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Research Article

Generalized Mandelbrot Sets of a Family of Polynomials $P_n(z) = \ = z^n + z + c;\ (n \geq 2)$

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In this paper, we study the general Mandelbrot set of the family of polynomials $\{P_n(z) = z^n + z + c;\ (n \geq 2)\}$, denoted by $\mathrm{GM}(P_n)$. We construct the general Mandelbrot set for these polynomials by the escaping method. We determine the boundaries, areas, fractals, and symmetry of the previous polynomials. On the other hand, we study some topological properties of $\mathrm{GM}(P_n)$. We prove that $\mathrm{GM}(P_n)$ is bounded and closed; hence, it is compact. Also, we characterize the general Mandelbrot set as a union of basins of attraction. Finally, we make a comparison between the properties of famous Mandelbrot set $M(z^2 + c)$ and our general Mandelbrot sets.

1. Introduction

In the complex dynamical systems, a most beautiful and complicated object, with a simple definition, was studied from 1980s, and it is the well-known Mandelbrot set (see Figure 1). In fact, this set has many properties, like fractals, chaos, self-similarity, and others [1–3]. The Mandelbrot set is used to study the dynamics of the quadratic polynomial $P(z) = z^2 + c$ [4, 5]. The Mandelbrot set, denoted by M, consists of all values of c for which the orbit of the critical point, 0, of P is bounded, i.e., $M(f) = \{c \in ¢\colon f^{(n)}(0)$ is bounded $\forall n \in N\}$ where $f^{(n)}(z) = f(f^{(n-1)}(z))$ [6]. Note that the critical points are important in studying the complex dynamics because of the following fact: if a function possesses an attracting periodic orbit, then the orbit of the critical points must converge to that periodic orbit [7–10].

There are a number of works that studied the general Mandelbrot sets, like the general Mandelbrot set of $(z^2 + c + \mathrm{Re}(z))$, the general Mandelbrot set of $(\overline{z} + c)$, the general Mandelbrot set of $(\sin z/c)$, and others [11–15].

The general Mandelbrot set, denoted by GM(f), is defined by $\mathrm{GM}(f) = \{c \in ¢\colon f^{(n)}(z_{cr})$ is bounded, z_{cr} is the critical point of $f\}$ [16, 17].

Wang et al. emphasized that the generalized Mandelbrot set contains a lot of information to construct the Julia set, which enters the state of chaos through the solution of fractal bifurcation and triple rotational bifurcation [18].

The paper [19] studies the structure topological inflexibility and the discontinuity evolution law of the generalized M–J sets generated from the complex mapping . Moreover, it studies the physical meaning of the generalized M–J sets. The researcher has other works in this field that the reader can find through the Internet.

In this paper, we construct the general Mandelbrot sets of the family of polynomials $P_n(z) = z^n + z + c,\ (n\ \geq 2)$, using the escaping algorithm. Moreover, we study their areas, their boundaries, and their fractals. Some topological properties of $\mathrm{GM}(P_n)$ were studied. We prove that $\mathrm{GM}(P_n)$ is closed and bounded and hence is compact, and we find a bound of $\mathrm{GM}(P_n)$, $\forall n \geq 2$. Also, we characterize $\mathrm{GM}(P_n)$ as a

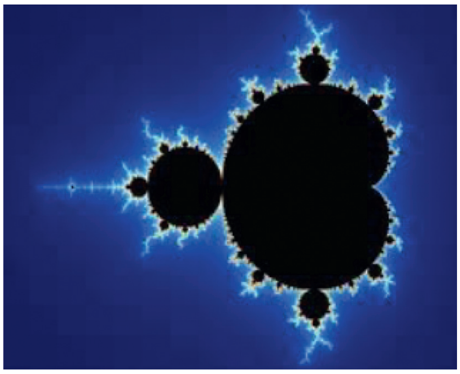

Figure 1: The Mandelbrot set.

union of basins of attraction of the attracting periodic point of P_n, $\forall n \geq 2$.

Finally, we construct (plot) the general Mandelbrot sets of $P_n(z)$, $n = 4,5,6,7,8,9$, and make a comparison of the properties of Mandelbrot set $M(z^2 + c)$ and our general Mandelbrot sets GM(P_n).

2. Definitions

(i) For any complex map g, the point z_{cr}, for which the eigenvalues of the Jacobi matrix are equal to zero, is called the **general critical point** of g [20].

(ii) **The general Mandelbrot set** of $f_c(z)$ is defined as follows:
$\mathrm{GM}(f) = \{c \in ⊄ \colon f_c^n(z_{cr}) \text{ is bounded } \forall n \in N\}$
where z_{cr} is critical point of f, and $f_c^n(z_{cr}) = f_c(f_c^{n-1}(z_{cr}))$ [21].
B$'$. The general Mandelbrot set of a function can be defined also as [20]

$$\mathrm{GM}(f) = \{c \in ⊄ \colon J_c(f) \text{ is conneced}\}. \tag{1}$$

(iii) A metric space (S, ρ) is said to be **connected** iff S is not a union $P \cup Q$ of any two disjoint closed sets; it is disconnected otherwise. A set A $\subseteq$ S is called **connected** iff (A, ρ) is connected as a subspace of (S, ρ), i.e., iff A is not a union of two disjoint sets $P, Q \neq \varnothing$ that are closed in (A, ρ), as a subspace of (S, ρ) [22, 23].

(iv) Let c be any complex number. The smallest closed set in the complex plane that contains all repelling periodic points of a family of complex function $\{f_\lambda(z)\}$ is called the **Julia** set of f_λ and is denoted as $J(f)$ [3].

(v) A **fixed point** x_0 (i.e., $f(x_0) = x_0$) is an attracting fixed point if $|f'(x_0)| < 1$. The point x_0 is a **repelling** fixed point if $|f'(x_0)| > 1$. Finally, if $|f'(x_0)| = 1$, the fixed point is called **neutral** [3].

(vi) Let f be a function and let x_0 be in the domain of f. Then,
$f(x_0)$ = the first iterate of x_0 for f
$f(f(x_0))$ = the second iterate of x_0 for f
More generally, $f^{(n)}(x_0)$ for the n-th iterate of x_0 for f
We call the sequence $\{f^{(n)}(x_0)\}_{n=0}^{\infty}$ of iterates of x_0 **the orbit** of x_0 [24].

(vii) If a fixed point P of f is attracting, then all points near to P are "attracted" toward P, in the sense that their iterates converge to P. The collection of all points whose iterates converge to P is called the basin of attraction of p [1].

(viii) The **basin of attraction** of the period point of period k, P_0, is the collection of all points whose iterates converges P_0 or f(P_0) or ... or $f^{(k-1)}(P_0)$ [1].

2.1. Classification of Fractals

2.1. Classification of Fractals. Fractals can be classified according to their self-similarity. There are three types of self-similarity found in fractals. **Exact self-similarity** is the strongest type of self-similarity, and the fractal appears identical at different scales, for example, the Sierpinski triangle and Koch snowflake exhibit exact self-similarity. The second type is the **quasi-self-similarity**; here a loose form of self-similarity and the fractal appears approximately (but not exactly) identical at different scales, i.e., the set contains small copies of the entire fractal in distorted and degenerate forms. Finally, **statistical self-similarity** is the weakest type of self-similarity. The fractal here has numerical or statistical measures which are preserved across scales [16, 25].

2.2. The Area. The problem of calculating the area of Mandelbrot set is not simple yet. However, there are several methods which can be used to calculate an approximated area for these sets. The methods that calculate the best approximation to that area are pixel counting method, Monte Carlo method and Grunwall method. We will use the Monte Carlo method by generating random points in the complex plane, and then these points are tested if they are in the general Mandelbrot set or not. The area of a region is then estimated from the ratio of points in the set to those outside it.

Remark 1. The approximate area of original Mandelbrot set using counting method is 1.4880, that using the Monte Carlo method is 1.5 ($\pm 0 \cdot 1$), and that using Gronwall's method is 1.7274 [6, 11, 26, 27].

3. The General Mandelbrot Set of the Polynomial $P_n(z) = = z^n + z + c$

In this section, we construct the general Mandelbrot sets of the polynomials, $P_n(z) = z^n + z + c$, $(n = 2, 3)$, denoted by GM(P_n). We determine the area, the fractal, and the boundary of GM(P_n), ($n = 2,3$). Also, we show that the Julia sets of P_2 and P_3 are connected for certain values of the parameter $c \in$ GM(P_n) and disconnected for a value of parameters which does not belong to MG(P_n).

Remark 2. For the general cases ($n > 3$), the same arguments, used in this section, can be applied with some more complicated details and have similar results.

3.1. *The General Mandelbrot Set of P_2, GM(P_2).*

To construct GM(P_2), we calculate the general critical points of $P_2(z) = z^2 + z + c$. We use the notations $z = x + iy$, $c = c_{re} + c_{im}$, and $f = u + iv$. Then, the equation has the following form:

$$P_2(z) = (x+iy)^2 + (x+iy) + c = x^2 + 2ixy - y^2 + x + iy + c. \tag{2}$$

Let $u = x^2 - y^2 + x + c_{re}$ and $v = 2xy + y + c_{im}$.

Its Jacobi matrix is $J = \begin{pmatrix} 2x+1 & -2y \\ 2y & 2x+1 \end{pmatrix}$.

Its characteristic polynomial is $\begin{vmatrix} 2x+1-\lambda & -2xy \\ 2xy & 2x+1-\lambda \end{vmatrix} = (2x+1-\lambda)^2 + 4y^2$. If $\lambda = 0$, then we get $(2x+1)^2 + 4y^2 = 0$.

If $y = 0$, then $(2x+1)^2 = 0$, and this follows $x = -1/2$. Thus, the point $(-1/2, 0)$ is a critical point for the polynomial $P_2(z)$. Also, if $x = 0$, then the other critical point is $(0, i/2)$. The construction of GM(P_2) is shown in Figures 2 and 3.

It is clear that the Mandelbrot set of this polynomial is similar to the original Mandelbrot set without deformations.

Let z_0 be a fixed point for the polynomial $P_n(z)$. Then, z_0 is attracting if $|P_n'(z_0)| < 1$. Thus, the solutions of the equation $|P_n'(z_0)| = 1$ are the boundary points of the stable (attracting) point. That is, the curve $|P_n'(z_0)| = 1$ bounds all the attracting fixed points of $P_n(z)$. Similarly, the boundary equation $|P_n^{(m)'}(z_0)| = 1$ bounds all the periodic points of period m which are attracting. In Sections 3.1.1. and 3.2.1. we use this criterion to draw the boundaries of GM (P_2) and GM (P_3), respectively, and it is clear that this criterion is true to draw the boundaries of GM (P_n), $\forall\, n$.

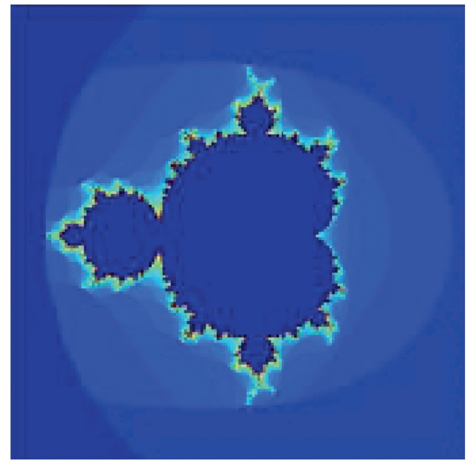

Figure 2: The general Mandelbrot set for the critical point $(-1/2, 0)$ of $P_2(z)$.

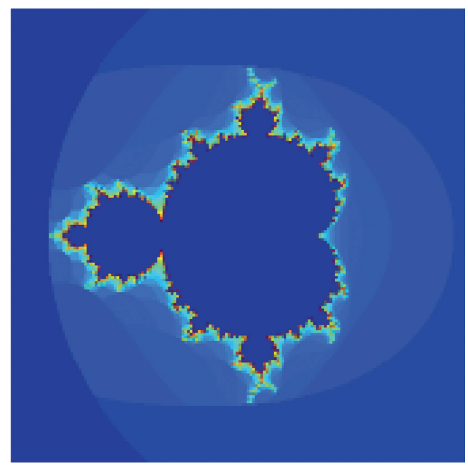

Figure 3: The general Mandelbrot for the critical point $(0, i/2)$ of $P_2(z)$.

3.1.1. *The Boundary of the Mandelbrot Set of the Fixed Point of P_2.*

For the polynomial P_2, the fixed points may be found by solving the equation $z^2 + z + c = \mathbf{z}$ which is

$$z^2 + c = 0 \tag{3}$$

The solution is $z_1 = \sqrt{c}i$ and $z_2 = -\sqrt{c}i$, i.e., z_1 is the first fixed point of P_2 and z_2 is the second fixed point of P_2.

The stability of each period point is determined from the derivative of the map at the point. Now,

$$\frac{\mathrm{d}P_2}{\mathrm{d}z} = 2z + 1, \tag{4}$$

$$2z + 1 = re^{i\theta}, \tag{5}$$

$$z = \frac{-1 + re^{i\theta}}{2}, \tag{6}$$

where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. Substituting (6) in (3), we have $(-1 + re^{i\theta}/2)^2 + c = 0$. The solution for c is

$$\left(\frac{-1 + re^{i\theta}}{2}\right)^2 + c = 0. \tag{7}$$

One of the fixed points, say z_1, is stable (attracting) as long as $|\mathrm{d}P_c/\mathrm{d}zz_1| < 1$. Therefore, the boundary of the point of period one is given by $||\mathrm{d}P_2/\mathrm{d}zz_1|z_1| = |2z_1 + 1| = r = 1$.

In the particular case, let $c = c_1 + ic_2$. Then, from (7), the boundary is given by the following parametric equations: $c_1 + ic_2 = -1/4 + \cos\theta/2 + i\sin\theta/2 - \cos 2\theta/4 - i\sin 2\theta/4$. Then,

$$c_1 = \frac{-1}{4} + \frac{\cos\theta}{2} - \frac{\cos 2\theta}{4}, \qquad c_2 = \frac{\sin\theta}{2} - \frac{\sin 2\theta}{4}. \tag{8}$$

We plot the curve in Figure 4.

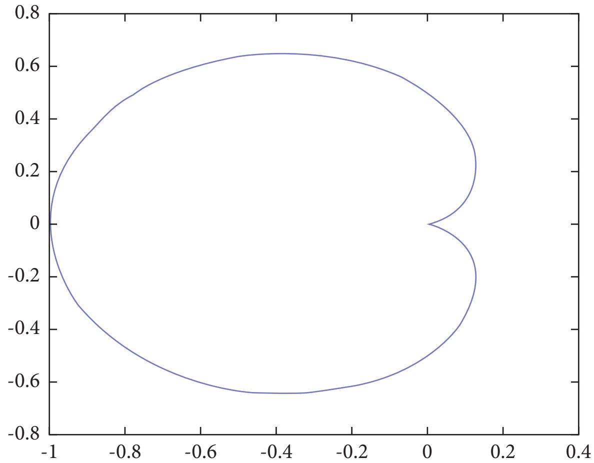

Figure 4: The boundary of the fixed point for $P_2(z)$.

3.1.2. The Fractals of GM(P_2). GM(P_2) admits, like the original Mandelbrot set, quasi-self-similarity. Also, it is similar around the x-axis.

3.1.3. The Area of GM(P_2). Using Monte Carlo method, the area of the general Mandelbrot set of $P_2(z)$ is 1.6456. Note that this area is very close to the area of the original Mandelbrot set, M(f).

3.1.4. The Julia Set of P_2. Applying the definition B', if we take $c = -0 \cdot 3 \in \mathrm{GM}(P_2)$, then the Julia set $J_c(P_2)$ is connected (Figure 5), and if $c = 0 \cdot 3 \notin \mathrm{GM}(P_2)$, then we obtain that $J_c(P_2)$ is disconnected (Figure 6).

3.2. The General Mandelbrot Set of the Polynomial $P_3(z)$, GM(P_3). Consider $P_3(z) = z^3 + z + c$. Using the notations $z = x + iy$, $c = c_{re} + c_{im}$ and $f = u + iv$, the equation has the following form:

$$\begin{aligned} P_3(z) &= (x+iy)^3 + (x+iy) + c \\ &= x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + c. \end{aligned} \tag{9}$$

Let $u = x^3 - 3xy^2 + x + c_{re}$ and $v = 3x^2y - y^3 + y + c_{im}$. Its Jacobi matrix is $J = \begin{pmatrix} 3x^2 - 3y^2 + 1 & -6xy \\ 6xy & 3x^2 - 3y^2 + 1 \end{pmatrix}$. Also, its characteristic polynomial is

$$\begin{aligned} &\begin{vmatrix} 3x^2 - 3y^2 + 1 - \lambda & -6xy \\ 6xy & 3x^2 - 3y^2 + 1 - \lambda \end{vmatrix} \\ &= \left(3x^2 - 3y^2 + 1 - \lambda\right)^2 + 36x^2y^2. \end{aligned} \tag{10}$$

Let $\lambda = 0$; then, we get $(3x^2 + 3y^2)^2 + 6x^2 - 6y^2 + 1 = 0$.
If $x = 0$, then we have $(3y^2)^2 - 6y^2 + 1 = 0, 9y^4 - 6y^2 + 1 = 0$.
Let $u = y^2$; then, $9u^2 - 6u + 1 = 0$, and this follows $(3u-1)^2 = 0$, $u = 1/3$.

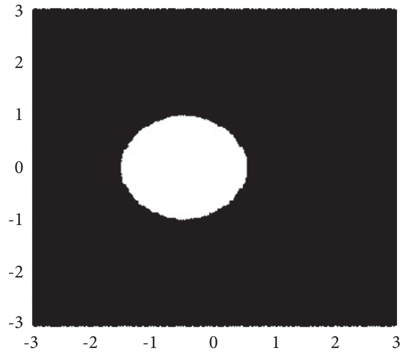

Figure 5: The Julia set of $P_2(z) = z^2 + z + c$ is connected where $(0.3 \in \mathrm{GM}(P_2))$.

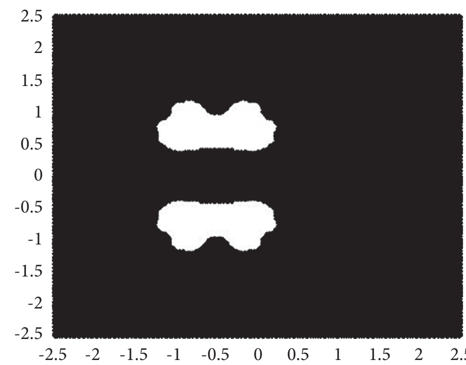

Figure 6: The Julia set of $P_2(z) = z^2 + z + c$ is disconnected where $(0.3 \notin \mathrm{GM}(P_2))$.

$y^2 = 1/3 \Rightarrow y = \pm 1/\sqrt{3}$. Thus, the points $(0, 1/\sqrt{3})$ and $(0, -1/\sqrt{3})$ are critical points for the polynomial $P_3(z)$. Figures 7 and 8 show the general Mandelbrot set of the critical points $(0, 1/\sqrt{3})$ and $(0, -1/\sqrt{3})$.

Also, if $y = 0$, the other critical points of $P_3(z)$ are the points $(i/\sqrt{3}, 0)$ and $(-i/\sqrt{3}, 0)$. The general Mandelbrot set GM(P_3) for these points is shown in Figures 9 and 10.

3.2.1. The Boundary of the Mandelbrot Set of the Fixed Point of $P_3(z)$. The fixed point of this polynomial may found by solving the following equation: $z^3 + z + c = z$, which implies $z^3 + c = 0$. Thus, $c = -z^3$.

Let $c = c_1 + ic_2$ and $z = x + iy$; by substituting them in (1), we get

$$c_1 + ic_2 = -x^3 + 3xy^2 - 3x^2yi + y^3i. \tag{11}$$

Now, substitute $x = r\cos(\theta)$ and $y = r\sin(\theta)$, and we get

$$\begin{aligned} c_1 + ic_2 = &-r^3\cos^3(\theta) + 3r^3\cos(\theta)\sin^2(\theta) - 3r^3 i\cos^2(\theta) \\ &+ r^3 i\sin^3(\theta). \end{aligned} \tag{12}$$

Now, the derivative of $P_3(z)$ is $dP_3/dz = 3z^2 + 1$.
The boundary consists of all z such that $|3z^2 + 1| = 1$.

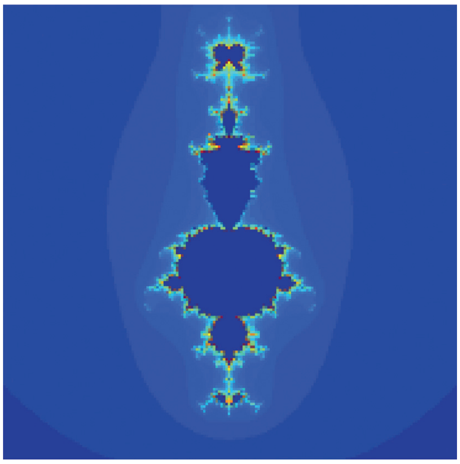

FIGURE 7: The general Mandelbrot set P_3 for the critical point $(0, 1/\sqrt{3})$.

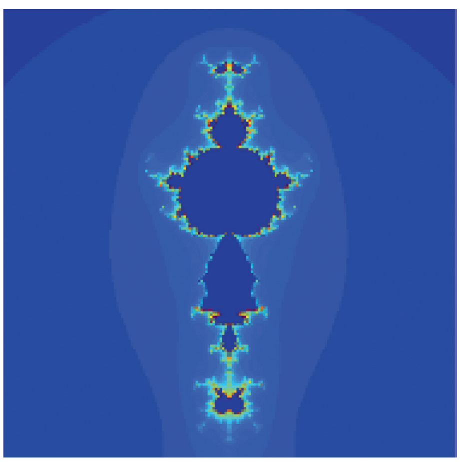

FIGURE 8: The general Mandelbrot set P_3 for the critical point $(0, -1/\sqrt{3})$.

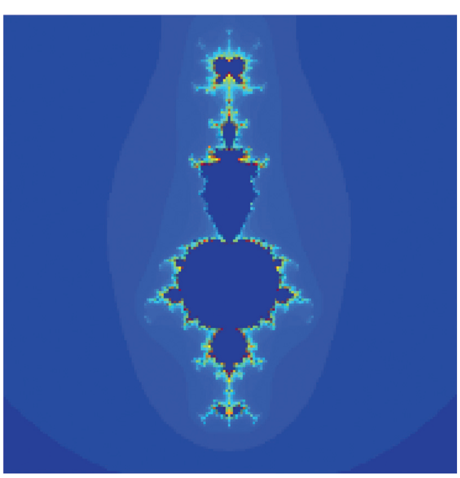

FIGURE 9: The general Mandelbrot set P_3 for the critical point $(i/\sqrt{3}, 0)$.

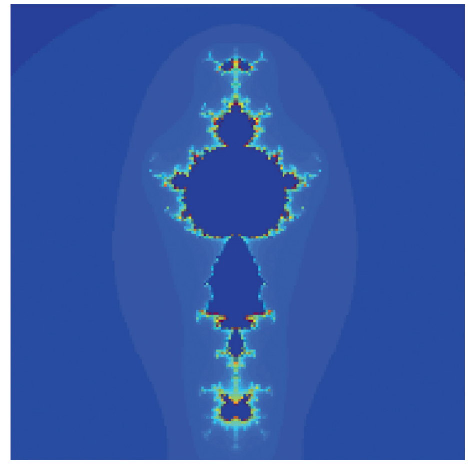

FIGURE 10: The general Mandelbrot set P_3 for the critical point $(-i/\sqrt{3}, 0)$.

Then, $= |z| = \sqrt{2/3}$. Substitute $r = \sqrt{2/3}$ in (9), and we get

$$\begin{aligned} c_1 &= -\left(\sqrt{\frac{2}{3}}\right)^3 \left(\cos^3(\theta) + 3\cos(\theta)\sin^2(\theta)\right), \\ c_2 &= \left(\sqrt{\frac{2}{3}}\right)^3 \left(-3\cos^2(\theta) + \sin^3(\theta)\right). \end{aligned} \tag{13}$$

We plot this for $0 \le \theta \le 2\pi$ (see Figure 11).

3.2.2. The Fractal of GM(P_3). The polynomial $P_3(z)$ admits quasi-self-similarity and is similar around y-axis (see Figures 7–10).

3.2.3. The Area of the Polynomial GM(P_3). Applying the Monte Carlo algorithm, we find that the area of $P_3(z)$ is 2.1952.

3.2.4. The Julia Set of the Polynomial P_3. Applying definition B$'$, if we take $c = 0\cdot 2i \in \mathrm{GM}(P_3)$, then we note that the Julia set $J_c(P_3)$ is connected (Figure 12) and for $c = 0\cdot 3 + 0\cdot 8i \notin \mathrm{GM}(P_3)$ then $J_c(P_3)$ is disconnected (Figure 13).

4. Topological Properties of General Mandelbrot Set of $P_n(z)$

In this section, we study some topological properties of GM (P_n), $n = 2,3, \ldots$.

Proposition 1. *The general Mandelbrot set of P_n, GM(P_n) is closed.*

Proof. Consider the complement of GM(P_n), i.e., $\mathbb{C}$-GM(P_n), $P_n(z) = z^n + z + c$.

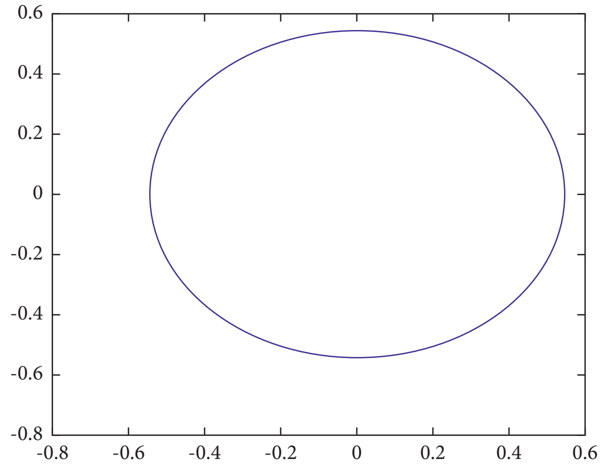

FIGURE 11: The boundary of fixed point of period one for $P_3(z)$.

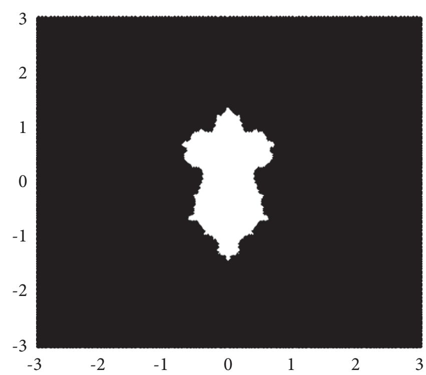

FIGURE 12: The Julia set of $P_3(z) = z^3 + z + c$ is connected where 0.2i ∈∈GM(P_3).

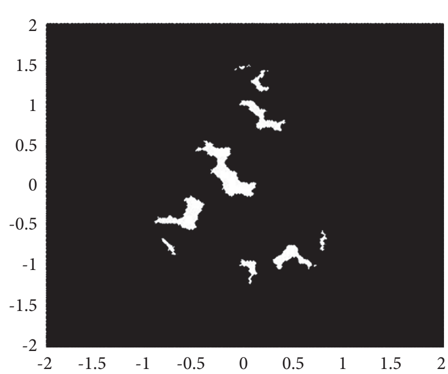

FIGURE 13: The Julia set of $P_3(z) = z^3 + z + c$ is connected where $0.3 + 0.8i$ ∉GM(P_3).

Let z_0 ∈¢-GM(P_n); then, by definition of GM(P_n), $P_n^{(k)}(z_0) \longrightarrow \infty$ as $k \longrightarrow \infty$. Since $P_n(z)$ is continuous, then there exists a neighborhood for z_0, say $B_{r_0}(z_0)$, such that $\forall z$ ∈∈$B_{r_0}(z_0)$, $P_n^{(k)}(z_0) \longrightarrow \infty$ as $k \longrightarrow \infty$, i.e., $B_r(z_0) \subseteq$ ¢− GM(P_n).

This implies that ¢-GM(P_n) is an open set in ¢. Therefore, GM(P_n) is closed. □

Proposition 2. *Let $P_n = z^n + z + c$, then there exists an integer k, such that $|P_n^{(m)}(z)| \longrightarrow \infty$ for $|z| > k$.*

Proof. Let $P_n(z) = z^n + z + c$; then, $|P_n(z)| \geq |z|^n - |z| - |c|$ (remember that $|a + b| \geq |a| - |b|$, $\forall a, b$ ∈¢).

Consider $g_n(z) = |z|^n - 2|z| - c$.

Since $|z|$ is positive, then whenever $|z|$ is large enough, we obtain that $g(z)$ is positive, i.e., $|z|^n - 2|z| - |c| \geq 0$. Choose k, an integer large enough, such that $g_n(z) > 0$, $\forall |z| > k$.

Thus, $|z|^n - |z| - |c| \geq |z|$. That is, $|P_n(z)| \geq |z|$, $\forall |z| > k$. Therefore, the function $|P_n(z)|$ is monotonically increasing.

Now, since P_n is continuous (polynomial), then $\forall z_0$ ∈¢ with $|z_0| > k$, there exists a neighborhood $B_r(z_0)$ such that $|P_n(z)| > |z|$, $\forall\, z \in B_r(z_0)$.

Therefore, $|P_n(z_0)|$ is unbounded for $|z_0| > k$.

Thus, the sequence $|P_n(z_0)| \longrightarrow \infty$, $\forall |z| > k$ (every unbounded monotonically increasing sequence is divergent). □

Proposition 3. *GM(P_n) is bounded.*

Proof. To prove GM(P_n) is bounded, we show that GM$(P_n) = \{c\colon |c| < k$, for some $k \in R^+\}$. Let z be a fixed point of P_n; then, $z^n + z + c = z \Rightarrow z^n + c = 0 \Rightarrow |z|^n = |c| \Rightarrow |z| = |c|^{1/n}$. Thus, $|z| = |c|^{1/n}$ is the equation boundary of all fixed points with $|P_n'(z)| < 1$.

The first derivative $P_n'(z) = nz^{n-1} + 1$.

If $nz^{n-1} + 1 = 0$, then $z_0 = \sqrt[n-1]{-1/n}$ is the critical point of P_n.

Now if $z_0 \notin$ GM(P_n), then $|P_n(z_0)| \geq |c|^{1/n}$. Thus,

$$\begin{gathered}\left|\left(\sqrt[n-1]{\frac{-1}{n}}\right)^n + \sqrt[n-1]{\frac{-1}{n}} + c\right| \geq |c|^{(1/n)}, \text{iff} \\ \left|\left(\sqrt[n-1]{\frac{-1}{n}}\right)^n + \sqrt[n-1]{\frac{-1}{n}}\right| + |c| \geq |c|^{(1/n)}, \text{iff} \qquad (14) \\ |c| - |c|^{(1/n)} + \left(\left(\sqrt[n-1]{\frac{-1}{n}}\right)^n + \sqrt[n-1]{\frac{-1}{n}}\right) \geq 0.\end{gathered}$$

Let $u = |c|^{1/n}$; then, $u^n = |c|$. Hence, the inequality becomes $u^n - u + A \geq 0$, where $A = \left(\left(\sqrt[n-1]{-1/n}\right)^n + \sqrt[n-1]{-1/n}\right)$. Now let $u_1, u_2, u_3, \ldots, u_n$ be the solutions of $u^n - u + A \geq 0$, and let $k = \max\{u_1, u_2, \ldots, u_n\}$. Thus, $|c| = k^n$. Therefore, $\forall c$ ∈¢, with $|c| > k$, then $c \notin$ GM(P_n).

Similarly, if z_0 is a periodic point of period m, then by the same argument and using Proposition 2, we obtain the same result. Thus, GM$(P_n) = \{c\colon |c| < k$ for some $k \in R^+\}$. Thus, GM(P_n) is bounded. □

Example 1. Consider the case $n = 2$, i.e., $P_2(z) = z^2 + z + c$.

The fixed point of P_2 satisfies $z^2 + z + c = z$.

$$\begin{aligned}&\Leftrightarrow z^2 = -c, \\ &\Leftrightarrow |z| = \sqrt{|c|}.\end{aligned} \qquad (15)$$

$P_2'(z) = 2z + 1$, and hence $z_0 = -1/2$ is the critical point of f.

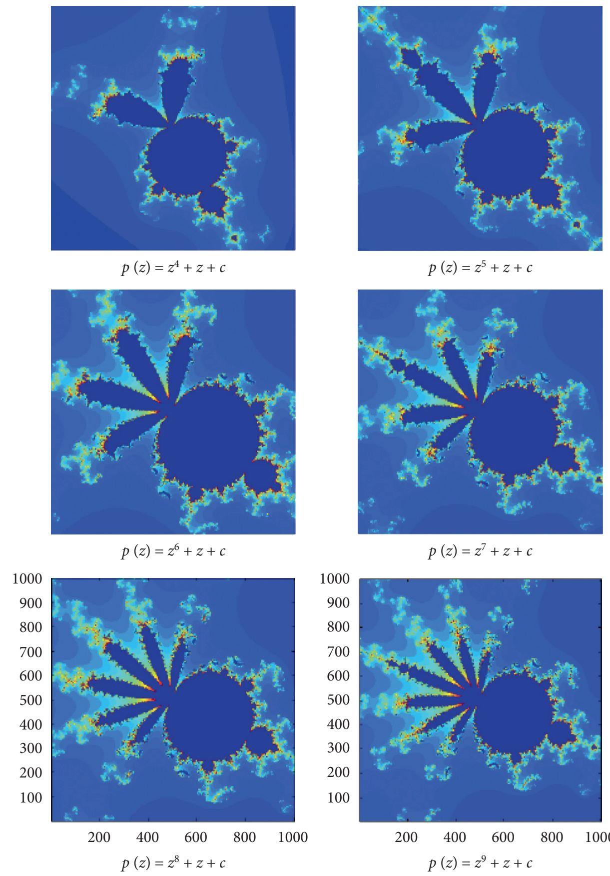

FIGURE 14: The general Mandelbrot set for $P_n(z) = z^n + z + c$, $n = 4, 5, \ldots, 9$.

Now, if

$$\begin{aligned} & \left|P_2(z_0)\right| \geq \sqrt{|c|} \\ & \Longleftrightarrow \left|\left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) + c\right| \\ & \geq \sqrt{|c|} \Longleftrightarrow \left|\frac{-1}{4} + c\right| \geq \sqrt{|c|} \\ & \Longleftrightarrow \left|\frac{-1}{4}\right| + |c| \geq \sqrt{|c|} \Longleftrightarrow |c| - \sqrt{c} + \left|\frac{-1}{4}\right| \geq 0. \end{aligned} \tag{16}$$

Let $u = \sqrt{c}$; then, $u^2 - u + 1/4 \geq 0$.

$$\begin{aligned} & \Longleftrightarrow \left(u - \frac{1}{2}\right)^2 \geq 0 \\ & \Longleftrightarrow u \geq \frac{1}{2}. \end{aligned} \tag{17}$$

Since $c = u^2$, we get $c = 1/4$. Thus, $|c| \geq |1/4|$. Now if z_0 is a periodic point of period m, then $|P_2^{(m)}(z_0)| > |P_2(z_0)|$ Proposition 2 and this implies that $z_0 \notin \mathrm{GM}(P_2)$ iff $|z_0| > 1/4$. Therefore, GM $(P_2) = \{c\colon |c| < |1/4|\}$, and GM (P_2) is bounded. From the above results, we conclude the following theorem.

Theorem 1. *Let $P_n(z) = z^n + z + c$ and let $GM(P_n)$ be the general Mandelbrot set of P_n, $n = 2, 3, \ldots$, and then $GM(P_n)$ is compact.*

Proof. By Proposition 1 and Proposition 3, we get that $\mathrm{GM}(P_n)$ is closed and bounded. Using Heine–Borel theorem, $\mathrm{GM}(P_n)$ is compact. □

Proposition 4. *Let $P_n(z) = z^n + z + c$; then, $P_n^{(m)}(z)$ has (at most) $n^m - 1$ attracting periodic orbits, $\forall$ $(m = 1, 2, 3, \ldots)$.*

Proof. $P_n(z) = z^n + z + c$ is a polynomial of degree n. Then, $P_n^{(m)}(z)$ is a polynomial of degree n^m. Now, $P_n^{(m)}$ is a polynomial of degree $n^m - 1$. Since z is a complex variable, then $P_n^{(m)'}(z) = 0$ has $n^m - 1$ distinct roots. That is, $P_n^{(m)}$ has $n^m - 1$ distinct critical points.

But the orbit of every critical point must lie in an attracting periodic orbit. Then, there are (at most) $n^m - 1$ periodic orbits.

The following theorem characterizes the general Mandelbrot set of our family as a union of basins of attractions. □

Table 1: Comparison between the Mandelbrot set generated with $P(z) = z2 + c$ and the general Mandelbrot sets of GM(P_2) and GM(P_3).

	Fractal	Symmetry	Area	Topological properties
M $(z^2 + c)$	Quasi-self-similarity	Around the x-axis	1.72	Compact
GM (P_2)	Quasi-self-similarity	Around the x-axis	1.6456	Compact
GM (P_3)	Quasi-self-similarity	Around the y-axis	2.1952	
G(MP_n)	Quasi-self-similarity	Around some diagonal		Compact

Theorem 2. *The general Mandelbrot set $GM(P_n)$ consists of the union of the basins of attraction of the attracting periodic orbits, and each basin of a periodic point contains all values of c whose orbits converge to that periodic point.*

Proof. By definition, $GM(P_n) = \{c\colon P_n^{(m)}(z_{cr})\ \text{is bounded}\}$. That is, $c \in \mathrm{GM}(P_n)$ iff the orbit of the critical points of $P_n(z_{cr})$ is bounded. But the orbit of any critical point must lie in the orbit of attracting period orbit and hence in a basin of an attracting periodic point. Thus, $\mathrm{GM}(P_n)$ is composed of basins of the attracting periodic orbits, and all the values of $c \in \mathrm{GM}(P_n)$ converge to these periodic orbits.

Now, we construct the general Mandelbrot set of the polynomials $P_n(z) = z^n + z + c$ for $n = 4, 5, \ldots, 9$ (see Figure 14).

$$\begin{aligned} P(z) &= z^9 + z + c, \\ P(z) &= z^8 + z + c, \\ P(z) &= z^7 + z + c, \\ P(z) &= z^6 + z + c, \\ P(z) &= z^5 + z + c, \\ P(z) &= z^4 + z + c. \end{aligned} \tag{18}$$

Note that for $n \geq 4$ in previous forms, the number of arms extended in the sets equals n−2 arm. Moreover, if we plot the GM of $P_n(z) = z^n + z + c$, then we find the same behavior. In a future work, we hope to analyze this behavior.

Finally, we give a comparison between the Mandelbrot set generated with $P(z) = z^2 + c$ and the general Mandelbrot sets of $\mathrm{GM}(P_2)$ and $\mathrm{GM}(P_3)$ (Table 1). □

5. Conclusions

(1) In this paper, we construct the general Mandelbrot sets of the family of polynomials $P_n(z) = z^n + z + c$, $(n \geq 2)$, using the escaping algorithm.

(2) The areas, boundaries, and fractals for $P_n(z)$ are studied and calculated for $n = 2,3$.

(3) We study some topological properties of GM(P_n). It is proved that GM(P_n) is closed and bounded and hence is compact; also, we find a bound of GM(P_n), $\forall n \geq 2$.

(4) GM (P_n) is characterized as a union of basins of attraction of the attracting periodic point of P_n, $\forall n \geq 2$.

(5) Finally, we plot the general Mandelbrot sets of $P_n(z)$, $n = 4,5,6,7,8,9$, and make a comparison of the properties of Mandelbrot set $M(z^2 + c)$ and our general Mandelbrot sets GM(P_n).

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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