Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2022, Article ID 2751666, 8 pages
https://doi.org/10.1155/2022/2751666

Research Article

Dividing the Perimeter of a Triangle into Unequal Proportions

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Received 25 April 2022; Accepted 9 May 2022; Published 9 June 2022

Academic Editor: Vladimir Mityushev

We fully describe the envelope of all line segments that divide the perimeter of a triangle into the ratio α: $(1-\alpha)$ as α varies from 0 to 1/2. If α is larger than the ratio of the longest side length to the perimeter, then the envelope is a 12-sided closed curve consisting of six line segments and six parabolic arcs. For other values of α, the envelope is the union of one to three parabolic arcs and possibly a 5- or 9-sided nonclosed curve consisting of line segments and parabolic arcs.

1. Introduction

Is there a point P where every line passing through P bisects the perimeter of a triangle? The answer is "*no*." These lines are not concurrent. Instead, they "wrap" around a curve called the *envelope* of perimeter-bisecting lines. Berele and Catoiu [1] show that the envelope of perimeter-bisecting lines, called the *perimeter-bisecting deltoid*, is a 6-sided closed curve consisting of three line segments and three parabolic arcs (see Figure 1).

In this article, we extend Berele and Catoiu's result to imbalanced proportions. Particularly, we classify and describe the envelope of all line segments that divide the perimeter of a triangle into the ratio α: $(1-\alpha)$, for $0<\alpha\leq 1/2$. For example, there are two possibilities for an equilateral triangle. If $\alpha\leq 1/3$, then the envelope of all line segments that divide the perimeter of a triangle into the ratio α: $(1-\alpha)$ is the union of three parabolic arcs. On the other hand, if $1/3<\alpha<1/2$, then the envelope is a 12-sided closed curve consisting of six line segments and six parabolic arcs (see Figure 2).

To tackle this problem, first, we reduce the problem of finding the envelope for a triangle to a smaller problem of finding the envelope for each angle. Then, we combine the results from the three angles together. The construction of "*single angle envelopes*" is given in Section 3, and the groundwork for combining single angle envelopes is given in Section 4 and Section 5. A complete description of the envelope of all line segments that divide the perimeter of a triangle into the ratio α: $(1-\alpha)$ is given in Section 6.

2. Preliminaries

We recall basic facts about the envelope of a family of curves. A one-parameter family of curves Γ is a collection of all curves $F(x,y,t)=0$ on the xy plane, where t is a parameter. In other words,

$$\Gamma=\{F(x,y,t)=0\colon t\in I\},\tag{1}$$

where I is an index set. In this article, we assume that the index set I is an interval, and the function F is sufficiently smooth. There are multiple definitions of the envelope of a family of curves. Here, we introduce two definitions that are equivalent in our interesting situations. The reader should refer to [2, 3] for facts and examples of the envelope of a one-parameter family of the curve.

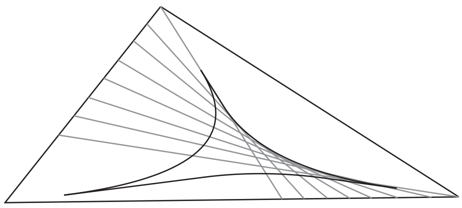

Figure 1: Every line tangent to the perimeter-bisecting deltoid bisects the perimeter of the triangle.

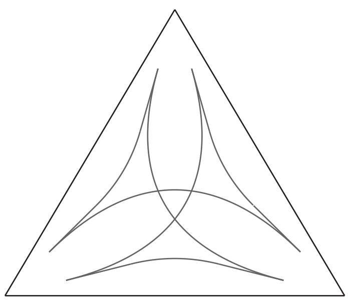

Figure 2: The envelope of all line segments that divide the perimeter of an equilateral triangle into the ratio α: $(1-\alpha)$, for $1/3<\alpha<1/2$.

Definition 1. The envelope of the family Γ is the set E_1 of points given by

$$E_1=\{(x,y)\colon F(x,y,t)=F_t(x,y,t)=0 \text{ for some } t\in I\}. \quad (2)$$

Informally, we "eliminate" t between the equations $F(x,y,t)=0$ and $F_t(x,y,t)=0$.

Definition 2. The envelope of the family Γ is a curve E_2 that touches each curve $F(x,y,t)=0$ at some point. More precisely, E_2 is the set of points $(x(t),y(t))$ that belong to $F(x,y,t)=0$ and where E_2 and $F(x,y,t)=0$ have the same tangent line there. The range of values of t may be smaller than the full interval where Γ is defined.

We are only concerned with a family of straight lines in this article. Definition 1 and Definition 2 are equivalent in this case.

Proposition 1 (Proposition 5 in [2]). *Assume that $F(x,y,t)=a(t)x+b(t)y+c(t)$, that is, all curves in Γ are straight lines. If $a'(t)b(t)-a(t)b'(t)\neq 0$, then $E_1=E_2$.*

3. Single Angle Envelopes

Our goal is to describe the envelope of all line segments that divide the perimeter of a triangle into the ratio α: $(1-\alpha)$. These segments are generated from two endpoints, M and N, which move along the sides of a triangle while dividing the perimeter into a fixed ratio. The trip is split into three parts where separations occur when either M or N meets a vertex. As a result, our problem reduces to studying the envelope of all segments $\overline{\mathrm{MN}}$ when M and N travel along each leg of a fixed angle $\angle O$ such that the sum OM + ON is constant. In this section, we show that such envelopes are parabolic arcs and provide some properties of the envelopes.

Definition 3. Let $\angle$AOC be a fixed angle that measures less than π and σ be a fixed positive real number. We define the single angle envelope $\mathscr{E}(\angle O,\sigma)$ to be the envelope of all segments $\overline{\mathrm{MN}}$ such that M is on the ray $\overrightarrow{\mathrm{OA}}$, N is on the ray $\overrightarrow{\mathrm{OC}}$, and OM + ON = σ. Since envelopes of congruent angles are congruent, we denote by $\mathscr{E}(\theta,\sigma)$ the envelope of any angle of measure θ and sum σ.

The term and concept of single angle envelopes were developed by Berele and Catoiu [1] when they studied the envelope of perimeter-bisecting lines of a triangle. Definition 3 and the statements of Lemma 1 and Lemma 2 are adapted from [1]. For completeness, we include alternate proofs of both lemmas here.

The next lemma shows that a single-angle envelope is a parabolic arc.

Lemma 1. *Let $\angle AOC$ be an angle of measure $2\phi<\pi$. In a coordinate system centered at O such that the y-axis is a symmetry axis of $\angle AOC$ when the point A is in the second quadrant and the point C is in the first quadrant (see Figure 3), the single angle envelope $\mathscr{E}(2\phi,\sigma)$ is a parabolic arc given by the following equation:*

$$y=\frac{\cos\phi}{2\sigma\sin^2\phi}x^2+\frac{\sigma\cos\phi}{2},\quad \text{when } x\in[-\sigma\sin\phi,\sigma\sin\phi]. \quad (3)$$

Proof. Let M be a point on the ray $\overrightarrow{\mathrm{OA}}$ and N be a point on the ray $\overrightarrow{\mathrm{OC}}$ such that OM + ON = σ. Let OM = t where $t\in[0,\sigma]$. Then, ON = $\sigma-t$. As vectors, $\overrightarrow{\mathrm{OM}}=t(-\sin\phi\,\vec{\imath}+\cos\phi\,\vec{\jmath})$ and $\overrightarrow{\mathrm{ON}}=(\sigma-t)(\sin\phi\,\vec{\imath}+\cos\phi\,\vec{\jmath})$. Hence, the segment $\overline{\mathrm{MN}}$ is part of the line given by the following equation:

$$y=\left(\frac{(\sigma-2t)\cos\phi}{\sigma\sin\phi}\right)(x+t\sin\phi)+t\cos\phi. \quad (4)$$

Let Γ be the collection of all such lines. We have that

$$\Gamma=\{F(x,y,t)=0\colon t\in[0,\sigma]\}, \quad (5)$$

where

$$F(x,y,t)=-\frac{2\cos\phi}{\sigma^2}t^2-\frac{2x\cos\phi}{\sigma^2\sin\phi}t+\frac{x\cos\phi}{\sigma\sin\phi}+\frac{2x\cos\phi}{\sigma}-\frac{y}{\sigma}. \quad (6)$$

The system of equations $F(x,y,t)=F_t(x,y,t)=0$ holds when

$$t=\frac{\sigma}{2}-\frac{x}{2\sin\phi}. \quad (7)$$

By substituting the above value of t into (4), we have that the envelope of Γ is given by the following equation:

$$y = \frac{\sigma \cos \phi}{2} + \frac{\cos \phi}{2\sigma \sin^2 \phi} x^2. \quad (8)$$

Since $t \in [0, \sigma]$, we have $x \in [-\sigma \sin \phi, \sigma \sin \phi]$. □

The following lemma provides a method for calculating the tangential points of a single angle envelope. □

Lemma 2. *Let $\angle AOC$ be an angle of measure $2\phi < \pi$ and σ be a fixed positive real number. Let M be a point on $\overline{OA}$ and N be a point on $\overline{OC}$ with $OM + ON = \sigma$. If $\overline{MN}$ is tangent to the single angle envelope $\mathscr{E}(\angle AOC, \sigma)$ at P, then*

$$(\mathrm{PM})(\mathrm{OM}) = (\mathrm{PN})(\mathrm{ON}). \quad (9)$$

Proof. Let $\mathrm{OM} = t$ and $\lambda = \mathrm{PM}/\mathrm{MN}$. Then, $\mathrm{ON} = \sigma - t$ and $\mathrm{PN}/\mathrm{MN} = 1 - \lambda$. We want to show that $(\mathrm{PM})(\mathrm{OM}) = (\mathrm{PN})(\mathrm{ON})$, which is equivalent to

$$\lambda t = (1 - \lambda)(\sigma - t). \quad (10)$$

From (7) and (8), we can write the coordinates of P in terms of t as

$$P = \left((\sigma - 2t) \sin \phi, \left(\sigma - 2t + \frac{2t^2}{\sigma} \right) \cos \phi \right). \quad (11)$$

Alternatively, as a vector,

$$\begin{aligned} \overrightarrow{\mathrm{OP}} &= (1 - \lambda)\overrightarrow{\mathrm{OM}} + \lambda \overrightarrow{\mathrm{ON}} \\ &= (-t + \lambda\sigma) \sin \phi \, \overrightarrow{\imath} + (t - 2\lambda t + \lambda\sigma) \cos \phi \, \overrightarrow{\jmath}. \end{aligned} \quad (12)$$

(11) and (12) describe the same point. We must have that

$$\begin{gathered} (-t + \lambda\sigma) \sin \phi = (\sigma - 2t) \sin \phi \\ \text{and} \\ \left(\sigma - 2t + \frac{2t^2}{\sigma} \right) \cos \phi = (t - 2\lambda t + \lambda\sigma) \cos \phi. \end{gathered} \quad (13)$$

Both equations imply (10) as wanted.

Proposition 1 implies that the single angle envelope $\mathscr{E}(\angle \mathrm{AOC}, \sigma)$ is tangent to the legs of the angle $\angle \mathrm{AOC}$. The locations of the tangential points can be calculated by (3) at the endpoints. We get the following lemma.

Lemma 3. *Let $\angle AOC$ be an angle of measure $2\phi < \pi$ and σ be a fixed positive real number. The single angle envelope $\mathscr{E}(\angle AOC, \sigma)$ is tangent to the ray $\overrightarrow{OA}$ at a point U and tangent to the ray $\overrightarrow{OC}$ at a point V where $OU = OV = \sigma$.*

4. α-Splitters

From now on, we will assume that $0 < \alpha \leq 1/2$, and we will use the following notation for the envelope of all line segments that divide the perimeter of a triangle into the ratio $\alpha\colon (1 - \alpha)$.

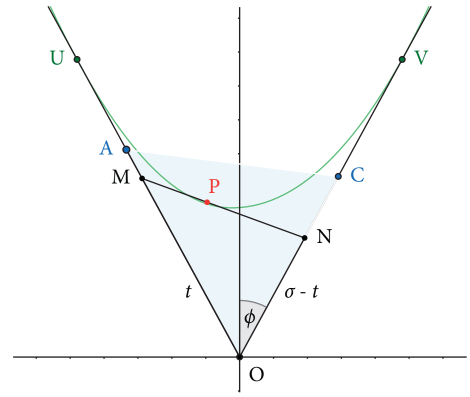

Figure 3: The single angle envelope $\mathscr{E}(2\phi, \sigma)$.

Notation 1. We denote the envelope of all line segments that divide the perimeter of a triangle into the ratio $\alpha\colon (1 - \alpha)$ by $\mathscr{D}_\alpha$.

We will also use the following notation for the side lengths of a triangle.

Notation 2. For a triangle $\Delta\mathrm{ABC}$, we write $a = \mathrm{BC}$, $b = \mathrm{CA}$, and $c = \mathrm{AB}$. We denote the perimeter of $\Delta\mathrm{ABC}$ by $p = a + b + c$.

Recall that our investigation on $\mathscr{D}_\alpha$ reduces to the study of single angle envelopes, which are parabolic arcs. For example, for a triangle ΔABC, every line segment that is tangent to the single angle envelopes $\mathscr{E}(\angle A, \alpha p)$ divides the perimeter of $\Delta\mathrm{ABC}$ into the ratio $\alpha\colon (1 - \alpha)$. However, the endpoints of such line segments are restricted to the sides of $\Delta\mathrm{ABC}$. As a result, not every point of $\mathscr{E}(\angle A, \alpha p)$ is a part of $\mathscr{D}_\alpha$. It is cut by cevians that divide the perimeter of a triangle into the ratio $\alpha\colon (1 - \alpha)$.

Definition 4. We call a cevian that divides the perimeter of a triangle into the ratio $\alpha\colon (1 - \alpha)$ an *α-splitter*. We call the endpoint of an α-splitter on the opposite side of a vertex an *α-splitting point*.

There can be up to two α-splitters at each vertex. Thus, a triangle has at most six α-splitters. We will use the following notation for the six α-splitting points.

Notation 3. For a triangle $\Delta\mathrm{ABC}$, we let A'_B and A'_C denote the α-splitting points on the side $\overline{\mathrm{BC}}$ such that $\mathrm{BA} + \mathrm{BA}'_B = \alpha p$ and $\mathrm{CA} + \mathrm{CA}'_C = \alpha p$, respectively. It follows that $\overline{\mathrm{AA}'_B}$ and $\overline{\mathrm{AA}'_C}$ are α-splitters. The notations for all α-splitting points are summarized. Figure 4 shows the locations of α-splitting points in a triangle.

$$\begin{aligned} A'_B &\in \overline{BC} \text{ with } BA + BA'_B = \alpha p; \\ A'_C &\in \overline{BC} \text{ with } CA + CA'_C = \alpha p; \\ B'_A &\in \overline{CA} \text{ with } AB + AB'_A = \alpha p; \\ B'_C &\in \overline{CA} \text{ with } CB + CB'_C = \alpha p; \\ C'_A &\in \overline{AB} \text{ with } AC + AC'_A = \alpha p; \\ C'_B &\in \overline{AB} \text{ with } BC + BC'_B = \alpha p. \end{aligned} \quad (14)$$

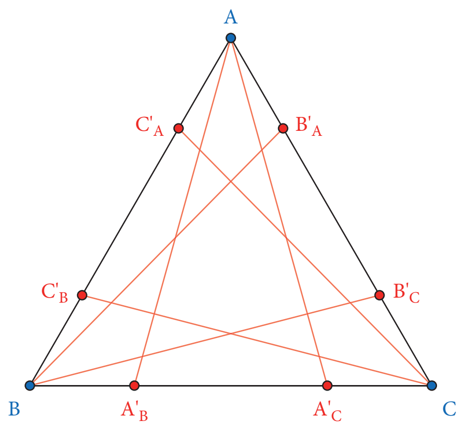

Figure 4: α-splitting points.

We note that when $\alpha = 1/2$, A'_B coincides with A'_C, B'_C coincides with B'_A, and C'_A coincides with C'_B. In plane geometry, a 1/2-splitter is simply called a *splitter*. The three splitters concur at a point known as the Nagel point [4].

The following lemma gives necessary and sufficient conditions for the existence of α-splitting points.

Lemma 3. *Let ΔABC be a triangle. The following statements hold.*

(1) The α-splitting points B'_C and C'_B exist if and only if $\alpha > a/p$

(2) The α-splitting points C'_A and A'_C exist if and only if $\alpha > b/p$

(3) The α-splitting points A'_B and B'_A exist if and only if $\alpha > c/p$

Proof. We give the proof of (1) here. The proofs of (2) *and* (3) are similar.

($\Rightarrow$) Assume that the α-splitting points B'_C and C'_B exist. It follows that $\mathrm{BC} + \mathrm{CB}'_C = \alpha p$ and $\mathrm{BC} + \mathrm{BC}'_B = \alpha p$, where $\mathrm{BC}'_B > 0$ and $\mathrm{CB}'_C > 0$. We then have $\alpha p > \mathrm{BC} = a$. Hence, $\alpha > a/p$.

($\Leftarrow$) Assume that $\alpha > a/p$. Then, $\alpha p > a = \mathrm{BC}$. Thus, there exists $x > 0$ such that $x + \mathrm{BC} = \alpha p$. From the triangle inequality, $\alpha \le 1/2 < (c + a)/p$. This means $x + \mathrm{BC} = \alpha p < \mathrm{AB} + \mathrm{BC}$. Thus, $x < \mathrm{AB}$. In particular, there exists a point $C'_B \in \overline{AB}$ such that $\mathrm{BC}'_B = x$; that is, $\mathrm{BC} + \mathrm{BC}'_B = \alpha p$. By a similar argument, there exists a point $B'_C \in \overline{AC}$ such that $\mathrm{BC} + \mathrm{CB}'_C = \alpha p$. The points C'_B and B'_C are α-splitting points. □

The following corollary determines the number of α-splitters based on the value of α. It follows directly from the proof of Lemma 3. □

Corollary 1. *Let ΔABC be a triangle such that $a \le b \le c$. Then, the following statements hold.*

(1) If $\alpha \le a/p$, then ΔABC has no α-splitters

(2) If $a/p < \alpha \le b/p$, then ΔABC has exactly two α-splitters: $\overline{BB'_C}$ and $\overline{CC'_B}$

(3) If $b/p < \alpha \le c/p$, then ΔABC has exactly four α-splitters: $\overline{BB'_C}$, $\overline{CC'_B}$, $\overline{CC'_A}$, and $\overline{AA'_C}$

(4) If $c/p < \alpha < 1/2$, then ΔABC has exactly six α-splitters: $\overline{BB'_C}$, $\overline{CC'_B}$, $\overline{CC'_A}$, $\overline{AA'_C}$, $\overline{AA'_B}$, and $\overline{BB'_A}$

5. Cutting Single Angle Envelopes with α-Splitters

As discussed earlier, the problem of finding $\mathcal{D}_\alpha$ reduces to studying the single angle envelopes $\mathcal{E}(\angle A, \alpha p)$ and $\mathcal{E}(\angle A, (1-\alpha)p)$. However, it is not necessary for all points of $\mathcal{E}(\angle A, \alpha p)$ and $\mathcal{E}(\angle A, (1-\alpha)p)$ to be $\mathcal{D}_\alpha$. They are cut by *α-splitters*. In this section, we find the locations of these cut points. The results in this chapter will serve as key lemmas for the classification of $\mathcal{D}_\alpha$ in Section 6.

5.1. Cutting $\mathcal{E}(\angle A, \alpha p)$ with α-Splitters. Figure 5 illustrates the results of Lemma 4 to Lemma 7.

Lemma 4. *Let ΔABC be a triangle. If the α-splitters $\overline{BB'_A}$ and $\overline{CC'_A}$ do not exist, then $\mathcal{E}(\angle A, \alpha p) \subset \mathcal{D}_\alpha$ and $\mathcal{E}(\angle A, \alpha p)$ is tangent to the sides $\overline{AB}$ and $\overline{AC}$ at the points U and V where $AU = AV = \alpha p$, respectively.*

Proof. From Lemma 1 and Lemma 3, $\mathcal{E}(\angle A, \alpha p)$ is a parabolic arc which is tangent to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at the points U and V where $\mathrm{AU} = \mathrm{AV} = \alpha p$, respectively. Because the α-splitters BB'_A and CC'_A do not exist, Lemma 4.5 implies that $\alpha p < \mathrm{AB}$ and $\alpha p < \mathrm{AC}$. This means that the points U and V are on the sides $\overline{\mathrm{AB}}$ and $\overline{\mathrm{AC}}$, respectively. Therefore, $\mathcal{E}(\angle A, \alpha p) \subset \mathcal{D}_\alpha$. □ □

Lemma 5. *Let ΔABC be a triangle. If the α-splitter $\overline{BB'_A}$ exists but the α-splitter $\overline{CC'_A}$ does not exist, then $\mathcal{E}(\angle A, \alpha p) \cap \mathcal{D}_\alpha$ is a parabolic arc with endpoints B_A^α on $\overline{BB'_A}$ and V on the side $\overline{AC}$ such that*

$$\frac{B_A^\alpha B}{B_A^\alpha B'_A} = \frac{\mathrm{AB}'_A}{\mathrm{AB}} = \frac{\alpha p - c}{c}, \qquad AV = \alpha p. \tag{15}$$

Proof. From Lemma 1 and Lemma 3, $\mathcal{E}(\angle A, \alpha p)$ is a parabolic arc that is tangent to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at the points U and V where $\mathrm{AU} = \mathrm{AV} = \alpha p$, respectively.

Since the α-splitter $\overline{\mathrm{BB}'_A}$ exists, Lemma 3 implies that $\mathrm{AB} < \alpha p$. Thus, the point U is not on the side $\overline{\mathrm{AB}}$. Therefore, the single angle envelope $\mathcal{E}(\angle A, \alpha p)$ is cut by the α-splitter $\overline{\mathrm{BB}'_A}$ at a point B_A^α. From Lemma 2, $(B_A^\alpha B)(\mathrm{AB}) = (B_A^\alpha B'_A)(\mathrm{AB}'_A)$. Hence,

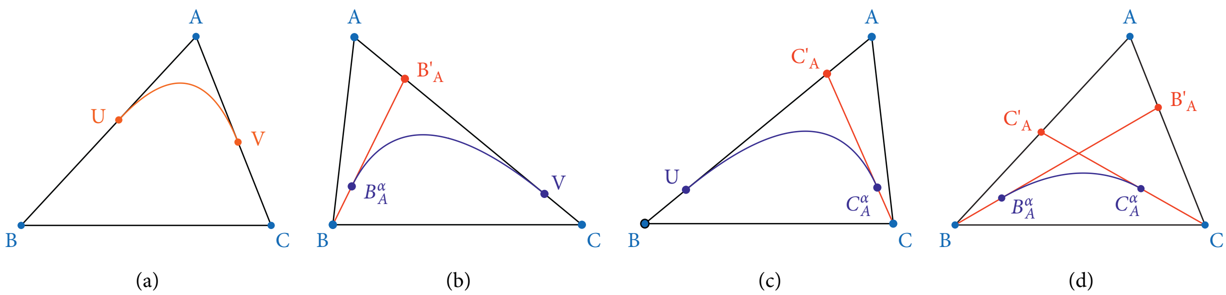

Figure 5: $\mathscr{E}(\angle A, \alpha p) \cap D_\alpha$ in (a) Lemma 5.1, (b) Lemma 5.2, (c) Lemma 5.3, and (d) Lemma 5.4.

$$\frac{B_A^\alpha B}{B_A^\alpha B_A'} = \frac{AB_A'}{AB} = \frac{\alpha p - c}{c}. \tag{16}$$

Because the α-splitter $\overline{CC_A'}$ does not exist, Lemma 4.5 implies that $\alpha p < AC$. Hence, the point V is on the side $\overline{AC}$ and $AV = \alpha p$. □

The proofs of Lemma 6 and Lemma 7 are similar to the proof of Lemma 5. □

Lemma 6. *Let ΔABC be a triangle. If the α-splitter $\overline{CC_A'}$ exists but the α-splitter $\overline{BB_A'}$ does not exist, then $\mathscr{E}(\angle A, \alpha p) \cap \mathscr{D}_\alpha$ is a parabolic arc with endpoints C_A^α on $\overline{CC_A'}$ and U on the side $\overline{AB}$ such that*

$$AU = \alpha p, \quad \frac{C_A^\alpha C}{C_A^\alpha C_A'} = \frac{AC_A'}{AC} = \frac{\alpha p - b}{b}. \tag{17}$$

Lemma 7. *Let ΔABC be a triangle. If the α-splitters $\overline{BB_A'}$ and $\overline{CC_A'}$ exist, then $\mathscr{E}(\angle A, \alpha p) \cap \mathscr{D}_\alpha$ is a parabolic arc with endpoints B_A^α on $\overline{BB_A'}$ and C_A^α on $\overline{CC_A'}$ such that*

$$\frac{B_A^\alpha B}{B_A^\alpha B_A'} = \frac{AB_A'}{AB} = \frac{\alpha p - c}{c}, \quad \frac{C_A^\alpha C}{C_A^\alpha C_A'} = \frac{AC_A'}{AC} = \frac{\alpha p - b}{b}. \tag{18}$$

5.2. Cutting $\mathscr{E}(\angle A, (1-\alpha)p)$ with α-Splitters

Lemma 8. *Let ΔABC be a triangle. If the α-splitters $\overline{BB_C'}$ and $\overline{CC_B'}$ do not exist, then $\mathscr{E}(\angle A, (1-\alpha)p)$ and $\mathscr{D}_\alpha$ are disjoint.*

Proof. Since the α-splitters $\overline{BB_C'}$ and $\overline{CC_B'}$ do not exist, Lemma 3 implies that $\alpha p < BC$, that is, $(1-\alpha)p > b + c$.

Recall that the single angle envelope $\mathscr{E}(\angle A, (1-\alpha)p)$ is the envelope of all segment $\overline{MN}$ such that M is on ray $\overrightarrow{AB}$ and N is on ray $\overrightarrow{AC}$ such that $AM + AN = (1-\alpha)p$. Suppose for the sake of contradiction that $M \in \overline{AB}$ and $N \in \overline{AC}$. It follows that $(1-\alpha)p = AM + AN \le b + c$. This is a contradiction. Therefore, there is no such segment $\overline{MN}$. In other words, $\mathscr{E}(\angle A, (1-\alpha)p)$ and $\mathscr{D}_\alpha$ are disjoint. □

The proof of the following lemma is similar to the proof of Lemma 5. Figure 6 illustrates the result of this lemma. □

Lemma 9. *Let ΔABC be a triangle. If the α-splitters $\overline{BB_C'}$ and $\overline{CC_B'}$ exist, then $\mathscr{E}(\angle A, (1-\alpha)p) \cap \mathscr{D}_\alpha$ is a parabolic arc with endpoints $B_C^{1-\alpha}$ on $\overline{BB_C'}$ and $C_B^{1-\alpha}$ on $\overline{CC_B'}$ such that*

$$\frac{B_C^{1-\alpha} B}{B_C^{1-\alpha} B_C'} = \frac{AB_C'}{AB} = \frac{(1-\alpha)p - c}{c}, \quad \frac{C_B^{1-\alpha} C}{C_B^{1-\alpha} C_B'} = \frac{AC_B'}{AC} = \frac{(1-\alpha)p - b}{b}. \tag{19}$$

6. Classifying the Envelopes

6.1. Statement of the Main Theorem. The following theorem completely describes the envelope of all line segments that divide the perimeter of a triangle into the ratio $\alpha : (1-\alpha)$. Figure 7 illustrates $\mathscr{D}_\alpha$ as α varies from 0 to 1/2. For convenience, we will denote a parabolic arc with endpoints P and Q by $\widehat{PQ}$.

Theorem 1. *Let ΔABC be a triangle such that $a \le b \le c$. Then, the following statements hold.*

(1) *If $\alpha \le a/p$, then $\mathscr{D}_\alpha$ is the union of three parabolic arcs. In particular,*

$$\mathscr{D}_\alpha = \mathscr{E}(\angle A, \alpha p) \cup \mathscr{E}(\angle B, \alpha p) \cup \mathscr{E}(\angle C, \alpha p). \tag{20}$$

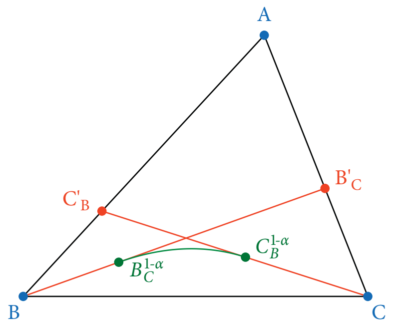

Figure 6: $\mathcal{E}(\angle A, (1-\alpha)p) \cap \mathcal{D}_\alpha$ in Lemma 5.6.

The single angle envelope $\mathcal{E}(\angle X, \alpha p)$ is tangent to the sides $\overline{XY}$ and $\overline{XZ}$ at the points U_X and V_X, respectively, such that $XU_X = XV_X = \alpha p$, when (X, Y, Z) is a permutation of (A, B, C).

(2) *If $a/p < \alpha \le b/p$, then $\mathcal{D}_\alpha$ is the union of $\mathcal{E}(\angle A, \alpha p)$ and a 5-sided nonclosed curve whose sides alternate between two line segments and three parabolic arcs. In particular,*

$$\mathcal{D}_\alpha = \left(\widehat{V_B C_B^\alpha} \cup \overline{C_B^\alpha C_B^{1-\alpha}} \cup \widehat{C_B^{1-\alpha} B_C^{1-\alpha}} \cup \overline{B_C^{1-\alpha} B_C^\alpha} \cup \widehat{B_C^\alpha U_C}\right) \cup \mathcal{E}(\angle A, \alpha p), \tag{21}$$

where

(a) *the single angle envelope $\mathcal{E}(\angle A, \alpha p)$ is tangent to the sides $\overline{AB}$ and $\overline{AC}$ at the points U_A and V_A, respectively, such that $AU_A = AV_A = \alpha p$*

(b) *$\widehat{V_B C_B^\alpha} \subset \mathcal{E}(\angle B, \alpha p)$ and $\widehat{B_C^\alpha U_C} \subset \mathcal{E}(\angle C, \alpha p)$ have endpoints V_B and U_C on the sides $\overline{AB}$ and $\overline{AC}$, respectively, with $BV_B = CU_C = \alpha p$, and*

(c) *$\widehat{C_B^{1-\alpha} B_C^{1-\alpha}} \subset \mathcal{E}(\angle A, (1-\alpha)p)$*

(3) *If $b/p < \alpha \le c/p$, then $\mathcal{D}_\alpha$ is a 9-sided nonclosed curve whose sides alternate between four line segments and five parabolic arcs. In particular,*

$$\begin{aligned}\mathcal{D}_\alpha = {} & \widehat{U_A C_A^\alpha} \cup \overline{C_A^\alpha C_A^{1-\alpha}} \cup \widehat{C_A^{1-\alpha} A_C^{1-\alpha}} \cup \overline{A_C^{1-\alpha} A_C^\alpha} \cup \widehat{A_C^\alpha B_C^\alpha} \\ & \cup \overline{B_C^\alpha B_C^{1-\alpha}} \cup \widehat{B_C^{1-\alpha} C_B^{1-\alpha}} \cup \overline{C_B^{1-\alpha} C_B^\alpha} \cup \widehat{C_B^\alpha V_B},\end{aligned} \tag{22}$$

where

(a) *$\widehat{U_A C_A^\alpha} \subset \mathcal{E}(\angle A, \alpha p)$ and $\widehat{C_B^\alpha V_B} \subset \mathcal{E}(\angle B, \alpha p)$ have endpoints U_A and V_B on the side $\overline{AB}$ with $AU_A = BV_B = \alpha p$*

(b) *$\widehat{A_C^\alpha B_C^\alpha} \subset \mathcal{E}(\angle C, \alpha p)$, and*

(c) *$\widehat{B_C^{1-\alpha} C_B^{1-\alpha}} \subset \mathcal{E}(\angle A, (1-\alpha)p)$ and $\widehat{C_A^{1-\alpha} A_C^{1-\alpha}} \subset \mathcal{E}(\angle B, (1-\alpha)p)$*

(4) *If $c/p < \alpha < 1/2$, then $\mathcal{D}_\alpha$ is a 12-sided closed curve whose sides alternate between six line segments and six parabolic arcs. In particular,*

$$\begin{aligned}\mathcal{D}_\alpha = {} & \widehat{B_A^\alpha C_A^\alpha} \cup \overline{C_A^\alpha C_A^{1-\alpha}} \cup \widehat{C_A^{1-\alpha} A_C^{1-\alpha}} \cup \overline{A_C^{1-\alpha} A_C^\alpha} \cup \widehat{A_C^\alpha B_C^\alpha} \\ & \cup \overline{B_C^\alpha B_C^{1-\alpha}} \cup \widehat{B_C^{1-\alpha} C_B^{1-\alpha}} \cup \overline{C_B^{1-\alpha} C_B^\alpha} \cup \widehat{C_B^\alpha A_B^\alpha} \\ & \cup \overline{A_B^\alpha A_B^{1-\alpha}} \cup \widehat{A_B^{1-\alpha} B_A^{1-\alpha}} \cup \overline{B_A^{1-\alpha} B_A^\alpha},\end{aligned} \tag{23}$$

where

(a) *$\widehat{B_A^\alpha C_A^\alpha} \subset \mathcal{E}(\angle A, \alpha p)$, $\widehat{A_B^\alpha C_B^\alpha} \subset \mathcal{E}(\angle B, \alpha p)$, and $\widehat{B_C^\alpha A_C^\alpha} \subset \mathcal{E}(\angle C, \alpha p)$, and*

(b) *$\widehat{B_C^{1-\alpha} C_B^{1-\alpha}} \subset \mathcal{E}(\angle A, (1-\alpha)p)$, $\widehat{C_A^{1-\alpha} A_C^{1-\alpha}} \subset \mathcal{E}(\angle B, (1-\alpha)p)$, and $\widehat{A_B^{1-\alpha} B_A^{1-\alpha}} \subset \mathcal{E}(\angle C, (1-\alpha)p)$*

The locations of endpoints of sides of $\mathcal{D}_\alpha$ can be identified exactly. In all cases, the endpoints Y_Z^α and $Y_Z^{1-\alpha}$ are on the α-splitter $\overline{YY_Z'}$ so that

$$\begin{aligned}\frac{Y_Z^\alpha Y}{Y_Z^\alpha Y_Z'} &= \frac{\alpha p - YZ}{YZ}, \\ \frac{Y_Z^{1-\alpha} Y}{Y_Z^{1-\alpha} Y_Z'} &= \frac{(1-\alpha)p - XY}{XY},\end{aligned} \tag{24}$$

when (X, Y, Z) is a permutation of (A, B, C).

We note that when $\alpha = 1/2$, $\mathcal{E}(\angle X, \alpha p)$ coincides with $\mathcal{E}(\angle X, (1-\alpha)p)$. As a result, six pairs of sides of $\mathcal{D}_\alpha$ coincide. In this case, we may consider $\mathcal{D}_\alpha$ as a 6-sided closed curve. Berele and Catoiu [1] call this curve the perimeter-bisecting deltoid.

6.2. Proof of theorem 1. Theorem 1 can be proven by combining the results of appropriate lemmas in Section 5 based on the existence of α-splitters, which is given in Corollary 1. For conciseness, we only give a detailed proof of Theorem 1 (2) here. Theorem 6.1 (1), (3), and (4) can be proven with similar methods.

Proof of (2).

Assume that $a/p < \alpha \le b/p$. From Corollary 1, the α-splitters $\overline{BB_C'}$ and $\overline{CC_B'}$ exist, but $\overline{AA_B'}$, $\overline{AA_C'}$, $\overline{BB_A'}$, and $\overline{CC_A'}$ do not exist.

First, we consider the angle $\angle A$. Because the α-splitters $\overline{BB_C'}$ and $\overline{CC_B'}$ exist, Lemma 4 implies that $\mathcal{E}(\angle A, \alpha p) \subset \mathcal{D}_\alpha$, and it is tangent to the sides $\overline{AB}$ and $\overline{AC}$ at the points U_A and V_A where $AU_A = AV_A = \alpha p$, respectively. Lemma 9 implies that $\mathcal{E}(\angle A, (1-\alpha)p) \cap \mathcal{D}_\alpha$ is a parabolic arc that is cut from $\mathcal{E}(\angle A, (1-\alpha)p)$ by $\overline{BB_C'}$ and $\overline{CC_B'}$. The endpoints are $B_C^{1-\alpha} \in \overline{BB_C'}$ and $C_B^{1-\alpha} \in \overline{CC_B'}$ such that

$$\begin{aligned}\frac{B_C^{1-\alpha} B}{B_C^{1-\alpha} B_C'} &= \frac{AB_C'}{AB} = \frac{(1-\alpha)p - AB}{AB}, \\ \frac{C_B^{1-\alpha} C}{C_B^{1-\alpha} C_B'} &= \frac{AC_B'}{AC} = \frac{(1-\alpha)p - AC}{AC}.\end{aligned} \tag{25}$$

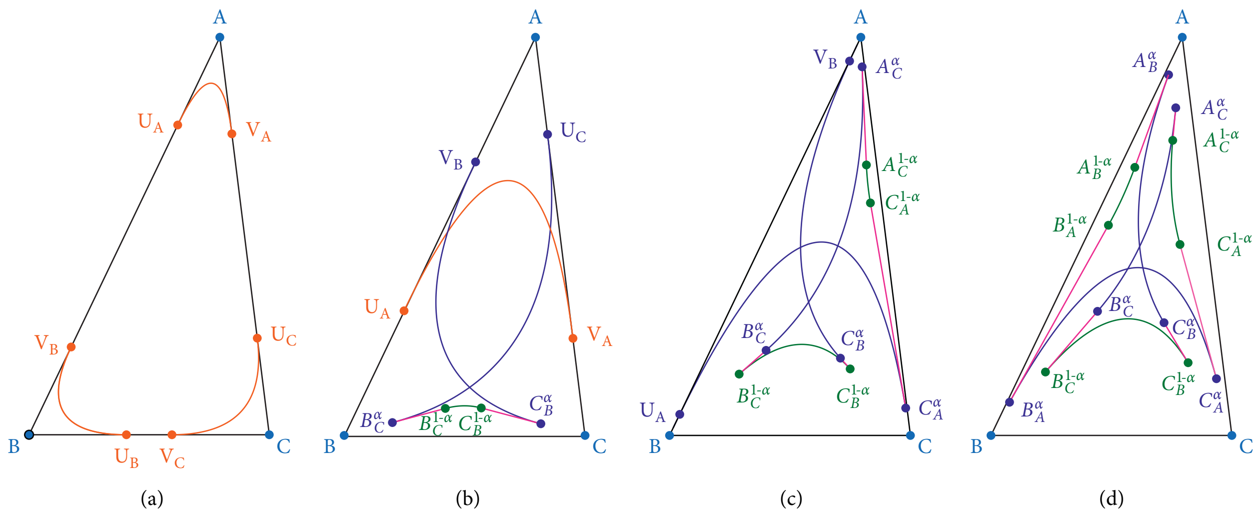

Figure 7: $\mathscr{D}_\alpha$ for a triangle ΔABC with $a \le b \le c$ when (a) $\alpha \le a/p$, (b) $a/p < \alpha \le b/p$, (c) $b/p < \alpha \le c/p$, and (d) $c/p < \alpha < 1/2$.

Next, we consider the angle $\angle B$. Since the α-splitter $\overline{CC_B'}$ exists, Lemma 6 implies that the intersection of $\mathscr{E}(\angle B, \alpha p)$ and $\mathscr{D}_\alpha$ is a parabolic arc which is cut from $\mathscr{E}(\angle B, \alpha p)$ by the α-splitter $\overline{CC_B'}$ and the side $\overline{AB}$. The endpoints are $C_B^\alpha \in \overline{CC_B'}$ and $V_B \in \overline{AB}$ such that

$$\begin{aligned} \frac{C_B^\alpha C}{C_B^\alpha C_B'} &= \frac{BC_B'}{BC} \\ &= \frac{\alpha p - BC}{BC}, \\ BV_B &= \alpha p. \end{aligned} \tag{26}$$

Because the α-splitters $\overline{AA_C'}$ and $\overline{CC_A'}$ do not exist, Lemma 5.5 implies that $\mathscr{E}(\angle B, (1-\alpha)p)$ and $\mathscr{D}_\alpha$ are disjoint.

Lastly, we consider the angle $\angle C$. Since the α-splitter $\overline{BB_C'}$ exists, Lemma 5.3 implies that the intersection of $\mathscr{E}(\angle C, \alpha p)$ and $\mathscr{D}_\alpha$ is a parabolic arc that is cut from $\mathscr{E}(\angle C, \alpha p)$ by the α-splitter $\overline{BB_C'}$ and the side $\overline{AC}$. The endpoints are $B_C^\alpha \in \overline{BB_C'}$ and $U_C \in \overline{AC}$ such that

$$\begin{aligned} \frac{B_C^\alpha B}{B_C^\alpha B_C'} &= \frac{CB_C'}{CB} \\ &= \frac{\alpha p - CB}{CB}, \\ CU_C &= \alpha p. \end{aligned} \tag{27}$$

Because the α-splitters $\overline{AA_B'}$ and $\overline{BB_A'}$ do not exist, Lemma 5.5 implies that $\mathscr{E}(\angle C, (1-\alpha)p)$ and $\mathscr{D}_\alpha$ are disjoint.

We have shown that $\mathscr{D}_\alpha$ contains parabolic arcs $\widehat{V_B C_B^\alpha}$, $\widehat{C_B^{1-\alpha} B_C^{1-\alpha}}$, and $\widehat{B_C^\alpha U_C}$, which are cut from appropriate single angle envelopes by α-splitters. Since C_B^α and $C_B^{1-\alpha}$ are two points on the same α-splitter, $\overline{CC_B'}$, which divides the perimeter of ΔABC into the ratio α: $(1-\alpha)$, the segment $\overline{C_B^\alpha C_B^{1-\alpha}}$ is also contained in $\mathscr{D}_\alpha$. Similarly, the segment $\overline{B_C^{1-\alpha} B_C^\alpha}$ is also a part of $\mathscr{D}_\alpha$. As a result, we have that

$$\left(\widehat{V_B C_B^\alpha} \cup \overline{C_B^\alpha C_B^{1-\alpha}} \cup \widehat{C_B^{1-\alpha} B_C^{1-\alpha}} \cup \overline{B_C^{1-\alpha} B_C^\alpha} \cup \widehat{B_C^\alpha U_C}\right) \subset \mathscr{D}_\alpha. \tag{28}$$

Therefore,

$$\mathscr{D}_\alpha = \left(\widehat{V_B C_B^\alpha} \cup \overline{C_B^\alpha C_B^{1-\alpha}} \cup \widehat{C_B^{1-\alpha} B_C^{1-\alpha}} \cup \overline{B_C^{1-\alpha} B_C^\alpha} \cup \widehat{B_C^\alpha U_C}\right) \cup \mathscr{E}(\angle A, \alpha p). \tag{29}$$

In other words, $\mathscr{D}_\alpha$ is the union of $\mathscr{E}(\angle A, \alpha p)$ and a 5-sided nonclosed curve whose sides alternate between two line segments and three parabolic arcs. This completes the proof of Theorem 1 (2).

6.3. Properties and Special Cases

In this section, we list some properties and special cases of $\mathscr{D}_\alpha$.

Corollary 2. *If $\alpha > a/p$, then adjacent sides of $\mathscr{D}_\alpha$ either connect smoothly or make a cusp.*

Proof. This follows directly from the properties of the envelopes of families of curves (see Proposition 1). ☐

Corollary 3. *Parabolic sides of $\mathscr{D}_\alpha$ have the angle bisectors as their axes of symmetry.*

Proof. The corollary follows directly from Lemma 1. ☐

The corollaries below are special cases of Theorem 1. ☐

Corollary 4 (equilateral case). *Let ΔABC be an equilateral triangle. Then, the following statements hold.*

(1) *If $\alpha \le 1/3$, then $\mathscr{D}_\alpha$ is the union of three parabolic arcs*

(2) *If $1/3 < \alpha < 1/2$, then $\mathscr{D}_\alpha$ is a 12-sided close curve whose sides alternate between six line segments and six parabolic arcs*

Corollary 5 (isosceles case). *Let ΔABC be an isosceles triangle.*

(1) If $a < b = c$, then the following statements hold

(a) If $\alpha \leq a/p$, then $\mathscr{D}_\alpha$ is the union of three parabolic arcs

(b) If $a/p < \alpha \leq b/p$, then $\mathscr{D}_\alpha$ is the union of $\mathscr{E}(\angle A, \alpha p)$ and a 5-sided nonclosed curve whose sides alternate between two line segments and three parabolic arcs

(c) If $b/p < \alpha < 1/2$, then $\mathscr{D}_\alpha$ is a 12-sided closed curve whose sides alternate between six line segments and six parabolic arcs

(2) If $a = b < c$, then the following statements hold

(a) If $\alpha \leq b/p$, then $\mathscr{D}_\alpha$ is the union of three parabolic arcs

(b) If $b/p < \alpha \leq c/p$, then $\mathscr{D}_\alpha$ is a 9-sided nonclosed curve whose sides alternate between four line segments and five parabolic arcs

(c) If $c/p < \alpha < 1/2$, then $\mathscr{D}_\alpha$ is a 12-sided closed curve whose sides alternate between six line segments and six parabolic arcs

7. Conclusion

The envelopes of area bisectors and perimeter bisectors of triangles, polygons, and convex regions have been topics of interest in plane geometry (e.g., see [1, 5–7]). These objects have potential applications in computational hydrodynamics and naval engineering [8]. The envelope of area bisectors of a triangle was generalized to imbalanced proportions by Middleton [9].

This work generalizes the envelope of perimeter bisectors. We fully describe the envelope of all line segments that divide the perimeter of a triangle into the ratio α: $(1 - \alpha)$, for $0 < \alpha \leq 1/2$. This is the very first step in an exciting avenue that will lead to interesting questions about the properties and generalizations of these objects.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

Theerakarn is partially supported by Srinakharinwirot University.

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