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Research Article

Biharmonic Curves in a Strict Walker 3-Manifold

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In this paper, we study the geometry of biharmonic curves in a strict Walker 3-manifold and we obtain explicit parametric equations for biharmonic curves and time-like biharmonic curves, respectively. We discuss the conditions for a speed curve to be a slant helix in a Walker manifold. We give an example of biharmonic curve for illustrating the main result.

1. Introduction

The study of submanifolds of a given ambiant space is a natural interesting problem which enriches our knowledge and understanding of the geometry of the space itself. Here, the ambient space we will consider is a Lorentzian three-manifold admitting a parallel null vector field called strict Walker manifold. It is known that Walker metrics have served as a powerful tool of constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. For details, see [1, 2].

As a generalization of Legendre curve, the notion of slant curves was introduced in [3]. A curve in a Walker 3-manifold is said to be slant if its tangent vector field has a constant angle with a position vector field.

Eells and Sampson [4] defined harmonic and biharmonic map between Riemannian manifolds. Jiang [5, 6] derived the first variation formula of the bienergy from the Euler–Lagrange equation. Harmonic maps are clearly biharmonic.

Nonharmonic biharmonic maps are called proper biharmonic maps. Chen and Ishikawa [7] showed nonexistence of proper biharmonic curves in Euclidean 3-space $\mathbb{E}^3$. Moreover, they classified all proper biharmonic curves in Minkowski 3-space $\mathbb{E}_1^3$ (see [8]).

Recently, Sasahara [9] introduced biharmonic maps between pseudo-Riemannian manifolds and studied proper biharmonic submanifolds in Lorentzian 3-space forms.

Lee [10] studies biharmonic curves in 3-dimensional Lorentzian Heisenberg space $(\mathbb{H}_3; g)$, and he shows that proper biharmonic space-like curve γ in Lorentzian Heisenberg space $(\mathbb{H}_3; g)$ is pseudohelix with some properties about their curvatures. Moreover, γ has the space-like normal vector field and is a slant curve. Finally, he found the parametric equations of them.

In [11], the authors study the nongeodesic nonnull biharmonic curves in 3-dimensional hyperbolic Heisenberg group with a semi-Riemannian metric of index 2. They prove that all of the nongeodesic nonnull biharmonic curves in such a 3-dimensional hyperbolic Heisenberg group are helices. Moreover, they obtain explicit parametric equations for nongeodesic nonnull biharmonic curves and nongeodesic space-like horizontal biharmonic curves, respectively.

And very recently, the author, in [12], studies proper biharmonic Frenet curves in 3-dimensional Lorentzian Sasakian space forms of constant holomorphic sectional curvature H. He gives a necessary and sufficient condition for a Frenet curve to be proper biharmonic in the Lorentzian Sasakian space forms. A classification of proper biharmonic

Frenet curves in 3-dimensional Lorentzian Heisenberg space is also given.

Motivated by the above works, in this paper, we study biharmonic curves in a strict Walker 3-manifold. We give a necessary and sufficient condition for curve in a Walker 3-manifold to be a slant helix.

The paper is organised as follows. In Section 2, we give some preliminaries tolls about biharmonic maps and Walker 3-manifold. In Section 3, we study biharmonic curves in a strict Walker 3-manifolds, and Section 4 talks about slant curves in this ambient space.

2. Preliminaries

2.1. Biharmonic Maps. In this section, we give some basic notions about biharmonic maps; for more details, see [11].

Let (M, g) and (N, h) be Riemannian manifolds and $\psi\colon M \longrightarrow N$ be a smooth map. The tension field of ψ (see [6]) is given by $\tau(\psi) = \operatorname{trace} \nabla d\psi$, where $\nabla d\psi$ is the second fundamental form of ψ defined by $\nabla d\psi(X, Y) = \nabla^{\psi}_X d\psi(Y) - d\psi(\nabla^M_X Y)$, $X, Y \in \Gamma(TM)$. For any compact domain $\Omega \subset M$, the bienergy is defined by

$$E_2(\psi) = \frac{1}{2}\int_{\Omega} |\tau(\psi)|^2 \nu_g. \tag{1}$$

Then, a smooth map ψ is called biharmonic map if it is a critical point of the bienergy functional for any compact domain $\Omega \subset M$. We have for the bienergy the following first variation formula:

$$\frac{\mathrm{d}}{\mathrm{d}t} E_2(\psi_t, \Omega)_{|t=0} = \int_{\Omega} \langle \tau_2(\psi), \omega \rangle \nu_g, \tag{2}$$

where ν_g is the volume element, ω is the variational vector field associated to the variation $\{\psi_t\}$ of ψ, and

$$\begin{aligned} \tau_2(\psi) &= -J(\tau_2(\psi)) \\ &= -\Delta^{\psi}\tau(\psi) - \operatorname{trace} R^N(d\psi, \tau(\psi))d\psi . \end{aligned} \tag{3}$$

$\tau_2(\psi)$ is called bitension field of ψ. Here, Δ^{ψ} is the rough Laplacian on the sections of the pull-back bundle $\psi^{-1}TN$ which is defined by

$$\Delta^{\psi} V = -\sum_{i=1}^{m} \left\{ \nabla^{\psi}_{e_i} \nabla^{\psi}_{e_i} V - \nabla^{\psi}_{\nabla^M_{e_i} e_i} V \right\}, \quad V \in \Gamma\left(\psi^{-1}TN\right), \tag{4}$$

where ∇^{ψ} is the pull-back connection on the pull-back bundle $\psi^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame on M. When the target manifold is semi-Riemannian manifold, the bienergy and bitension field can be defined in the same way.

Let M be a semi-Riemannian manifold and $\gamma\colon I \longrightarrow M$ be a nonnull curve parametrized by arc length. By using the definition of the tension field, we have

$$\begin{aligned} \tau(\gamma) &= \nabla^{\gamma}_{\partial/\partial s} d\gamma\left(\frac{\partial}{\partial s}\right) \\ &= \nabla_T T, \end{aligned} \tag{5}$$

where $T = \gamma'$. In this case, biharmonic equation for the curve γ reduces to

$$\tau_2(\gamma) = \nabla^3_T T - R(T, \nabla_T T)T = 0. \tag{6}$$

2.2. Walker Manifold. A Walker n-manifold is a pseudo-Riemannian manifold, which admits a field of null parallel r-planes, with $r \le n/2$. The canonical forms of the metrics were investigated by A. G. Walker [1]. Walker has derived adapted coordinates to a parallel plan field. Hence, the metric of a three-dimensional Walker manifold (M, g^{ε}_f) with coordinates (x, y, z) is expressed as

$$g^{\varepsilon}_f = \mathrm{d}x \circ \mathrm{d}z + \varepsilon \mathrm{d}y^2 + f(x, y, z)\mathrm{d}z^2. \tag{7}$$

And its matrix form is

$$\begin{aligned} g^{\varepsilon}_f &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix} \text{with inverse} \left(g^{\varepsilon}_f\right)^{-1} \\ &= \begin{pmatrix} -f & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{8}$$

For some function $f(x, y, z)$, $\varepsilon = \pm 1$, and thus, $D = \operatorname{Span} \partial_x$ as the parallel degenerate line field. Notice that when $\varepsilon = 1$ and $\varepsilon = -1$, the Walker manifold has signature $(2, 1)$ and $(1, 2)$, respectively, and, therefore, is Lorentzian in both cases.

It follows after a straightforward calculation that the Levi-Civita connection of any metric (1) is given by

$$\begin{aligned} \nabla_{\partial_x} \partial z &= \frac{1}{2} f_x \partial_x, \\ \nabla_{\partial_y} \partial z &= \frac{1}{2} f_y \partial_x, \\ \nabla_{\partial_z} \partial z &= \frac{1}{2}(f f_x + f_z)\partial_x + \frac{1}{2} f_y \partial_y - \frac{1}{2} f_x \partial_z, \end{aligned} \tag{9}$$

where ∂_x, ∂_y, and ∂_z are the coordinate vector fields ∂/∂_x, ∂/∂_y, and ∂/∂_z, respectively. Hence, if (M, g^{ε}_f) is a strict Walker manifolds, i.e., $f(x, y, z) = f(y, z)$, then the associated Levi-Civita connection satisfies

$$\begin{aligned} \nabla_{\partial_y} \partial z &= \frac{1}{2} f_y \partial_x, \\ \nabla_{\partial_z} \partial z &= \frac{1}{2} f_z \partial_x - \frac{\varepsilon}{2} f_y \partial_y. \end{aligned} \tag{10}$$

Note that the existence of a null parallel vector field (i.e., $f = f(y, z)$) simplifies the nonzero components of the Christoffel symbols and the curvature tensor of the metric g^{ε}_f as follows:

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2} f_y,$$
$$\Gamma^1_{33} = \frac{1}{2} f_z, \qquad (11)$$
$$\Gamma^2_{33} = -\frac{\varepsilon}{2} f_y.$$

Starting from local coordinates (x, y, z) for which (7) holds, it is easy to check that

$$e_1 = \partial_y,$$
$$e_2 = \frac{2-f}{2\sqrt{2}}\partial_x + \frac{1}{\sqrt{2}}\partial_z, \qquad (12)$$
$$e_3 = \frac{2+f}{2\sqrt{2}}\partial_x - \frac{1}{\sqrt{2}}\partial_z,$$

are local pseudo-orthonormal frame fields on (M, g_f^ε), with $g_f^\varepsilon(e_1, e_1) = \varepsilon$, $g_f^\varepsilon(e_2, e_2) = 1$, and $g_f^\varepsilon(e_3, e_3) = -1$. Thus, the signature of the metric g_f^ε is $(\varepsilon, 1, -1)$.

Proposition 1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g defined above, we have*

$$\nabla_{e_i} e_j = \begin{bmatrix} 0 & \frac{1}{4} f_y (e_2 + e_3) & -\frac{1}{4} f_y (e_2 + e_3) \\ \frac{1}{4} f_y (e_2 + e_3) & -\frac{\varepsilon}{4} f_y e_1 & \frac{\varepsilon}{4} f_y e_1 \\ -\frac{1}{4} f_y (e_2 + e_3) & \frac{\varepsilon}{4} f_y e_1 & -\frac{\varepsilon}{4} f_y e_1 \end{bmatrix}. \qquad (13)$$

The curvature tensor field of ∇ *is given by*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \qquad (14)$$

where $X, Y, Z \in \Gamma(M)$. *If we denote by*

$$R_{ijk} = R(e_i, e_j) e_k, \qquad (15)$$

where the indices i, j, and k take the values 1, 2, *and* 3, *then the nonzero components of the curvature tensor field are*

$$R_{121} = -R_{131} = -\frac{1}{4} f_{yy}(e_2 + e_3),$$
$$R_{122} = -R_{123} = -R_{132} = R_{133} = \frac{\varepsilon}{4} f_{yy} e_1. \qquad (16)$$

The vector product of u and v in (M, g_f^ε) *with respect to the metric* g_f^ε *is the vector denoted by* $u \times_f v$ *in M defined by*

$$g_f^\varepsilon(u \times_f v, w) = \det(u, v, w), \qquad (17)$$

for all vector w in M, where $\det(u, v, w)$ *is the determinant function associated to the canonical basis of* $\mathbb{R}^3$. *If* $u = (u_1, u_2, u_3)$ *and* $v = (v_1, v_2, v_3)$, *then, by using (17), we have*

$$u \times_f v = \left(\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} - f \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \right) \vec{i} - \varepsilon \begin{vmatrix} u_1 & v_1 \\ u_3 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_2 & v_2 \\ u_3 & v_3 \end{vmatrix} \vec{k}. \qquad (18)$$

Lemma 1. *The Walker cross product in M has the following properties:*

(1) The Walker cross product is bilinear and antisymmetric

(2) $X \times_f Y$ *is perpendicular to both X and Y*

(3) The frame defined in (5) verifies the following: $e_1 \times_f e_2 = -e_3$, $e_2 \times_f e_3 = -e_1$, *and* $e_3 \times_f e_1 = e_2$

3. Biharmonic Curves in Walker 3-Manifold

Let $\alpha: I \subset \mathbb{R} \longrightarrow (M, g_f^\varepsilon)$ be a curve parametrized by its arc length s.

The Frenet frame of α is the vectors T, N, and B along α, where T is the tangent, N is the principal normal, and B is the binormal vector. They satisfied the Frenet formulas:

$$\begin{cases} \nabla_T T(s) = \varepsilon_2 \kappa(s) N(s), \\ \nabla_T N(s) = -\varepsilon_1 \kappa T(s) - \varepsilon_3 \tau B(s), \\ \nabla_T B(s) = \varepsilon_2 \tau(s) N(s), \end{cases} \qquad (19)$$

where κ and τ are, respectively, the curvature and the torsion of the curve α, with $\varepsilon_1 = g_f(T; T)$, $\varepsilon_2 = g_f(N; N)$, and $\varepsilon_3 = g_f(B, B)$.

From (10), we obtain

$$\nabla_T^3 T = (-3\varepsilon_1\varepsilon_2\kappa'\kappa)T + \left(-\varepsilon_1\kappa^3 + \varepsilon_2\kappa'' - \varepsilon_3\kappa\tau^2\right)N - \varepsilon_2\varepsilon_3(2\kappa'\tau + \kappa\tau')B. \qquad (20)$$

Using (16) and Lemma 1, we obtain

$$R(T, \nabla_T T)T = \frac{\varepsilon_2 \kappa f_{yy}(B_2 - B_3)}{4} \cdot [(B_2 - B_3)N + (N_2 - N_3)B], \qquad (21)$$

where $T = T_1 e_1 + T_2 e_2 + T_3 e_3$, $N = N_1 e_1 + N_2 e_2 + N_3 e_3$, and $B = B_1 e_1 + B_2 e_2 + B_3 e_3$.

Hence, we obtain

$$\tau_2(\gamma) = (-3\varepsilon_1\varepsilon_2\kappa'\kappa)T + \left(\varepsilon_1\kappa^3 - \varepsilon_2\kappa'' + \varepsilon_3\kappa\tau^2 - \frac{\varepsilon_2\kappa f_{yy}(B_2 - B_3)^2}{4}\right)N$$
$$+\left(\frac{\varepsilon_2\kappa f_{yy}(B_3 - B_2)(N_2 - N_3)}{4} + \varepsilon_2\varepsilon_3\left(2\kappa'\tau + \kappa\tau'\right)\right)B. \quad (22)$$

Theorem 1. *Let $\alpha\colon I \subset \mathbb{R} \longrightarrow (M, g_f^{\varepsilon})$ be a curve parametrized by its arc length s. Then, α is a nongeodesic biharmonic curve if and only if*

$$\begin{cases} \kappa = \text{Constant} \neq 0, \\ \varepsilon_1\varepsilon_3\kappa^2 + \tau^2 = \dfrac{\varepsilon_2\varepsilon_3 f_{yy}(B_2 - B_3)^2}{4}, \\ \tau' = f_{yy}(N_2 - N_3)(B_2 - B_3), \end{cases} \quad (23)$$

where $B_2 \neq B_3$.

Proof 1. From (23), it follows that it is biharmonic if and only if

$$\begin{cases} \kappa'\kappa = 0, \\ \varepsilon_1\kappa^3 - \varepsilon_2\kappa'' + \varepsilon_3\kappa\tau^2 - \dfrac{\varepsilon_2\kappa f_{yy}(B_2 - B_3)^2}{4} = 0, \\ \dfrac{\varepsilon_2\kappa f_{yy}(B_2 - B_3)(N_2 - N_3)}{4} + \varepsilon_2\varepsilon_3(2\kappa\tau' + \kappa\tau') = 0. \end{cases} \quad (24)$$

The first equation of (24) shows that $\kappa = \textit{constant} \neq 0$. Then, the second equation (24) becomes

$$\varepsilon_1\kappa^3 + \varepsilon_3\kappa\tau^2 - \frac{\varepsilon_2\kappa f_{yy}(B_2 - B_3)^2}{4} = 0. \quad (25)$$

Since $\kappa \neq 0$, we obtain

$$\varepsilon_1\varepsilon_3\kappa^2 + \tau^2 = \frac{\varepsilon_2\varepsilon_3 f_{yy}(B_2 - B_3)^2}{4}. \quad (26)$$

Since κ = constant, the third equation also becomes

$$\frac{\varepsilon_2\kappa f_{yy}(B_3 - B_2)(N_2 - N_3)}{4} + \varepsilon_2\varepsilon_3\kappa\tau' = 0. \quad (27)$$

That implies also

$$-\frac{\varepsilon_2 f_{yy}(B_2 - B_3)^2(N_2 - N_3)}{4} + \varepsilon_2\varepsilon_3\tau'(B_2 - B_3) = 0. \quad (28)$$

Then, we have

$$\tau' = \frac{\left(\varepsilon_1\varepsilon_3\kappa^2 + \tau^2\right)(N_2 - N_3)}{4\varepsilon_2\varepsilon_3(B_2 - B_3)}. \quad (29)$$

Since $f_{yy} = \varepsilon_1\varepsilon_3\kappa^2 + \tau^2/4\varepsilon_2\varepsilon_3(B_2 - B_3)^2$, we obtain

$$\tau' = f_{yy}(N_2 - N_3)(B_2 - B_3). \quad (30)$$

□

Corollary 1. *Let $\alpha\colon I \subset \mathbb{R} \longrightarrow (M, g_f^{\varepsilon})$ be a time-like curve parameterized by its arc length s. We suppose that $f(y,z) = A_1(z)y + A_2(z)$, where $A_i, i = 1, 2$, are functions of z. Then, α is a nongeodesic biharmonic curve if and only if*

$$\begin{cases} \kappa = \text{Constant} \neq 0, \\ \tau = \text{Constant} \neq 0, \\ \tau^2 = \varepsilon_3\kappa^2. \end{cases} \quad (31)$$

Remark 1. In the condition of the corollary, the binormal must be space-like.

4. Slant Helix in Walker 3-Manifold

Let (M, g_f) be a three-dimensional strict Walker manifold.

Definition 1. A unit speed curve α is called a slant helix if there exists a nonzero constant vector field U in M such that the function $g_f(N(s), U)$ is constant, where N is the normal of α.

We remark that, in Walker manifold like in Minkowski ambient space, we cannot define the angle between two vectors (except that both vectors are of time-like type). For this reason, we avoid to say about the angle between the vector fields $N(s)$ and U. We have the following result in the three-dimensional Walker manifold.

Theorem 2. *Let α be a speed unit curve in M. Then, α is a slant helix if only if*

$$\frac{\kappa^2(\tau/\kappa)'}{\left(\varepsilon_1\tau^2 + \varepsilon_3\kappa^2\right)^{3/2}} = \text{constant}, \quad (32)$$

with $\varepsilon_1\tau^2 + \varepsilon_3\kappa^2 \neq 0$.

Proof 2. Let α be a slant helix. Let U be a vector field such that the function $g_f(N(s); U)$ = constant = b. There exist smooth functions a_1 and a_3 such that

$$U = a_1T(s) + bN(s) + a_3B(s), \quad (33)$$

where a_1 and a_3 are functions of s and (T, N, B) is the Frenet frame of the curve α.

Differentiating (33) with respect to s, one obtains

$$\begin{aligned}\nabla_T U &= a_1' T(s) + a_1 \nabla_T T + b \nabla_T N + a_3' B + a_3 \nabla_T B \\ &= a_1' T(s) + a_1 \varepsilon_2 k N - b \varepsilon_1 k T - b \varepsilon_3 \tau B + a_3' B + a_3 \varepsilon_2 \tau N.\end{aligned} \tag{34}$$

Then, $\nabla_T U = 0$ implies that

$$\begin{cases} a_1' - b\varepsilon_1 k = 0, \\ a_1 \varepsilon_2 k + a_3 \varepsilon_2 \tau = 0, \\ a_3' - b \varepsilon_3 \tau = 0. \end{cases} \tag{35}$$

From the second equation of (35), we obtain $a_1 = -(\tau/k) a_3$.

Moreover,

$$g_f(U;U) = \varepsilon_1 a_1^2 + \varepsilon_2 b^2 + \varepsilon_3 a_3^2 = m, \tag{36}$$

where m is a constant. According to $a_1 = -(\tau/k)a_3$, we obtain

$$\begin{aligned} g_f(U;U) &= \varepsilon_1 a_3^2 \left(\frac{\tau}{k}\right)^2 + \varepsilon_2 b^2 + \varepsilon_3 a_3^2 \\ &= \text{constant}, \end{aligned} \tag{37}$$

and then, we get $a_3^2(\varepsilon_1 (\tau/k)^2 + \varepsilon_3) = \text{constant} - \varepsilon_2 b^2$.

We denote

$$3a_3^2 \left(\varepsilon_1 \left(\frac{\tau}{k}\right)^2 + \varepsilon_3 \right) = \varepsilon m^2; \quad m > 0,\ \varepsilon \in \{-1, 0, 1\}. \tag{38}$$

If $\varepsilon = 0$, then we have $a_3 = 0$ and $a_1 = 0$. Therefore, $b = 1/\varepsilon_1 k a_1'$. Then, $U = 0$, which is a contradiction. So, we have $\varepsilon = 1$ or $\varepsilon = -1$.

The third equation of (35), $a_3' - b\epsilon_3 \tau = 0$, implies that $\mathrm{d}/\mathrm{d}s(a_3) = b\varepsilon_3\tau$. And using the fact that $a_3 = \pm m/\sqrt{\varepsilon_1 (\tau/k)^2 + \varepsilon_3}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\pm m}{\sqrt{\sqrt{\varepsilon_1 (\tau/k)^2 + \varepsilon_3}}} \right) = b \varepsilon_3 \tau. \tag{39}$$

That is,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\sqrt{\sqrt{\varepsilon_1 (\tau/k)^2 + \varepsilon_3}}} \right) = \pm \frac{b \varepsilon_3 \tau}{m}. \tag{40}$$

Finally,

$$\frac{k^2 (\tau/k)'}{\left(\varepsilon_1 \tau^2 + \varepsilon_3 k^2\right)^{3/2}} = \pm \frac{b}{m}. \tag{41}$$

Conversely, assume that

$$\frac{k^2 (\tau/k)'}{\left(\varepsilon_1 \tau^2 + \varepsilon_3 k^2\right)^{3/2}}, \tag{42}$$

is a constant, which we denote by b.

Now, we define the vector U by

$$U = \frac{-(\varepsilon_1/\varepsilon_3)\tau}{\sqrt{\varepsilon_3 k^2 + \varepsilon_1 \tau^2}} T + bN + \frac{(\varepsilon_3/\varepsilon_1)k}{\sqrt{\varepsilon_3 k^2 + \varepsilon_1 \tau^2}} B. \tag{43}$$

A differentiation of U together with the Frenet equations gives $\nabla_T U = 0$, that is, U is a constant vector. On the contrary, $g_f(N(s); U) = 1$, and this means that α is a slant helix. This concludes the proof of the theorem. □

Example 1. Let $\gamma\colon I \subset \mathbb{R} \longrightarrow (M, g_f^{\varepsilon})$; the speed curve is given by

$$\gamma(s) = \left(\frac{9s^3}{s+6}, \frac{3s^2}{2}, \frac{3s^3}{2} \right). \tag{44}$$

An easy computation gives that the curvature and the torsion of the curve γ are $\kappa^2 = \tau^2 = 9$. Then, by Theorem 1, the curve γ is a helix.

By equations (21)–(23), we have $\tau_2 = 0$, and then, the curve γ is biharmonic [13].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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