24 + 14 + 4)$. However, the sum along every face diagonal is not necessary to be the same as the magic sum (for example, $18 + 7 + 26 \neq 42$). Examples of bent triagonals are $\{2, 14, 27\}, \{2, 14, 18\}, \{2, 14, 4\}, \{2, 14, 24\}, \{2, 14, 1\}, \{2, 14, 10\}$, and $\{1, 14, 26\}$. The sum along each bent triagonal is not required to be the same as the magic sum.

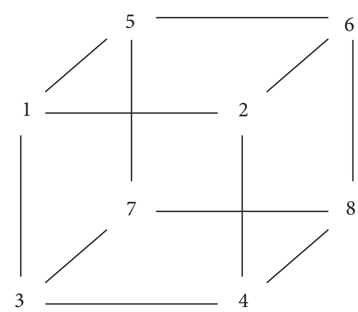

Figure 1: An example of cubes of order 2 but not magic.

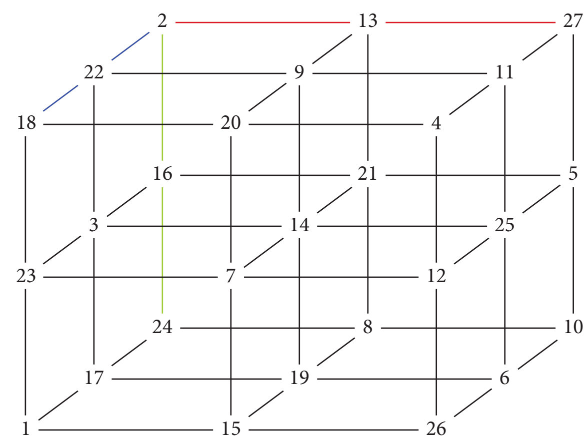

Figure 2: An example of magic cubes of order 3.

The above example shows that the sum of entries on some lines of a magic cube needs not be the same as the magic sum, for example, on bent triagonals or broken triagonals. This leads to a study of specific types of magic cubes. We shall give definitions of the types discussed in this work.

Definition 5. A *bent triagonal* magic cube is a magic cube such that the sum of all entries in each bent triagonal equals the magic sum. A *pantriagonal* magic cube is a magic cube such that the sum of all entries in each broken triagonal equals the magic sum.

Example 5. It is clear that the cube of order 3 in example 4 is not pantriagonal. In fact, the smallest pantriagonal magic cube possible is of order 4.

Definition 6. A *semi-pantriagonal* magic cube of order n is a magic cube of order n such that the sum of all entries in each 2-segment broken triagonals which each segment contains $n/2$ cells when n is even and $(n-1)/2$ cells (the center cell is added) or $(n+1)/2$ cells (the center cell is subtracted) when n is odd equals the magic sum.

Example 6. From Figure 2, the sum of all entries in each 2-segment broken triagonals in which each segment contains $(3+1)/2 = 2$ cells when the center cell is subtracted equals the magic sum. In fact, $13+25+3+15-14 = 11+21+7+17-14 = 20+25+3+8-14 = 22+7+21+6-14 = 42$. Clearly, the sum of all entries in each 2-segment broken triagonals which each segment contains $(3-1)/2 = 1$ cell when the center cell is added also equals the magic sum because they are main triagonals in this case.

Clearly, all pantriagonal magic cubes are semi-pantriagonal. In a cube of order 4, there are 12 broken triagonals that consist of 2-segments where each segment contains 2 cells.

Definition 7. An *associated* magic cube of order n is a magic cube of order n such that every pair on the opposite sides of the center of the cube sums to the same number.

Example 7. The cube of order 3 in example 4 is associated because $2+26 = 22+6 = 18+10 = 20+8 = 4+24 = 11+17 = 27+1 = 13+15 = 9+19 = 16+12 = 3+35 = 23+5 = 7+21 = 28$.

Definition 8. A *compact* magic cube of order n is a magic cube of order n such that every $1\times2\times2$, $2\times1\times2$ or $2\times2\times1$ block, parallel to the side of the cube, sums to the magic sum.

Example 8. The cube of or order 3 in example 4 is not compact because $18+20+23+7 = 68 \neq 42$.

3. Magic Cubes of Order 4

We first consider a cube of order 4 whose entries belong to F_2^6. The points on each row (left to right), each column (back to front), and each horizontal plane (top to bottom), starting with 0 by using base-2 notation, are placed accordingly (see Figure 3). In [2], Alder and Li showed that one can transform a magic cube of order m into a magic square of order $2m$ by taking the rows of the cube one at a time and using them to fill up the square. Figure 4 is the resulting square of order 8 converted from the cube of order 4 when each pair of adjacent rows (no repeated row used) of the cube of Figure 3 is put into one row of the square.

4. Affine Magic Cubes of Order 4

On the finite-dimensional vector space F_2^6 over F_2, there are $2^6 = 64$ elements in F_2^6 and $\dim F_2^6 = 6$ with the standard basis $\{e_1 = 100000, e_2 = 010000, e_3 = 001000, e_4 = 000100, e_5 = 000010, e_6 = 000001\}$. From Figure 3, we observe that the first pillar is $\{000000, 100000, 010000, 110000\} = \text{span}\{e_1, e_2\}$, the first column is $\{000000, 001000, 000100, 001100\} = \text{span}\{e_3, e_4\}$, the first row is $\{000000, 000010, 000001, 000011\} = \text{span}\{e_5, e_6\}$, and the main triagonal is $\{000000, 101010, 010101, 111111\} = \text{span}\{e_1+e_3+e_5, e_2+e_4+e_6\}$. The following subspaces of F_2^6 and their parallel affine subspaces in Figure 3 are noticed:

(1) $\text{span}\{e_1, e_2\}$ and the affine subspaces parallel to it correspond to each pillar.

(2) $\text{span}\{e_3, e_4\}$ and the affine subspaces parallel to it correspond to each column.

(3) $\text{span}\{e_5, e_6\}$ and the affine subspaces parallel to it correspond to each row.

(4) $\text{span}\{e_1+e_3+e_5, e_2+e_4+e_6\}$ and the affine subspaces parallel to it correspond to 4 main triagonals and 12 broken triagonals consisting of 2-segments where each segment containing 2 cells.

(5) $\text{span}\{e_1+e_3+e_5, e_1+e_2+e_4+e_6\}$, $\text{span}\{e_1+e_3+e_5, e_2+e_3+e_4+e_6\}$, $\text{span}\{e_1+e_3+e_5, e_2+e_4+e_5+e_6\}$ and the affine subspaces parallel to them correspond to 2-segment and 3-segment broken triagonals.

(6) $\text{span}\{e_1+e_2+e_3+e_4+e_5+e_6\}$ and the affine subspaces parallel to it correspond to each pair on the opposite sides of the center of the cube.

(7) $\text{span}\{e_2, e_4\}$, $\text{span}\{e_2, e_6\}$, $\text{span}\{e_4, e_6\}$, $\text{span}\{e_2, e_3+e_4\}$, $\text{span}\{e_2, e_5+e_6\}$, $\text{span}\{e_4, e_5+e_6\}$, $\text{span}\{e_1+e_2, e_3+e_4\}$, $\text{span}\{e_1+e_2, e_5+e_6\}$, $\text{span}\{e_3+e_4, e_5+e_6\}$ and the affine subspaces parallel to them correspond to $1\times2\times2$, $2\times1\times2$, or $2\times2\times1$ block, parallel to the sides of the cube.

(8) $\text{span}\{e_1+e_3+e_6, e_2+e_4+e_6\}$, $\text{span}\{e_1+e_4+e_5, e_2+e_4+e_6\}$, $\text{span}\{e_1+e_4+e_6, e_2+e_4+e_6\}$, $\text{span}\{e_2+e_3+e_5, e_2+e_4+e_6\}$, $\text{span}\{e_2+e_3+e_6, e_2+e_4+e_6\}$, $\text{span}\{e_2+e_4+e_5, e_2+e_4+e_6\}$ and the affine subspaces parallel to them correspond to all bent triagonals in the cube.

Let $\{f_1, f_2, f_3, f_4, f_5, f_6\}$ be the dual basis of $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. Then, $f_i(e_j)$ is 1 if $i = j$ and 0 otherwise. Moreover, for any linear functional $\psi \in F_2^{6*}$, there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in F_2$ such that $\psi = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 + \alpha_5 f_5 + \alpha_6 f_6$. In [5], Henrich mentioned that each linear functional on F_2^4 either is constant or takes the values 0 and 1 equally often. The following proposition is an analogous result for nonzero linear functionals on F_2^6:

Proposition 1. *Let ψ be a linear functional on F_2^6 with values in F_2 and E a subspace of F_2^6 which is not contained in the null space of ψ. Then, on any affine subspace parallel to E, the functional ψ takes the values 0 and 1 equally often.*

Proof. Let $\widetilde{E}$ be an affine subspace parallel to E. Then, there exists a vector $v \in F_2^6$ such that $\widetilde{E} = E + v$. Let $A = \{u \in \widetilde{E} \mid \psi(u) = 0\}$ and $B = \{u \in \widetilde{E} \mid \psi(u) = 1\}$. Since E is not in the null space of ψ, there is $y \in E$ such that $\psi(y) = 1$. Then, $y + v \in \widetilde{E}$ and $\psi(y+v) = \psi(y) + \psi(v) = 1 + \psi(v)$.

Case 1: If $\psi(v) = 1$, then $\psi(y+v) = 1+1 = 0$, and hence $y + v \in A$. Define a map $\phi\colon A \longrightarrow B$ by

$$\phi(a) = a + y, \tag{2}$$

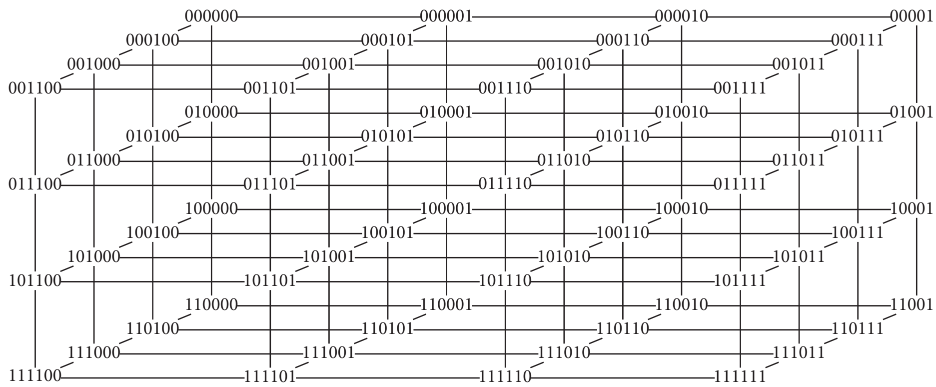

Figure 3: Cube of order 4 in base 2.

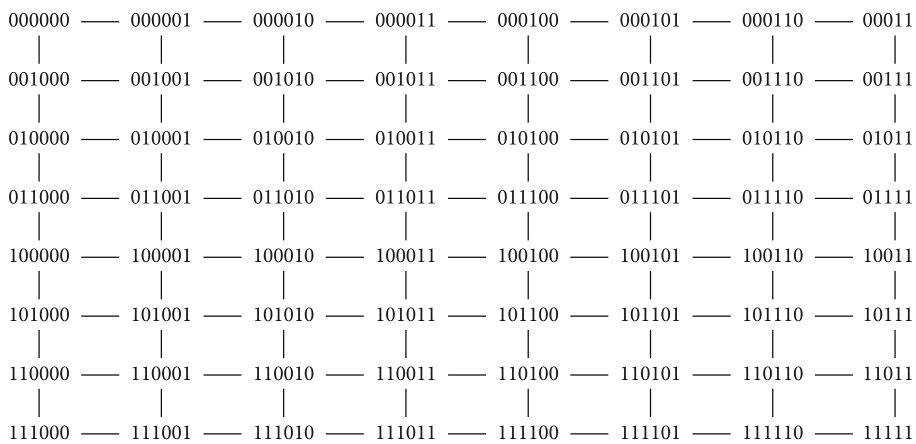

Figure 4: Square of order 8.

for all $a \in A$. Note that for all $a \in A$, $a + y \in \widetilde{E}$, and $\psi(a + y) = \psi(a) + \psi(y) = 0 + 1 = 1$, hence $a + y \in B$. It is easy to see that ϕ is injective. To show that ϕ is surjective, let $b \in B$. Then, $b + y \in \widetilde{E}$ such that $\psi(b + y) = \psi(b) + \psi(y) = 1 + 1 = 0$, and hence $b + y \in A$. Also, we have $\phi(b + y) = b + y + y = b$. Thus, ϕ is surjective and hence bijective.

Case 2: If $\psi(v) = 0$, then $\psi(y + v) = 1 + 0 = 1$, and hence $y + v \in B$. Define a map $\phi\colon B \longrightarrow A$ by

$$\phi(b) = b + y, \tag{3}$$

for all $b \in B$. Note that for all $b \in B$, $b + y \in \widetilde{E}$ and $\psi(b + y) = \psi(b) + \psi(y) = 1 + 1 = 0$, hence $b + y \in A$. It is easy to see that ϕ is injective. To show that ϕ is surjective, let $a \in A$. Then, $a + y \in \widetilde{E}$ such that $\psi(a + y) = \psi(a) + \psi(y) = 0 + 1$, and hence $a + y \in B$. Also, we have $\phi(a + y) = a + y + y = a$. Thus, ϕ is surjective and hence bijective.

Since A and B are finite, the number of elements in A must be equal to the number of elements in B from both cases. Therefore, ψ takes the values 0 and 1 equally often on $\widetilde{E}$. □

Definition 9. Let $V_1, V_2, \ldots, V_6$ be affine functions on F_2^6 with values in F_2. The map W from F_2^6 to $\mathbb{Z}$ given by

$$W(x) = \sum_{j=1}^{6} 2^{6-j} V_j(x) + 1, \tag{4}$$

is an *affine cube*.

In the discussion of a particular affine cube W, we shall denote the linear part of V_j by ψ_j.

Definition 10. Let $V_1, V_2, \ldots, V_6$ be affine functions on F_2^6 with values in F_2, and let W be the affine cube from F_2^6 to $\mathbb{Z}$ determined by all V_j. Let E be a subspace of F_2^6. Then, E is *magic* for W if each linear functional ψ_j of V_j is nonzero on E for all $j = 1, \ldots, 6$.

Proposition 2. *Let $V_1, V_2, \ldots, V_6$ be affine functions on F_2^6 with values in F_2 , and let E be a subspace of F_2^6 . Then, E is magic for the affine cube W determined by all V_j if and only if W has uniform sums on the affine subspaces parallel to E .*

Proof. Let E be a subspace of F_2^6 with dimension d. Suppose E is magic. Let $\widetilde{E}$ be an affine subspace parallel to E. By Proposition 1, each linear functional ψ_j takes the values 0 and 1 each 2^{d-1} times on $\widetilde{E}$. Since each affine function V_j is either $\psi_j + 0$ or $\psi_j + 1$, V_j takes the values 0 and 1 each 2^{d-1} times on $\widetilde{E}$. Thus, the sum in $\mathbb{Z}$ of V_j over $\widetilde{E}$ is 2^{d-1}. Hence, we have that the sum of W over $\widetilde{E}$ is

$$32 \cdot 2^{d-1} + 16 \cdot 2^{d-1} + 8 \cdot 2^{d-1} + 4 \cdot 2^{d-1} + 2 \cdot 2^{d-1} + 2^{d-1} + 2^{d}. \quad (5)$$

To prove the converse, suppose E is not magic. Let i be the smallest index such that ψ_i is zero on E. Since ψ_i is nonzero on F_2^6, there exists a vector $v \in F_2^6$ such that $\psi_i(v) \neq 0$. Set $\widetilde{E} = E + v$. If $V_i = \psi_i + 0$, then the sum in $\mathbb{Z}$ of V_i over E is 0 but the sum in $\mathbb{Z}$ of V_i over $\widetilde{E}$ is 2^d. Then, the sum of $2^{6-i}V_i$ over E is 0 and the sum of $2^{6-i}V_i$ over $\widetilde{E}$ is 2^{6+d-i}. This implies the sum of W over E is less than the sum of W over $\widetilde{E}$ because the sum in $\mathbb{Z}$ of V_j is between 0 and 2^d and $2^{6-i} > 2^{6-j}$ for all $j > i$. On the other hand, we can show that if $V_i = \psi_i + 1$, the sum of W over E is greater than the sum of W over $\widetilde{E}$. □

Definition 11. Let $V_1, V_2, \ldots, V_6$ be affine functions on F_2^6 with values in F_2 and let W be the affine cube from F_2^6 to $\mathbb{Z}$ determined by all V_j. Then, W is *nonsingular* if $\psi_1, \ldots, \psi_6$ are linearly independent.

Proposition 3. *Let $V_1, V_2, \ldots, V_6$, be affine functions on F_2^6 with values in F_2. If the affine cube W from F_2^6 to $\mathbb{Z}$ determined by all V_j is nonsingular, then the values of $W(x)$ are exactly $\{1, \ldots, 64\}$ with no omissions or duplications.*

Proof. Since $W(x) \in \mathbb{Z}, 1 \leq W(x) \leq 64$ for $x \in F_2^6$. Since there are 64 elements in F_2^6, it is sufficient to show that $W(x)$ has a nonrepeating value. Suppose x and y are elements of F_2^6 such that $W(x) = W(y)$. Then $V_j(x) = V_j(y)$ for $j = 1, \ldots, 6$ and hence $\psi_j(x) = \psi_j(y)$ for $j = 1, \ldots, 6$. However, W is nonsingular implies that $\psi_1, \ldots, \psi_6$ are linearly independent, and therefore, span F_2^{6*}. Let $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the dual basis of $\{\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6\}$ for F_2^6. Then, $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6$ and $y = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5 + \beta_6 e_6$ for some $\alpha_j, \beta_j \in F_2$ for $j = 1, \ldots, 6$. Since $\psi_j(x) = \psi_j(y)$ for $j = 1, \ldots, 6$, we have $\alpha_j = \beta_j$ for $j = 1, \ldots, 6$. It follows that $x = y$. □

Analogously to the affine magic squares, we will give the definition of affine magic cubes of order 4. Since we only study the magic cube of order 4, we shall omit "of order 4" for convenience.

Definition 12. An *affine magic cube* is an affine cube that is nonsingular and of which the subspaces $\text{span}\{e_1, e_2\}$, $\text{span}\{e_3, e_4\}$, $\text{span}\{e_5, e_6\}$, and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$ are magic.

In [3], it is shown that an affine magic square is quasi-pandiagonal or pandiagonal. The following theorem is the analog for affine magic cubes.

Theorem 1. *Let A be an affine magic cube. Then*

(1) A is semi-pantriagonal.

(2) A is pantriagonal if and only if $\text{span}\{e_1 + e_3 + e_5, e_1 + e_2 + e_4 + e_6\}$, $\text{span}\{e_1 + e_3 + e_5, e_2 + e_3 + e_4 + e_6\}$, and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_5 + e_6\}$ are magic.

Proof. Let V_j, $j = 1, \ldots, 6$, be affine functions on F_2^6 with values in F_2 which determine A. Let W be the affine cube from F_2^6 to $\mathbb{Z}$ determined by V_j, $j = 1, \ldots, 6$. Suppose W is an affine magic cube. This implies the subspaces $\text{span}\{e_1, e_2\}$, $\text{span}\{e_3, e_4\}$, $\text{span}\{e_5, e_6\}$, and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$ are magic. By Proposition 2, W has uniform sums on the affine subspaces parallel to $\text{span}\{e_1, e_2\}$, $\text{span}\{e_3, e_4\}$, $\text{span}\{e_5, e_6\}$, and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$. Note that the subspace $\text{span}\{e_1, e_2\}$ corresponds to the first pillar of A, and the affine subspaces parallel to it correspond to the other pillars. Likewise, $\text{span}\{e_3, e_4\}$ corresponds to the first column of A, and the affine subspaces parallel to it correspond to the other columns. Moreover, $\text{span}\{e_5, e_6\}$ corresponds to the first row of A, and the affine subspaces parallel to it correspond to the other rows. Finally, $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$ corresponds to the main triagonal of A, and the affine subspaces parallel to it correspond to the 4 main triagonals and 12 broken triagonals consist of 2-segments where each segment containing 2 cells of A. Thus, A is semi-pantriagonal. By Proposition 2, the subspaces $\text{span}\{e_1 + e_3 + e_5, e_1 + e_2 + e_4 + e_6\}$, $\text{span}\{e_1 + e_3 + e_5, e_2 + e_3 + e_4 + e_6\}$, and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_5 + e_6\}$ are all magic if and only if W has uniform sums on the affine subspaces parallel to $\text{span}\{e_1 + e_3 + e_5, e_1 + e_2 + e_4 + e_6\}$, $\text{span}\{e_1 + e_3 + e_5, e_2 + e_3 + e_4 + e_6\}$ and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_5 + e_6\}$ which correspond to 2-segment and 3-segment broken triagonals, i.e., A is pantriagonal by definition. □

The following results follow directly from Proposition 2 and the definitions of associated, compact, and bent triagonal magic cubes.

Theorem 2. *Let A be an affine magic cube. Then*

(1) A is associated if and only if $\text{span}\{e_1 + e_2 + e_3 + e_4 + e_5 + e_6\}$ is magic.

(2) A is compact if and only if $\text{span}\{e_2, e_4\}$, $\text{span}\{e_2, e_6\}$, $\text{span}\{e_4, e_6\}$, $\text{span}\{e_2, e_3 + e_4\}$, $\text{span}\{e_2, e_5 + e_6\}$, $\text{span}\{e_4, e_5 + e_6\}$, $\text{span}\{e_1 + e_2, e_3 + e_4\}$, $\text{span}\{e_1 + e_2, e_5 + e_6\}$ and $\text{span}\{e_3 + e_4, e_5 + e_6\}$ are magic.

(3) A is bent triagonal if and only if $\text{span}\{e_1 + e_3 + e_6, e_2 + e_4 + e_6\}$, $\text{span}\{e_1 + e_4 + e_5, e_2 + e_4 + e_6\}$, $\text{span}\{e_1 + e_4 + e_6, e_2 + e_4 + e_6\}$, $\text{span}\{e_2 + e_3 + e_5, e_2 + e_4 + e_6\}$, $\text{span}\{e_2 + e_3 + e_6, e_2 + e_4 + e_6\}$. and $\text{span}\{e_2 + e_4 + e_5, e_2 + e_4 + e_6\}$ are magic.

5. Applications

The results we have derived in this work can be applied to study some features of magic cubes. In this section, we will discuss some well-known magic cubes of order 4 by applying our results to these affine magic cubes.

5.1. Hermann Schubert's Magic Cube. The following magic cube of order 4 was published by Hermann Schubert in Germany in 1898 [1]. It is the first published magic cube of order 4 by the now-accepted rules.

By subtracting 1 from each number in Figure 5 and writing them in base 2, we obtain the magic cube on F_2^6 as in

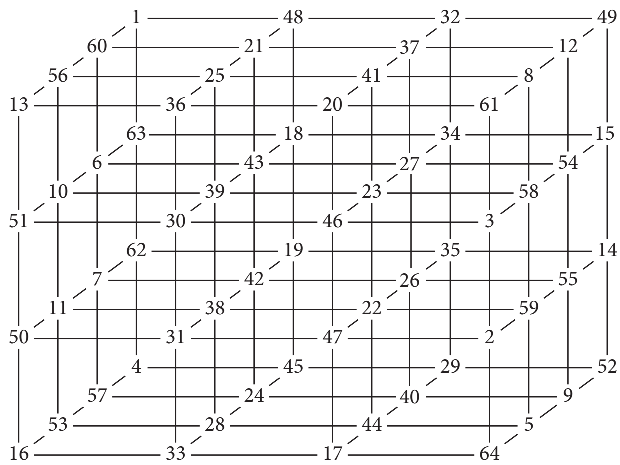

FIGURE 5: Schubert 1898.

Figure 6. Next, we would like to find affine functions V_1, V_2, V_3, V_4, V_5, and V_6 which define Figure 6. Since 000000 needs to be matched with 000000, the following can be derived:

$$\begin{gathered} V_1(000000) = 0, V_2(000000) = 0, V_3(000000) = 0, \\ V_4(000000) = 0, V_5(000000) = 0, V_6(000000) = 0. \end{gathered} \tag{6}$$

Hence, the affine functions must be in the following forms:

$$\begin{gathered} V_1 = \psi_1, V_2 = \psi_2, V_3 = \psi_3, \\ V_4 = \psi_4, V_5 = \psi_5, V_6 = \psi_6, \end{gathered} \tag{7}$$

where, ψ_j is the linear part of V_j for $j = 1, \ldots, 6$. Note that in this case V_j is a linear functional for $j = 1, \ldots, 6$. To find linear parts $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$, and ψ_6, we consider mappings of basis elements e_1, e_2, e_3, e_4, e_5, and e_6 of F_2^6 in Table 1.

Thus, the values of $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$, and ψ_6 of the basis are as in Table 2.

Hence, the affine functions which determine Figure 6 are

$$\begin{aligned} V_1 &= f_1 + f_2 + f_3 + f_4 + f_6, \\ V_2 &= f_1 + f_2 + f_3 + f_4 + f_5, \\ V_3 &= f_1 + f_2 + f_4 + f_5 + f_6, \\ V_4 &= f_1 + f_2 + f_3 + f_5 + f_6, \\ V_5 &= f_2 + f_3 + f_4 + f_5 + f_6, \\ V_6 &= f_1 + f_3 + f_4 + f_5 + f_6, \end{aligned} \tag{8}$$

and the corresponding number is $W(x) = \sum_{j=1}^{6} 2^{6-j} V_j(x) + 1$. Note that W is an affine function, and the linear part ψ_j of each V_j are as follows:

$$\begin{aligned} \psi_1 &= f_1 + f_2 + f_3 + f_4 + f_6, \\ \psi_2 &= f_1 + f_2 + f_3 + f_4 + f_5, \\ \psi_3 &= f_1 + f_2 + f_4 + f_5 + f_6, \\ \psi_4 &= f_1 + f_2 + f_3 + f_5 + f_6, \\ \psi_5 &= f_2 + f_3 + f_4 + f_5 + f_6, \\ \psi_6 &= f_1 + f_3 + f_4 + f_5 + f_6. \end{aligned} \tag{9}$$

It is easy to see that $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$, and ψ_6 are linearly independent. Thus, W is nonsingular. Since each ψ_j is nonzero on the subspaces $\operatorname{span}\{e_1, e_2\}, \operatorname{span}\{e_3, e_4\}$, span $\{e_5, e_6\}$ and $\operatorname{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$, $\operatorname{span}\{e_1, e_2\}$, span $\{e_3, e_4\}, \operatorname{span}\{e_5, e_6\}$ and $\operatorname{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$ are magic. Therefore, Schubert's cube is an affine magic cube, and hence semi-pantriagonal. This cube is also associated because $\operatorname{span}\{e_1 + e_2 + e_3 + e_4 + e_5 + e_6\}$ is magic. However, it is not pantriagonal, not compact, and also not bent triagonal because the subspaces $\operatorname{span}\{e_1 + e_3 + e_5, e_1 + e_2 + e_4 + e_6\}$, span $\{e_2, e_3 + e_4\}$ and $\operatorname{span}\{e_1 + e_3 + e_6, e_2 + e_4 + e_6\}$ are not magic.

5.2. Hermann Weidemann's Magic Cube. The following magic cube of order 4 by Hermann Weidemann was first published in 1922 [5].

To find the affine functions V_1, V_2, V_3, V_4, V_5, and V_6, which define the Modified Weidemann, obtained by subtracting one from each number in Figure 7 and writing them in base 2, we observe that 000000 needs to map to 011000. Thus,

$$\begin{gathered} V_1(000000) = 0, V_2(000000) = 1, V_3(000000) = 1, \\ V_4(000000) = 0, V_5(000000) = 0, V_6(000000) = 0. \end{gathered} \tag{10}$$

Hence, the affine functions must be in the following forms:

$$\begin{gathered} V_1 = \psi_1 + 0, V_2 = \psi_2 + 1, V_3 = \psi_3 + 1, \\ V_4 = \psi_4 + 0, V_5 = \psi_5 + 0, V_6 = \psi_6 + 0, \end{gathered} \tag{11}$$

where, ψ_j is the linear part of V_j, for $j = 1, \ldots, 6$. Note that V_2 and V_3 are not linear in this case. To find linear parts $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$, and ψ_6, we consider mappings of basis elements e_1, e_2, e_3, e_4, e_5, and e_6 of F_2^6 in Table 3.

Hence, the affine functions which determine the Modified Weidemann are

$$\begin{aligned} V_1 &= f_2 + f_4 + f_5 + 0, \\ V_2 &= f_1 + f_2 + f_3 + f_4 + f_6 + 1, \\ V_3 &= f_1 + f_3 + f_6 + 1, \\ V_4 &= f_1 + f_2 + f_4 + f_5 + f_6 + 0, \\ V_5 &= f_1 + f_4 + f_5 + 0, \\ V_6 &= f_2 + f_3 + f_4 + f_6 + 0, \end{aligned} \tag{12}$$

and the corresponding number is $W(x) = \sum_{j=1}^{6} 2^{6-j} V_j(x) + 1$. Note that W is an affine function and the linear part ψ_j of each V_j are as follows:

$$\begin{aligned} \psi_1 &= f_2 + f_4 + f_5, \\ \psi_2 &= f_1 + f_2 + f_3 + f_4 + f_6, \\ \psi_3 &= f_1 + f_3 + f_6, \\ \psi_4 &= f_1 + f_2 + f_4 + f_5 + f_6, \\ \psi_5 &= f_1 + f_4 + f_5, \\ \psi_6 &= f_2 + f_3 + f_4 + f_6. \end{aligned} \tag{13}$$

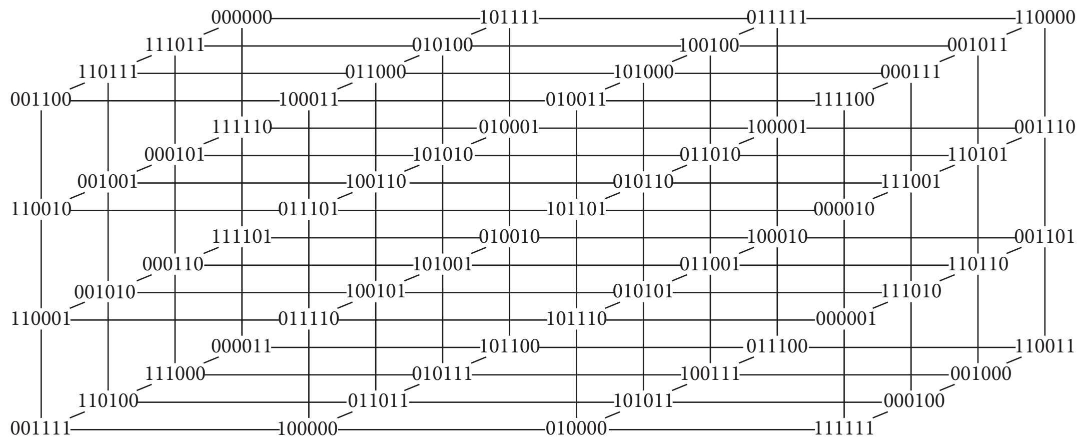

Figure 6: Modified Schubert.

Table 1: Mappings of basis elements to the Modified Schubert.

Basis	Image	$V_1 = \psi_1$	$V_2 = \psi_2$	$V_3 = \psi_3$	$V_4 = \psi_4$	$V_5 = \psi_5$	$V_6 = \psi_6$
100000	111101	1	1	1	1	0	1
010000	111110	1	1	1	1	1	0
001000	110111	1	1	0	1	1	1
000100	111011	1	1	1	0	1	1
000010	011111	0	1	1	1	1	1
000001	101111	1	0	1	1	1	1

Table 2: Linear parts of each basis element.

Basis	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6
100000	1	1	1	1	0	1
010000	1	1	1	1	1	0
001000	1	1	0	1	1	1
000100	1	1	1	0	1	1
000010	0	1	1	1	1	1
000001	1	0	1	1	1	1

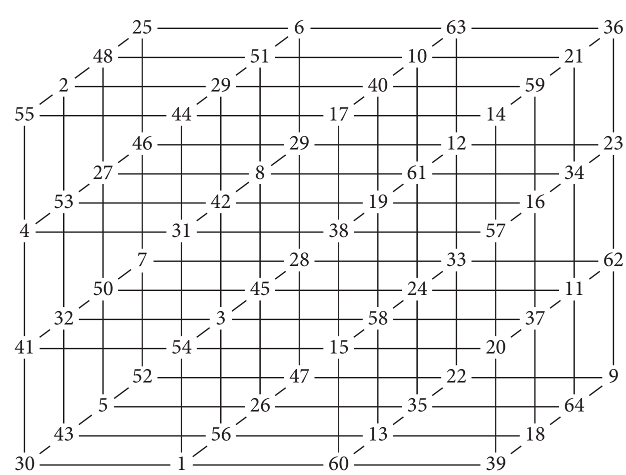

Figure 7: Weidemann 1922.

Table 3: Mappings of basis elements to the Modified Weidemann.

Basis	Image	$V_1 = \psi_1 + 0$	$V_2 = \psi_2 + 1$	$V_3 = \psi_3 + 1$	$V_4 = \psi_4 + 0$	$V_5 = \psi_5 + 0$	$V_6 = \psi_6 + 0$
100000	000110	0	0	0	1	1	0
010000	101101	1	0	1	1	0	1
001000	000001	0	0	0	0	0	1
000100	101111	1	0	1	1	1	1
000010	111110	1	1	1	1	1	0
000001	000101	0	0	0	1	0	1

It is easy to see that $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$, and ψ_6 are linearly independent. Thus, W is nonsingular. Since each ψ_j is nonzero on the subspaces $\text{span}\{e_1, e_2\}, \text{span}\{e_3, e_4\}$, span $\{e_5, e_6\}$ and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$, $\text{span}\{e_1, e_2\}$, span $\{e_3, e_4\}, \text{span}\{e_5, e_6\}$ and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$ are magic. Therefore, Weidemann's cube is an affine magic cube and hence semi-pantriagonal. However, it is not pantriagonal, not associated, not compact, and also not bent triagonal because the subspaces $\text{span}\{e_1 + e_3 + e_5, e_1 + e_2 + e_4 + e_6\}$, are not magic.

5.3. John Hendricks 1972's Magic Cube. John Hendricks published this cube in 1972 when he introduced pantriagonal magic cubes [6].

The affine functions which determine the Modified Hendricks 1972 obtained by subtracting one from each number in Figure 8 and writing them in base 2 are

$$\begin{aligned} V_1 &= f_2 + f_4 + f_5, \\ V_2 &= f_2 + f_4 + f_5 + f_6, \\ V_3 &= f_2 + f_3 + f_6, \\ V_4 &= f_2 + f_3 + f_4 + f_6, \\ V_5 &= f_1 + f_4 + f_6, \\ V_6 &= f_1 + f_2 + f_4 + f_6, \end{aligned} \tag{14}$$

and the corresponding number is $W(x) = \sum_{j=1}^{6} 2^{6-j} V_j(x) + 1$. Note that W is an affine function. In this case, we have that $V_j = \psi_j$ is a linear function for $j = 1, \ldots, 6$. Since $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$, and ψ_6 are linearly independent, W is nonsingular. It is easy to see that $\text{span}\{e_1, e_2\}$, span $\{e_3, e_4\}, \text{span}\{e_5, e_6\}$, and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$ are magic. Thus, Hendrick 1972's cube is an affine magic cube and hence semi-pantriagonal. It is pantriagonal and also compact because $\text{span}\{e_1 + e_3 + e_5, e_1 + e_2 + e_4 + e_6\}$, $\text{span}\{e_1 + e_3 + e_5, e_2 + e_3 + e_4 + e_6\}$, $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_5 + e_6\}$, $\text{span}\{e_2, e_4\}\text{span}\{e_2, e_6\}$, $\text{span}\{e_4, e_6\}$, $\text{span}\{e_2, e_3 + e_4\}$, $\text{span}\{e_2, e_5 + e_6\}$, $\text{span}\{e_4, e_5 + e_6\}$, $\text{span}\{e_1 + e_2, e_3 + e_4\}$, $\text{span}\{e_1 + e_2, e_5 + e_6\}$ and $\text{span}\{e_3 + e_4, e_5 + e_6\}$ are magic. However, it is neither bent triagonal nor associated because $\text{span}\{e_1 + e_3 + e_6, e_2 + e_4 + e_6\}$ and $\text{span}\{e_1 + e_2 + e_3 + e_4 + e_5 + e_6\}$ are not magic.

5.4. John Hendricks 1999's Magic Cube. In this last example, we consider the magic cube by John Hendricks appearance in his 1999 book [7]. It has the unique feature of containing bent triagonals.

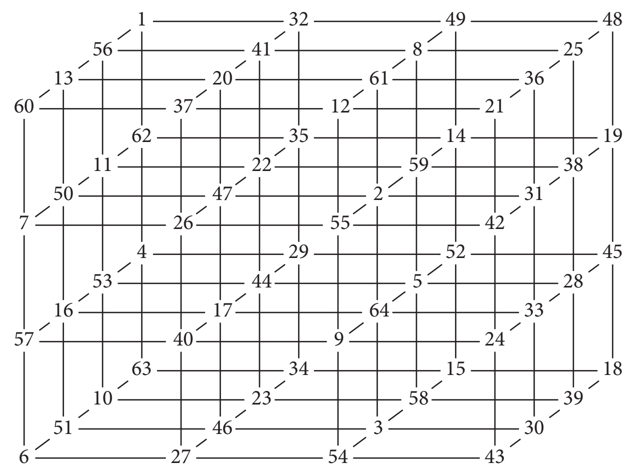

Figure 8: Hendricks 1972.

The affine functions which determine the Modified Hendricks 1999 obtained by subtracting one from each number in Figure 9 and writing them in base 2 are

$$\begin{aligned} V_1 &= f_1 + f_3 + f_5 + f_6 + 1, \\ V_2 &= f_1 + f_3 + f_6 + 1, \\ V_3 &= f_1 + f_3 + f_4 + f_5 + 0, \\ V_4 &= f_1 + f_4 + f_5 + 1, \\ V_5 &= f_1 + f_2 + f_3 + f_5 + 1, \\ V_6 &= f_2 + f_3 + f_5 + 1, \end{aligned} \tag{15}$$

and the corresponding number is $W(x) = \sum_{j=1}^{6} 2^{6-j} V_j(x) + 1$. Note that W is an affine function and the linear part ψ_j of each V_j are as follows:

$$\begin{aligned} \psi_1 &= f_1 + f_3 + f_5 + f_6, \\ \psi_2 &= f_1 + f_3 + f_6, \\ \psi_3 &= f_1 + f_3 + f_4 + f_5, \\ \psi_4 &= f_1 + f_4 + f_5, \\ \psi_5 &= f_1 + f_2 + f_3 + f_5, \\ \psi_6 &= f_2 + f_3 + f_5. \end{aligned} \tag{16}$$

Then $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5$, and ψ_6 are linearly independent, and so W is nonsingular. Since $\text{span}\{e_1, e_2\}, \text{span}\{e_3, e_4\}$, $\text{span}\{e_5, e_6\}$, and $\text{span}\{e_1 + e_3 + e_5, e_2 + e_4 + e_6\}$ are magic, Hendrick 1999s cube is an affine magic cube and hence semi-pantriagonal. It is bent triagonal because $\text{span}\{e_1+$

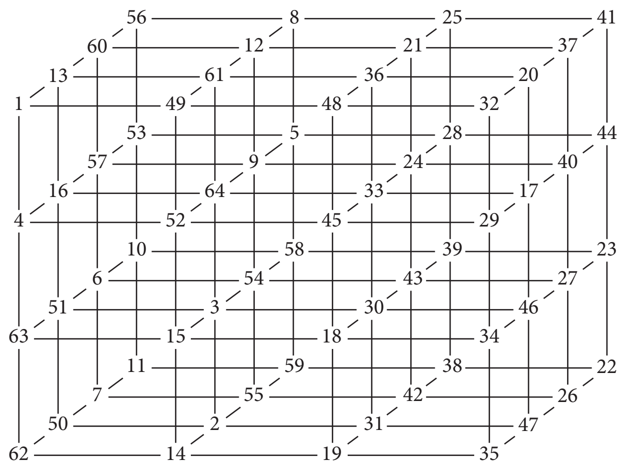

Figure 9: Hendricks 1999.

$e_3+e_6, e_2+e_4+e_6\}$, $\text{span}\{e_1+e_4+e_5, e_2+e_4+e_6\}$, $\text{span}\{e_1+e_4+e_6, e_2+e_4+e_6\}$ $\text{span}\{e_2+e_3+e_5,\ e_2+e_4+e_6\}$, $\text{span}\{e_2+e_3+e_6, e_2+e_4+e_6\}$,

and $\text{span}\{e_2+e_4+e_5, e_2+e_4+e_6\}$ are magic. However, it is not pantriagonal, not associated, and also not compact because $\text{span}\{e_1+e_3+e_5, e_1+e_2+e_4+e_6\}$, $\text{span}\ \{e_1+e_2+e_3+e_4+e_5+e_6\}$, $\text{span}\{e_2, e_4\}$ are not magic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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