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Research Article

A Note about Young's Inequality with Different Measures

Saba Mehmood, Eridani Eridani, and Fatmawati Fatmawati

Department of Mathematics, Faculty of Science and Technology, Universitas Airlangga, Surabaya 60115, Indonesia

Correspondence should be addressed to Eridani Eridani; eridani@fst.unair.ac.id

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The key purpose of this paper is to work on the boundedness of generalized Bessel–Riesz operators defined with doubling measures in Lebesgue spaces with different measures. Relating Bessel decaying the kernel of the operators is satisfying some elementary properties. Doubling measure, Young's inequality, and Minköwski's inequality will be used in proofs of boundedness of integral operators. In addition, we also explore the relation between the parameters of the kernel and generalized integral operators and see the norm of these generalized operators which will also be bounded by the norm of their kernel with different measures.

1. Introduction

Suppose we are given $K_\alpha\colon (0,\infty) \longrightarrow (0,\infty)$, with $K_\alpha(t)\colon = t^{\alpha-n}$, $0<\alpha<n$. Fractional integral operator, one of integral operators, is often studied since early last century. This operator is defined by

$$T_\alpha f(x)\colon = \int_{\mathbb{R}^n} K_\alpha(|x-y|) f(y) \mathrm{d}y, \tag{1}$$

for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, with $1 \leq p < \infty$. Here, K_α is called fractional integral kernel or Riesz kernel [1].

Studies about Riesz potentials T_α were started since 1920's. Hardy-Littlewood [2,3] proved the boundedness of Riesz potentials on Lebesgue spaces for $n=1$. After 50's, Hardy-Littlewood and Sobolev [4] proved the boundedness of T_α for $n \in N$. [5] (see also D. Edmunds [[6], Chapter 6]) Kokilashvili had a complete description of nondoubling measure μ guaranteeing the boundedness of fractional integral operator T_α from $L^p(\mu, X)$ to $L^q(\mu, X)$, $1<p<q<1$. We notice that this result was derived in [7] for potentials on Euclidean spaces. In [4], theorems of Sobolev and Adams type for fractional integrals defined on quasimetric measure spaces were established.

Some two-weight norm inequalities for fractional operators on $\mathbb{R}^n$ with nondoubling measure were studied in [8]. The boundedness of the Riesz potential in Lebesgue and Morrey spaces defined on Euclidean spaces was studied in Peter and Adams's paper [9,10]. The same problem for fractional integrals on $\mathbb{R}^n$ with nondoubling measure was investigated by Sawano in [11]. Eridani ([12], Theorem 4, Theorem 3.1, Theorem 3.3) established the boundedness of fractional integral operators and mention the necessary and sufficient conditions for the boundedness of maximal operators.

Since 1930s, some researchers [2,3] have studied the boundedness of T_α on some function spaces.

Theorem 1 (Hardy-Littlewood-Sobolev) (see [4]). *If $f \in L^p(\mathbb{R}^n)$, with $1<p<n/\alpha$, then there exists $C_{p,q}>0$ such that*

$$\left\|T_\alpha f\colon L^q(\mathbb{R}^n)\right\| \leq C_{p,q}\left\|f\colon L^p(\mathbb{R}^n)\right\|, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \tag{2}$$

From now on, $C, C_{p,q}, C_{p,q,s} > 0$ will be serve as a positive constant, not necessarily the same one.

The purpose of this paper is to work on the boundedness of generalized Bessel–Riesz operators defined in Lebesgue spaces with different measures. Role of Young's inequality and Minköwski's inequality will be used in proofs of boundedness of integral operators.

Here, we define

$$\|f\colon L^p\| := \left[\int_{\mathbb{R}^n} |f(t)|^p \mathrm{d}t\right]^{1/p}, \quad 1 \le p < \infty, \tag{3}$$

and $f \in L^p = L^p(\mathbb{R}^n)$, as a collection of f such that $\|f\colon L^p\| < \infty$. Next, for a given $K_{\alpha,\gamma}\colon (0,\infty) \longrightarrow (0,\infty)$, with

$$K_{\alpha,\gamma}(t)\colon = \frac{t^{\alpha-n}}{[1+t]^{\gamma}}, \quad 0 < \alpha < n, 0 \le \gamma. \tag{4}$$

We define Bessel-Riesz operator as

$$T_\alpha f(x)\colon = \int_{\mathbb{R}^n} K_{\alpha,\gamma}(|x-y|) f(y)\mathrm{d}y, \tag{5}$$

For every $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, with $1 \le p < \infty$. Here, $K_{\alpha,\gamma}$ is called Bessel–Riesz kernel.

The origin of Bessel–Riesz operators is Schröndinger equation. The Schrödinger equation is a linear partial differential equation that describes the wave function or state function of a quantum-mechanical system. In quantum mechanics, the analogue of Newton's law is Schrödinger's equation. In 1999, Kazuhiro Kurata, Seiichi Nishigaki, and Satoko Sugano [13] studied boundedness of integral operators on Lebesgue and generalized Morrey spaces and its application to estimate in Morrey spaces for the Schrödinger operator $L_2 = -\Delta + V(x) + W(x)$ with nonnegative $V \in (\mathrm{RH})_\infty$ (reverse Hölders class) and small perturbed potentials W.

In a recent previous year 2016, Idris et al. ([14], Theorem 6, pp. 3) presented the boundedness of Bessel–Riesz operators. They obtained results that were similar to Chiarenza–Frasca's result [15] for the boundedness of fractional integral operators.

Eridani et al. [12] presented the boundedness of fractional integral operators defined on quasimetric measure spaces. Moreover, Idris et al. [14] have investigated the boundedness of generalized Morrey spaces with weight and presented the boundedness of these operators on Lebesgue spaces and Morrey spaces for Euclidean spaces.

Since Euclidean spaces are the simplest example of measure metric spaces. Kurata et al. [16] have investigated the boundedness of Bessel–Riesz operators on generalized Morrey spaces with weight. The boundedness of these operators on Lebesgue spaces in Euclidean settings will be proved using Young's inequality and Minköwski's inequality.

Moreover, we will also find the norm of the generalized Bessel–Riesz operators bounded by the norm of the kernels. Saba et al. [17] used Young's inequality, to prove the boundedness of Bessel–Riesz operators on Lebesgue spaces in measure metric spaces, which are easy consequences of Young's inequality. The second consequence after the fact Bessel–Riesz operator is bounded on Lebesgue spaces; we entered to next phenomena of Morrey spaces. In Young's inequality, we have the best constant known as 1. But at this point, we still have no information about the best constant in Morrey spaces. So in this paper, we move towards generalized Bessel–Riesz operators in Lebesgue spaces.

We will also mention the case when the measure satisfies the doubling condition. The derived conditions are simultaneously necessary and sufficient for appropriate inequalities that were derived in [18,19]. We also have the following result [14] about the boundedness of such an operator as follows:

Theorem 2 (Idris–Gunawan–Lindiarni–Eridani). If $f \in L^p$ and $K_{\alpha,\gamma} \in L^q$, then there exists $C_{p,q,s} > 0$ such that

$$\left\|T_{\alpha,\gamma} f\colon L^s\right\| \le C_{p,q,s}\left\|K_{\alpha,\gamma}\colon L^q\right\| \cdot \left\|f\colon L^p\right\|, \quad \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1. \tag{6}$$

For some functions $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and $g\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, we define

$$(f \star g)(x)\colon = \int_{\mathbb{R}^n} g(x-y) f(y)\mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{7}$$

We know that $f \star g$ is a generalization of T_α and $T_{\alpha,\gamma}$. Moreover, the above result is a particular case of the following [20].

Theorem 3 (Young's inequality). If $f \in L^p$ and $g \in L^q$, then there exists $C^*_{p,q,s} > 0$ such that

$$\left\|f \star g\colon L^s\right\| \le C^*_{p,q,s}\left\|g\colon L^q\right\| \cdot \left\|f\colon L^p\right\|, \quad \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1. \tag{8}$$

Kurata et al. [16] have shown that $W \cdot T_{\alpha,\gamma}$ is bounded on generalized Morrey spaces where W is any real functions. For applications of the above operators in Euclidean spaces setting, see [16].

2. The Kernel

For $1 \le s < \infty, 0 < \gamma < \infty$, and $\mathbb{R}^+\colon = (0,\infty)$, we define functions $\rho\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, with the following conditions:

$$(\rho I).\ \frac{1}{2} \le \frac{a}{b} \le 2 \Rightarrow \frac{1}{C_1} \le \frac{\rho(a)}{\rho(b)} \le C_1;$$

$$(\rho II).\ \int_0^r \frac{\rho(t)^s}{t^{n(s-1)+1}} dt + \int_r^\infty \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}} dt < \infty, \quad r > 0; \tag{9}$$

$$(\rho III).\ \int_0^1 \frac{\rho(t)^s}{t^{n(s-1)+1}} dt + \int_1^\infty \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}} dt \le C_2 < \infty.$$

From (ρI) we will have

$$\int_r^{2r} \frac{\rho(t)}{t} \mathrm{d}t \sim \rho(r). \tag{10}$$

That is, there exists $0 < c_1 \le c_2$ such that

$$c_1\rho(r) \le \int_r^{2r} \frac{\rho(t)}{t} \mathrm{d}t \le c_2\rho(r), \quad r > 0. \tag{11}$$

Since

$$\frac{1}{r^{s\gamma}}\int_0^1 \frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t \geq \int_r^1 \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}}\,dt, \quad 0<r<1,$$
$$\int_1^r \frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t \leq r^{s\gamma}\int_1^\infty \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}}\,\mathrm{d}t, \quad 1<r<\infty, \tag{12}$$

then (ρII) and (ρIII) are equivalent, with the following explanations.

It is easy to see that (ρII)$\Rightarrow$(ρIII). Suppose (ρIII) is true. Then, for $0<r<1$, we have

$$\int_0^r \frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t \leq \int_0^1 \frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t<\infty,$$
$$\int_r^\infty \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}}\,\mathrm{d}t = \int_r^1+\int_1^\infty \leq C_2+\int_r^1$$
$$\leq C_2+\frac{1}{r^{s\gamma}}\int_0^1 \frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t<\infty. \tag{13}$$

On the other hand, for $1<r<\infty$, then

$$\int_r^\infty \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}}\,\mathrm{d}t<\int_1^\infty \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}}\,\mathrm{d}t<C_2<\infty,$$
$$\int_0^r \frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t = \int_0^1+\int_1^r \leq C_2+r^{s\gamma}\int_1^\infty \frac{\rho(t)^s}{t^{n(s-1)+s\gamma+1}}\,\mathrm{d}t<\infty, \tag{14}$$

and we already prove that both of the conditions are equivalent.

We define

$$K_{\rho,\gamma}(|x|) := \frac{\rho(|x|)}{|x|^n[1+|x|]^\gamma}, \quad x\in\mathbb{R}^n. \tag{15}$$

Using (ρII) with $1\leq s<\infty$, and any $R>0$, then

$$\begin{aligned}
\int_{\mathbb{R}^n}\left|K_{\rho,\gamma}(|x|)\right|^s\mathrm{d}x &= \int_{0\leq|x|}\frac{\rho(|x|)^s}{|x|^{\mathrm{sn}}[1+|x|]^{s\gamma}}\,\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}}\int_{2^kR\leq|x|<2^{k+1}R}\frac{\rho(|x|)^s}{|x|^{\mathrm{sn}}[1+|x|]^{s\gamma}}\,\mathrm{d}x \\
&\sim \sum_{k=-\infty}^{-1}\frac{\rho\left(2^kR\right)^s}{\left(2^kR\right)^{n(s-1)}\left[1+2^kR\right]^{s\gamma}}+\sum_{k=0}^{\infty}\frac{\rho\left(2^kR\right)^s}{\left(2^kR\right)^{n(s-1)}\left[1+2^kR\right]^{s\gamma}} \\
&\leq \sum_{k=-\infty}^{-1}\frac{\rho\left(2^kR\right)^s}{\left(2^kR\right)^{n(s-1)}}+\sum_{k=0}^{\infty}\frac{\rho\left(2^kR\right)^s}{\left(2^kR\right)^{n(s-1)+s\gamma}} \\
&\sim \sum_{k=-\infty}^{-1}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t+\sum_{k=0}^{\infty}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)^s}{t^{1+n(s-1)+s\gamma}}\,\mathrm{d}t \\
&= \int_0^R\frac{\rho(t)^s}{t^{n(s-1)+1}}\,\mathrm{d}t+\int_R^\infty\frac{\rho(t)^s}{t^{1+n(s-1)+s\gamma}}\,\mathrm{d}t<\infty.
\end{aligned} \tag{16}$$

By (ρII), then $K_{\rho,\gamma}\in L^s$, for $1\leq s<\infty$.

If $k\in\{-1,-2,-3,\ldots\}$, then $0<2^k<1$ and $[1+2^k]^{n\gamma}<2^{n\gamma}$. On the other hand, we can also have for $k\in\{0,1,2,3,\ldots\}$, then $1\leq 2^k$, and $2^k<1+2^k<2^{k+1}$.

Finally, we will have

$$\begin{aligned}
&\int_0^1 \frac{\rho(t)^s}{t^{n(s-1)+1}}\mathrm{d}t + \int_1^\infty \frac{\rho(t)^s}{t^{1+n(s-1)+s\gamma}}\mathrm{d}t \\
&= \sum_{k=-\infty}^{-1}\int_{2^k}^{2^{k+1}} \frac{\rho(t)^s}{t^{n(s-1)+1}}\mathrm{d}t + \sum_{k=0}^{\infty}\int_{2^k}^{2^{k+1}} \frac{\rho(t)^s}{t^{1+n(s-1)+s\gamma}}\mathrm{d}t \\
&\sim \sum_{k=-\infty}^{-1}\frac{\rho\left(2^k\right)^s}{\left(2^k\right)^{n(s-1)}} + \sum_{k=0}^{\infty}\frac{\rho\left(2^k\right)^s}{\left(2^k\right)^{n(s-1)+s\gamma}} \\
&\leq C\left(\sum_{k=-\infty}^{-1}\frac{\rho\left(2^k\right)^s\left(2^k\right)^n}{\left(2^k\right)^{\mathrm{ns}}\left[1+2^k\right]^{n\gamma}} + \sum_{k=0}^{\infty}\frac{\rho\left(2^k\right)^s\left(2^k\right)^n}{\left(2^k\right)^{\mathrm{ns}}\left[1+2^k\right]^{n\gamma}}\right) \\
&= C\sum_{k\in\mathbb{Z}}\int_{2^k\leq|x|<2^{k+1}}\frac{\rho(|x|)^s}{|x|^{\mathrm{sn}}[1+|x|]^{s\gamma}}\mathrm{d}x \\
&= C\int_{0\leq|x|}\frac{\rho(|x|)^s}{|x|^{\mathrm{sn}}[1+|x|]^{s\gamma}}\mathrm{d}x = \int_{\mathbb{R}^n}\left|K_{\rho,\gamma}(|x|)\right|^s\mathrm{d}x.
\end{aligned} \tag{17}$$

After the above estimates, we conclude.

Lemma 1. *For $1\leq s<\infty$, we will have*

$$\int_{\mathbb{R}^n}\left|K_{\rho,\gamma}(|x|)\right|^s\mathrm{d}x \sim \int_0^1 \frac{\rho(t)^s}{t^{n(s-1)+1}}\mathrm{d}t + \int_1^\infty \frac{\rho(t)^s}{t^{1+n(s-1)+s\gamma}}\mathrm{d}t. \tag{18}$$

With the same technique as used to estimate the kernel defined by equation (15) or every $y\in\mathbb{R}^n$, we also have

$$\int_{\mathbb{R}^n}\left|K_{\rho,\gamma}(|x-y|)\right|^s\mathrm{d}x \sim \int_0^1 \frac{\rho(t)^s}{t^{n(s-1)+1}}\mathrm{d}t + \int_1^\infty \frac{\rho(t)^s}{t^{1+n(s-1)+s\gamma}}\mathrm{d}t. \tag{19}$$

As a classical example, we can consider the following:

$$\rho(t) := t^{[\alpha+n(s-1)]/s}, \quad 1\leq\frac{[\alpha+1]}{\gamma}<s<\infty. \tag{20}$$

If we consider

$$T_{\rho,\gamma}f(x) := \int_{\mathbb{R}^n}K_{\rho,\gamma}(|x-y|)f(y)\mathrm{d}y, \quad x\in\mathbb{R}^n, \tag{21}$$

then as a consequences of Young's inequality, we also have the following.

Lemma 2. *If $1/q=1/p+1/s-1$ and $f\in L^p$, then there exists $C>0$ such that*

$$\left\|T_{\rho,\gamma}f\colon L^q\right\|\leq C\left\|K_{\rho,\gamma}\colon L^s\right\|\cdot\left\|f\colon L^p\right\|, \quad 1<p,q,r<\infty. \tag{22}$$

From the above lemma, there exists $C>0$ such that

$$\begin{aligned}
\left\|T_{\rho,\gamma}f\colon L^p\right\| &\leq C\left\|K_{\rho,\gamma}\colon L^p\right\|\cdot\left\|f\colon L^1\right\|, \text{ or } \left\|T_{\rho,\gamma}f\colon L^p\right\| \\
&\leq C\left\|K_{\rho,\gamma}\colon L^1\right\|\cdot\left\|f\colon L^p\right\|, \quad 1\leq p<\infty.
\end{aligned} \tag{23}$$

Suppose μ is an arbitrary measure on $\mathbb{R}^n$. We define $\mu\in$ (GC) (growth condition), if and only if there exists $C_2>0$ such that

$$\mu(B(a,R))\leq C_2R^n, \tag{24}$$

ror every open balls $B(a,R)\colon = \{x\in\mathbb{R}^n\colon |x-a|<R\}$ in $\mathbb{R}^n$. For more information about this kind of measure, see [21].

Based on the above definitions, we try to estimate

$$\left\|K_{\rho,\gamma}\colon L^s(\mu)\right\| := \left(\int_{\mathbb{R}^n}\left|K_{\rho,\gamma}(|x|)\right|^s d\mu(x)\right)^{1/s}, \quad 1\leq s<\infty. \tag{25}$$

For $1\leq s<\infty$, and $R>0$, we consider the following:

$$\begin{aligned}
\int_{|x|<R}\frac{\rho(|x|)^s}{|x|^{\mathrm{sn}}[1+|x|]^{s\gamma}}d\mu(x) &= \sum_{k=-\infty}^{-1}\int_{2^kR\leq|x|<2^{k+1}R}\frac{\rho(|x|)^s}{|x|^{\mathrm{sn}}[1+|x|]^{s\gamma}}d\mu(x) \\
&\leq C\sum_{k=-\infty}^{-1}\frac{\rho\left(2^kR\right)^s\mu\left(B\left(0,2^kR\right)\right)}{\left(2^kR\right)^{sn}\left[1+2^kR\right]^{s\gamma}} \leq C\sum_{k=-\infty}^{-1}\frac{\rho\left(2^kR\right)^s}{\left(2^kR\right)^{(s-1)n}} \\
&\leq C\sum_{k=-\infty}^{-1}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)^s}{t^{(s-1)n+1}}\mathrm{d}t = C\int_0^R\frac{\rho(t)^s}{t^{(s-1)n+1}}\mathrm{d}t,
\end{aligned} \tag{26}$$

and also

$$\begin{aligned}\int_{R\le|x|}\frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}d\mu(x) &= \sum_{k=1}^{\infty}\int_{2^kR\le|x|<2^{k+1}R}\frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}d\mu(x)\\ &\le C\sum_{k=0}^{\infty}\frac{\rho(2^kR)^s\mu(B(0,2^kR))}{(2^kR)^{sn}[1+2^kR]^{s\gamma}}\le C\sum_{k=0}^{\infty}\frac{\rho(2^kR)^s}{(2^kR)^{(s-1)n+s\gamma}}\\ &\le C\sum_{k=0}^{\infty}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}dt = C\int_R^{\infty}\frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}dt,\end{aligned}\tag{27}$$

where letter $C>0$ shall always denote a constant, not necessarily the same one.

At this point, for $1\le s<\infty$, and $\mu\in$ (GC), we define

$$\left\|K_{\rho,\gamma}\colon L^s(\mu)\right\| = \sup_{R>0}\left(\int_0^R\frac{\rho(t)^s}{t^{(s-1)n+1}}dt+\int_R^{\infty}\frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}dt\right)^{1/s}.\tag{28}$$

3. Main Results

For any measure ν on $\mathbb{R}^n$, any measurable functions $f\colon\mathbb{R}^n\longrightarrow\mathbb{R}$, and $x\in\mathbb{R}^n$, we define

$$T_{\rho,\gamma}f(x) := \int_{\mathbb{R}^n}K_{\rho,\gamma}(|x-y|)f(y)d\nu(y).\tag{29}$$

Before we state our main results, about the boundedness of $T_{\rho,\gamma}$, we consider the following simple result [13].

Lemma 3 (Minköwski's inequality). *Suppose $1\le p<\infty$, and we are given $F\colon\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}$. For any measure ν and μ on $\mathbb{R}^n$, then*

$$\begin{aligned}&\left[\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}F(x,y)d\nu(y)\right|^p d\mu(x)\right]^{1/p}\\ &\le\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}|F(x,y)|^p d\mu(x)\right]^{1/p}d\nu(y).\end{aligned}\tag{30}$$

Now, we state the following:

Theorem 4. *Suppose $\mu\in$ (GC) and ν is any measure on $\mathbb{R}^n$.*

If $f\in L^1(\nu)$ and $K_{\rho,\gamma}\in L^s(\mu)$, then there exists $C_s>0$ such that

$$\left\|T_{\rho,\gamma}f\colon L^s(\mu)\right\|\le C_s\left\|K_{\rho,\gamma}\colon L^s(\mu)\right\|\cdot\left\|f\colon L^1(\nu)\right\|,\quad 1\le s<\infty.\tag{31}$$

Proof. By Minköwski's inequality, with $1\le s<\infty$, then

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left|T_{\rho,\gamma}f(x)\right|^s d\mu(x)\right)^{1/s} &= \left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}K_{\rho,\gamma}(|x-y|)f(y)d\nu(y)\right|^s d\mu(x)\right)^{1/s}\\ &\le\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|K_{\rho,\gamma}(|x-y|)f(y)\right|^s d\mu(x)\right)^{1/s}d\nu(y)\\ &\le\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|K_{\rho,\gamma}(|x-y|)\right|^s d\mu(x)\right)^{1/s}|f(y)|d\nu(y)\\ &\le C\left\|K_{\rho,\gamma}\colon L^s(\mu)\right\|\cdot\left\|f\colon L^1(\nu)\right\|.\end{aligned}\tag{32}$$

As a corollary of the above result, we also have □

Corollary 1. *Suppose $\mu\in$ (GC) and ν is any measure on $\mathbb{R}^n$.*

If $f\in L^1(\nu)$ and $K_{\rho,\gamma}\in L^1(\mu)$, then there exists $C>0$ such that

$$\left\|T_{\rho,\gamma}f\colon L^1(\mu)\right\|\le C\left\|K_{\rho,\gamma}\colon L^1(\mu)\right\|\cdot\left\|f\colon L^1(\nu)\right\|.\tag{33}$$

Suppose we have the

$$\frac{1}{s}=\frac{1}{p}+\frac{1}{q}-1,\text{ or }1=\frac{1}{s}+1-\frac{1}{p}+1-\frac{1}{q}=\frac{1}{s}+\frac{1}{p'}+\frac{1}{q'}.\tag{34}$$

It is noted that

$$\begin{aligned}p&=q'\left(1-\frac{p}{s}\right),\\ q&=p'\left(1-\frac{q}{s}\right).\end{aligned}\tag{35}$$

After the above simple calculations, we have

Theorem 5. *Suppose $\mu\in$ (GC) and ν any measure on $\mathbb{R}^n$ such that*

$$\nu(B) \leq C^{*}\mu(B), \tag{36}$$

for some $C^{*} > 0$ and for every open balls B on $\mathbb{R}^n$.

If $f \in L^p(\nu)$ and $K_{\rho,\gamma} \in L^q(\mu)$, then there exists $C > 0$ such that

$$\left\| T_{\rho,\gamma} f \colon L^s(\mu) \right\| \leq C \left\| K_{\rho,\gamma} \colon L^q(\mu) \right\| \cdot \left\| f \colon L^p(\nu) \right\|, \quad \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1. \tag{37}$$

Proof. With Hölder's inequality, we start with the following:

$$\begin{aligned}
\left|T_{\rho,\gamma} f(x)\right| &\leq \int_{\mathbb{R}^n} |f(y)|^{p/s+(1-p/s)} \left|K_{\rho,\gamma}(|x-y|)\right|^{q/s+(1-p/s)} d\nu(y) \\
&\leq \left[\int_{\mathbb{R}^n} \left|K_{\rho,\gamma}(|x-y|)\right|^q |f(y)|^p d\nu(y)\right]^{1/s} \left[\int_{\mathbb{R}^n} |f(y)|^{q'(1-p/s)} d\nu(y)\right]^{1/q'} \\
&\quad \times \left[\int_{\mathbb{R}^n} \left|K_{\rho,\gamma}(|x-y|)\right|^{p'(1-q/s)} d\nu(y)\right]^{1/p'} \\
&= \left[\int_{\mathbb{R}^n} \left|K_{\rho,\gamma}(|x-y|)\right|^q |f(y)|^p d\nu(y)\right]^{1/s} \left[\int_{\mathbb{R}^n} |f(y)|^p d\nu(y)\right]^{1/q'} \\
&\quad \times \left[\int_{\mathbb{R}^n} \left|K_{\rho,\gamma}(|x-y|)\right|^q d\nu(y)\right]^{1/p'}.
\end{aligned} \tag{38}$$

If we define

$$T^{p,q}_{\rho,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\rho,\gamma}(|x-y|)^q |f(y)|^p d\nu(y), \tag{39}$$

then for every x, we come to

$$\left|T_{\rho,\gamma} f(x)\right|^s \leq \left\| f \colon L^p(\nu) \right\|^{sp/q'} \cdot \left|T^{p,q}_{\rho,\gamma} f(x)\right| \left[\int_{\mathbb{R}^n} \left|K_{\rho,\gamma}(|x-y|)\right|^q d\nu(y)\right]^{s/p'}. \tag{40}$$

Now, we want to estimate the right hand side, especially, for $x \in \mathbb{R}^n$ and $R > 0$, we will have

$$\int_{\mathbb{R}^n} \left|K_{\rho,\gamma}(|x-y|)\right|^q d\nu(y) = \int_{|x-y|<R} + \int_{|x-y|\geq R}. \tag{41}$$

We start with

$$\begin{aligned}
\int_{|x-y|<R} &= \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \\
&\sim C \sum_{k=-\infty}^{-1} K_{\rho,\gamma}\left(2^k R\right)^q \nu\left(B\left(x, 2^{k+1} R\right)\right) \\
&\leq C \sum_{k=-\infty}^{-1} \frac{\rho\left(2^k R\right)^q}{\left(2^k R\right)^{(q-1)n}} \leq C \int_0^R \frac{\rho(t)^q}{t^{(q-1)n+1}} \mathrm{d}t \\
&\leq C \left\| K_{\rho,\gamma} \colon L^q(\mu) \right\|^q,
\end{aligned} \tag{42}$$

and also

$$\begin{aligned}
\int_{|x-y|\geq R} &\sim C \sum_{k=0}^{\infty} K_{\rho,\gamma}\left(2^k R\right)^q \nu\left(B\left(x, 2^{k+1} R\right)\right) \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho\left(2^k R\right)^q}{\left(2^k R\right)^{nq-n+q\gamma}} \leq C \int_R^{\infty} \frac{\rho(t)^q}{t^{n(q-1)+q\gamma+1}} dt \\
&\leq C \left\| K_{\rho,\gamma} \colon L^q(\mu) \right\|^q.
\end{aligned} \tag{43}$$

Up to now, for every $x \in \mathbb{R}^n$, we already have

$$\left|T_{\rho,\gamma} f(x)\right|^s \leq C \left\| K_{\rho,\gamma} \colon L^q(\mu) \right\|^{qs/p'} \cdot \left\| f \colon L^p(\nu) \right\|^{qs/p'} \cdot \left|T^{p,q}_{\rho,\gamma} f(x)\right|. \tag{44}$$

By the previous fact, we know that

$$\left\| T_{\rho,\gamma} f \colon L^1(\mu) \right\| \leq C_1 \left\| K_{\rho,\gamma} \colon L^1(\mu) \right\| \cdot \left\| f \colon L^1(\nu) \right\|, \tag{45}$$

and after this, we come to

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|T^{p,q}_{\rho,\gamma} f(x)\right| d\mu(x) &\leq C_1 \left\| \left|K_{\rho,\gamma}\right|^q \colon L^1(\mu) \right\| \cdot \left\| |f|^p \colon L^1(\nu) \right\| \\
&= C_1 \left\| K_{\rho,\gamma} \colon L^q(\mu) \right\|^q \cdot \left\| f \colon L^p(\nu) \right\|^p.
\end{aligned} \tag{46}$$

And finally,

$$\begin{aligned}\left\|T_{\rho,\gamma}f\colon L^s(\mu)\right\|^s \le C\left\|K_{\rho,\gamma}\colon L^q(\mu)\right\|^{qs/p'} \\ \cdot \left\|f\colon L^p(\nu)\right\|^{sp/q'}\int_{\mathbb{R}^n}\left|T^{p,q}_{\rho,\gamma}f(x)\right|d\mu(x) \\ \le C\left\|K_{\rho,\gamma}\colon L^q(\mu)\right\|^{q+qs/p'}\cdot\left\|f\colon L^p(\nu)\right\|^{p+sp/q'}.\end{aligned} \tag{47}$$

Since $q + qs/p' = s = p + sp/q'$, then we are done.
As a corollary, we also have: □

Corollary 2. *Suppose $\mu \in (GC)$ and ν any measure on $\mathbb{R}^n$ such that*

$$\nu(B) \le C^*\mu(B), \tag{48}$$

for some $C^* > 0$ and for every open balls B on $\mathbb{R}^n$.

If $f \in L^s(\nu)$ and $K_{\rho,\gamma} \in L^1(\mu)$, then there exists $C > 0$ such that

$$\left\|T_{\rho,\gamma}f\colon L^s(\mu)\right\| \le C\left\|K_{\rho,\gamma}\colon L^1(\mu)\right\|\cdot\left\|f\colon L^s(\nu)\right\|, \quad 1 \le s < \infty. \tag{49}$$

4. Pointwise Multiplier Operators

Suppose $f \in L^p(\mu)$ and $g \in L^{p'}(\mu)$, with $1 < p < \infty$ and $1 = 1/p + 1/p'$. Then, by Hölder inequality, we will have $\|f\cdot g\colon L^1(\mu)\| \le \|f\colon L^p(\mu)\|\cdot\|g\colon L^{p'}(\mu)\|$.

Next, we consider a pointwise multipliers operator W by

$$W\colon f \mapsto W\cdot f, \text{ with } [W\cdot f](x) \coloneqq W(x)\cdot f(x), \quad x \in \mathbb{R}^n. \tag{50}$$

So, from Hölder inequality, if $W \in L^{p'}(\mu)$, then W is a bounded operators from $L^p(\mu)$ to $L^1(\mu)$, with $\|W\cdot f\colon L^1(\mu)\| \le \|W\colon L^{p'}(\mu)\|\cdot\|f\colon L^p(\mu)\|, \quad 1 < p < \infty$.

Another example is the following equation. Suppose, we consider a fractional integral operators T_α, and we define

$$\begin{aligned}W\cdot T_\alpha\colon f \mapsto W\cdot T_\alpha f, \text{ with } [W\cdot T_\alpha f](x) \\ \coloneqq W(x)\cdot T_\alpha f(x), \quad x \in \mathbb{R}^n.\end{aligned} \tag{51}$$

Again, since $1 < p < n/\alpha$ and $1/q + \alpha/n = 1/p$, in view of Hölder inequality, then

$$\begin{aligned}\left\|W\cdot T_\alpha f\colon L^p\right\| \le \left\|W\colon L^{n/\alpha}\right\|\cdot\left\|T_\alpha f\colon L^q\right\| \\ \le C\left\|W\colon L^{n/\alpha}\right\|\cdot\left\|f\colon L^p\right\|.\end{aligned} \tag{52}$$

So, if $W \in L^{n/\alpha}$, then $W\cdot T_\alpha\colon L^p \longrightarrow L^p$ is a bounded operator.

From our main results, we also have:

Corollary 3. *Suppose $\mu \in (GC)$ and ν is any measure on $\mathbb{R}^n$. If $f \in L^1(\nu), W \in L^{s'}(\mu)$, and $K_{\rho,\gamma} \in L^s(\mu)$, then*

$$W\cdot T_{\rho,\gamma}\colon L^1(\nu) \longrightarrow L^1(\mu), \tag{53}$$

is a bounded operator. That is, for every $s \in [1,\infty]$, there exists $C_s > 0$ such that

$$\left\|W\cdot T_{\rho,\gamma}f\colon L^1(\mu)\right\| \le C_s\left\|W\colon L^{s'}(\mu)\right\|\cdot\left\|K_{\rho,\gamma}\colon L^s(\mu)\right\|\cdot\left\|f\colon L^1(\nu)\right\|. \tag{54}$$

Corollary 4. *Suppose $\mu \in (GC)$ and ν any measure on $\mathbb{R}^n$ such that*

$$\nu(B) \le C^*\mu(B), \tag{55}$$

for some $C^* > 0$ and for every open balls B on $\mathbb{R}^n$.

If $f \in L^s(\nu), W \in L^{s'}(\mu)$, and $K_{\rho,\gamma} \in L^1(\mu)$, then

$$W\cdot T_{\rho,\gamma}\colon L^s(\nu) \longrightarrow L^1(\mu), \tag{56}$$

is a bounded operator. That is, for every $s \in [1,\infty]$, there exists $C_s > 0$ such that

$$\left\|W\cdot T_{\rho,\gamma}f\colon L^1(\mu)\right\| \le C\left\|K_{\rho,\gamma}\colon L^1(\mu)\right\|\cdot\left\|W\colon L^{s'}(\mu)\right\|\cdot\left\|f\colon L^s(\nu)\right\|. \tag{57}$$

Next, this will be served as our last corollary.

Corollary 5. *Suppose $\mu \in (GC)$ and ν any measure on $\mathbb{R}^n$ such that*

$$\nu(B) \le C^*\mu(B), \tag{58}$$

for some $C^* > 0$ and for every open balls B on $\mathbb{R}^n$.

If $f \in L^p(\nu), W \in L^{q'}(\mu)$, and $K_{\rho,\gamma} \in L^q(\mu)$, then

$$W\cdot T_{\rho,\gamma}\colon L^p(\nu) \longrightarrow L^p(\mu), \quad \frac{1}{s} + \frac{1}{q'} = \frac{1}{p}, \tag{59}$$

is a bounded operator. That is, there exists $C > 0$ such that

$$\left\|W\cdot T_{\rho,\gamma}f\colon L^p(\mu)\right\| \le C\left\|K_{\rho,\gamma}\colon L^q(\mu)\right\|\cdot\left\|W\colon L^{q'}(\mu)\right\|\cdot\left\|f\colon L^p(\nu)\right\|. \tag{60}$$

5. Conclusions

From the result of this study, we have seen the boundedness of the generalized Bessel–Riesz operators in Lebesgue spaces, in the framework of different measures (Theorem 3.1, Theorem 5) by using, as tools, an estimate of the Lebesgue norm of the Bessel–Riesz kernel (Theorem 2), the Young inequality (Theorem 3), and functions with doubling conditions. In the future, we shall continue this study to prove the boundedness of generalized Bessel–Riesz operators on Morrey spaces and generalized Morrey spaces.

Data Availability

The data used to support the findings of this study are currently under embargo while the research findings are commercialized. Requests for data, after the publication of this article, will be considered by the corresponding author.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Vol. 2, Princeton university press, New Jersey, United States, 1970.

[2] G. H. Hardy and J. E. Littlewood, "Some properties of fractional integrals. I," *Mathematische Zeitschrift*, vol. 27, no. 1, pp. 565–606, 1928.

[3] G. H. HardyHardy and J. E. Littlewood, "Some properties of fractional integrals. II," *Mathematische Zeitschrift*, vol. 34, no. 1, pp. 403–439, 1932.

[4] S. Sobolev, "On a theorem of functional analysis," *Eleven Papers on Analysis*, vol. 34, pp. 39–68, 1963.

[5] V. Kokilashvili and M. Alexander, "Fractional integrals on measure spaces," *Fractional calculus and Applied Analysis*, vol. 4, pp. 1–24, 2001.

[6] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, "Bounded and Compact Integral Operators," *Math. Appl.",* vol. 543, p. 3440, 2002.

[7] V. Kokilashvili, "Weighted estimates for classical integral operators," *Nonlinear Analysis, Function Spaces and Applications*, Vieweg+ Teubner Verlag, vol. 4, pp. 86–103, Wiesbaden, 1990.

[8] J. García-CuervaGarcía-Cuerva and A. E. Gatto, "Boundedness properties of fractional integral operators associated to non-doubling measures," *Studia Mathematica*, vol. 162, no. 3, pp. 245–261, 2004.

[9] J. Peetre, "On the theory of *Lp,λ* spaces," *Journal of Functional Analysis*, vol. 4, no. 1, pp. 71–87, 1969.

[10] D. R. Adams, "A note on Riesz potentials," *Duke Mathematical Journal*, vol. 42, no. 4, pp. 765–778, 1975.

[11] Y. Sawano and H. Tanaka, "Morrey Spaces for non-doubling measures," *Acta Mathematica Sinica, English Series*, vol. 21, no. 6, pp. 1535–1544, 2005.

[12] A. Eridani, V. Kokilashvili, and A. Meskhi, "Morrey spaces and fractional integral operators," *Expositiones Mathematicae*, vol. 27, no. 3, pp. 227–239, 2009.

[13] L. Grafakos, "Classical Fourier Analysis," *Graduate Texts in Mathematics*, Vol. 2, Springer, New York, 2008.

[14] M. Idris, H. Gunawan, J. Lindiarni, and Eridani, "The boundedness of Bessel-Riesz operators on Morrey spaces," *AIP Conference Proceedings*, vol. 1729, no. 1, p. 020006, 2016.

[15] F. Chiarenza, "Morrey spaces and Hardy-Littlewood maximal function," *Rend. Mat. Appl.*vol. 3, no. 7, pp. 273–279, 1987.

[16] K. Kurata, S. Nishigaki, and S. Sugano, "Boundedness of integral operators on generalized Morrey spaces and its application to Schrodinger operators," *Proceedings of the American Mathematical Society*, vol. 128, no. 4, pp. 1125–1134, 2000.

[17] S. Mehmood, Eridani, and Fatmawati, "Morrey spaces and boundedness of Bessel-Riesz operators," *AIP Conference Proceedings*, vol. 2329, no. 1, Article ID 030001, 2021.

[18] V. I. Burenkov and H. V. Guliyev, "Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces," *Studia Mathematica*, vol. 163, no. 2, pp. 157–176, 2004.

[19] V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, "Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces," *Journal of Computational and Applied Mathematics*, vol. 208, no. 1, pp. 280–301, 2007.

[20] F. Jones, *Lebesgue Integration on Euclidean Space*, Jones and Bartlett Learning, Massachusetts, United States, 2001.

[21] G. H. Eridani and E. Nakai, "On generalized fractional integral operators," *Scientiae Mathematicae Japonicae*, vol. 60, no. 3, pp. 539–550, 2004.