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Research Article

Daubechies Wavelet Scaling Function Approach to Solve Volterra's Population Model

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In this paper, we focus on a collocation approach based on Daubechies wavelet scaling functions for approximating the solution of Volterra's model of population growth of a species with a closed system. We present that the integral and derivative terms, which appear in Volterra's model of the population, will be computed exactly in dyadic points. Utilizing this collocation technique, Volterra's population model reduces into a system of nonlinear algebraic equations. In addition, an error bound for our method will be explored. The numerical results demonstrate the applicability and accuracy of our method.

1. Introduction

Daubechies wavelet scaling functions have become a popular method for applications in signal analysis and approximation[1–6]. The Daubechies wavelet and their scaling functions have some features such as orthogonality, vanishing moment, and compact support, and applying them in numerical analysis makes sense. The main issue with Daubechies scaling functions is the lack of closed form, and their value is not known except in dyadic points. Despite this issue, because of their refinement relation, we are able to compute their derivation and integral in dyadic intervals accurately [7]. Adding this feature to their previously mentioned remarkable properties makes the result obtained with them reliable and more accurate than other methods.

Volterra's population model for population growth of a species within a closed system is given [8, 9].

$$\frac{\mathrm{d}p}{\mathrm{d}\tilde{t}} = ap - bp^2 - cp\int_0^{\tilde{t}} p(x)\mathrm{d}x, \quad p(0) = p_0, \tag{1}$$

where positive constants a, b and c are the birthday rate, the crowding coefficient, and the toxicity coefficient, respectively. p_0 is the initial population and $p(\tilde{t})$ denotes the population time $\tilde{t}$. The additional integral term indicates the accumulated toxicity of the species. Dimensionless variables are introduced as follows by applying the scale and time population:

$$t = \frac{\tilde{t}c}{b}, \qquad u = \frac{pb}{a}, \tag{2}$$

to obtain the nondimensional problem,

$$\kappa\frac{\mathrm{d}u}{\mathrm{d}t} = u - u^2 - u\int_0^t u(x)\mathrm{d}x, \quad u(0) = u_0, \tag{3}$$

where $u(t)$ is the scaled population of identical individuals at the time t. The trivial solution for equation (3) is $u(t) = 0$, and the nontrial analytical solution given in [10] shows that $u(t) > 0$ for all t if $u_0 > 0$.

Several numerical and analytical approximations for solving equation (3) have been reported such as the successive approximation method [8], singular perturbation method [9], Euler and Runge Kutta methods [10]. In [11], a domian decomposition method and the sinc-Galerkin method were compared, and in [12], the series solution method, decomposition method, and Pade approximation were compared. The spectral methods were also applied to solve equation (3) in [13-15]. In [16], the rational Chebyshev

functions and the Hermite function collocation approach are used to solve equation (3). Equation (3) in [16]is converted to a non-linear ODE. In most of these methods (see [11, 17]), for computing the integral in (3), quadrature approximation methods are used, but in our method, we compute it exactly.

This paper is organized as follows: in Section 2, multiresolution analysis and scaling functions of the Daubechies wavelet are introduced. In Section 3, computations of the integral and the derivation of the scaling functions of the Daubechies wavelet are illustrated. In Section 4, the application of these scaling functions to construct a numerical method for solving (2) is extended, and error analysis is investigated. Finally, in Section 5, the numerical results and the conclusion are presented.

2. Multiresolution Analysis and Daubechies Wavelets

Multiresolution analysis is applied to represent a given function f into different scale components [4, 5]. In order to do this, the basis function ϕ that can be scaled to give multiple resolutions of the original function is introduced. In order to achieve representation of functions in the Hilbert space $L^2(\mathbb{R})$ at different levels of resolution, a nested sequence of closed subspaces V_j was sought as shown in the following equation:

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z}, \tag{4}$$

with the following properties [2, 5]:

(1). $\overline{\cup_{j\in\mathbb{Z}} V_j} = L_2(\mathbb{R})$
(2). $\cap_{j\in\mathbb{Z}V_j} = \{0\}$
(3). $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$, $j \in \mathbb{Z}$
(4). $f(x) \in V_0 \Leftrightarrow f(x-k) \in V_0$, $k \in \mathbb{Z}$
(5). There exists $\phi \in V_0$ in which the set $\{\phi(x-k)$: $k \in \mathbb{Z}\}$ is an orthonormal basis of V_0

Then, these properties $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ form an orthonormal basis for the space V_j. For each j, since V_j is a proper subspace of V_{j+1}, there is some space in V_{j+1} called W_j, which when combined with V_j gives us V_{j+1}. This space W_j is called the *wavelet subspace* and is orthogonal complementary to V_j in V_{j+1}. In other words,

$$\begin{aligned} V_j \cap W_j &= \{0\}, \quad j \in \mathbb{Z}, \\ V_j \perp W_j, &\quad j \in \mathbb{Z}, \end{aligned} \tag{5}$$

and so

$$V_{j+1} = V_j \oplus W_j, \tag{6}$$

where $\oplus$ represents a *direct sum*. It follows, then, we have

$$L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j. \tag{7}$$

Now, let us introduce a wavelet function $\psi(x)$ that $\psi(x-k)$ forms an orthonormal basis of the subspace W_0, i.e., $W_0 = \text{span}\{\psi(x-k)$: $k \in \mathbb{Z}\}$. Then, we obtain

$$\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x - k\right), \quad j, k \in \mathbb{Z}, \tag{8}$$

which is an orthonormal basis for W_j. Furthermore, since W_0 and V_0 are contained in the space V_1, we can express the scaling function $\phi(x)$ and the wavelet function $\psi(x)$ in terms of the scaling function at the next higher scale as shown in the following equations:

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x-k), \tag{9}$$

$$\psi(x) = \sum_{k\in\mathbb{Z}} g_k \phi(2x-k). \tag{10}$$

Equation (9) is referred to as the *dilation equation*, and the parameters h_k are known as *filter coefficients*. Equation (10) is referred to as the *wavelet equation,* and the parameters g_k are known as *wavelet parameters*.

2.1. Daubechies Scaling Coefficients. In order to determine scaling coefficients, Daubechies imposed some useful conditions as follows:

(1) A scaling function must be able to exactly represent polynomials of order smaller than p.
(2) $\int_0^{2p-1} \phi(x)\mathrm{d}x = 1$.
(3) Orthogonality of a scaling function to its integer translates and orthogonality of a scaling function to the Daubechies wavelet are shown in the following equations:

$$\begin{aligned} \int_0^{2p-1} \phi(x-k)\phi(x-l)\mathrm{d}x &= \delta_{k,l} \quad l, k \in \mathbb{Z}, \\ \int_0^{2p-1} \phi(x)\psi(x-l)\mathrm{d}x &= 0, \quad l \in \mathbb{Z}. \end{aligned} \tag{11}$$

These conditions conclude the following results

$$\sum_{k=0}^{D-1} (-1)^k k^m h_k = 0, \quad m = 0, 1, \ldots, p-1, \tag{12}$$

$$\sum_{k=0}^{2p-1} h_k = 2, \tag{13}$$

$$\sum_{k=0}^{D-1} h_k h_{2m+k} = 2\delta_{0,m}, \quad k = 1-p, \ldots, p-1, \tag{14}$$

$$h_k = 0, \quad \forall k \notin \{0, 1, \ldots, 2p-1\}. \tag{15}$$

Imposing equations (12-15) to filter coefficients enables us to compute $\{h_k\}_{k=0}^{2p-1}$, so dilation equation (9) for the Daubechies scaling function will be as follows:

$$\phi(x) = \sum_{k=0}^{2p-1} h_k \phi(2x-k). \tag{16}$$

In [18], the exact values of h_k are evaluated. In [19], it is proved that for a fixed p, there exists only one linear

independent scaling function ϕ which satisfies the dilation equation with these filter coefficients. The support of ϕ lies in $[0, 2p-1]$, and if we denote $D = 2p$, then the parameter D is called the wavelet genus or Daubechies' number [1, 6]. The relation between h_k and g_k is shown in the following equation [5]:

$$g_k = (-1)^k h_{D-k-1}. \tag{17}$$

In Table 1, filter coefficients of the Daubechies scaling function with p vanishing moments have been shown.

2.2. Computing Daubechies Scaling Functions and Their Derivative at Integers and Dyadic Points. Generally, scaling functions of the Daubechies wavelet do not have a closed form. Instead, they have to be generated recursively from dilation equation (10). The scaling function ϕ has support for the interval $[0, D-1]$. Putting $x = 0, \ldots, D-1$ in (10) yields a homogeneous linear system of equations. For the Daubechies scaling functions, we have $\phi(0) = 0$ and $\phi(D-1) = 0$, and so, we will discard $\phi(D-1)$ in our computations, but for reasons which are explained below, we keep $\phi(0)$. In a matrix form, it becomes

$$\begin{bmatrix} h_0 & 0 & 0 & \ldots & 0 & 0 \\ h_2 & h_1 & h_0 & \ldots & 0 & 0 \\ h_4 & h_3 & h_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & h_{D-3} & h_{D-4} \\ 0 & 0 & 0 & \ldots & h_{D-1} & h_{D-2} \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \ldots \\ \phi(D-3) \\ \phi(D-2) \end{bmatrix} = \begin{bmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \ldots \\ \phi(D-3) \\ \phi(D-2) \end{bmatrix},$$
$$M\phi = \phi. \tag{18}$$

Thus, the vector of integer values of the scaling function ϕ is the eigenvector of M corresponding to the eigenvalue 1. As in all eigenvalue problems, the solution to the following system:

$$(M - I)\phi = 0, \tag{19}$$

for obtaining ϕ, the solution is not unique. In order to obtain a unique solution, the normalizing condition is as follows:

$$\sum_{k=0}^{D-1} \phi(k) = 1, \tag{20}$$

which will be imposed on (19) [4]. So, by considering $\phi(0) = \phi(D-1) = 0$, from (20),we have

$$\sum_{k=1}^{D-2} \phi(k) = 1. \tag{21}$$

Therefore, from (20) and (21), we obtain the following equation:

$$M_{\text{new}}\phi_{\text{new}} = \mathbf{e}_1, \tag{22}$$

where M_{new} is

Table 1: Filter coefficients of the Daubechies scaling function.

k	h_k
	$p = 2$
0	0.6830127018922193
1	1.1830127018922192
2	0.3169872981077807
3	−0.1830127018922192
	$p = 3$
0	0.4704672077841636
1	1.1411169158314436
2	0.6503650005262323
3	−0.1909344155683274
4	−0.1208322083103963
5	0.0498174997368837
	$p = 4$
0	0.3258034280512982
1	1.0109457150918286
2	0.8922001382609015
3	−0.0395750262356447
4	−0.2645071673690398
5	0.0436163004741772
6	0.0465036010709818
7	−0.0149869893303614
	$p = 5$
0	0.2264189825835584
1	0.8539435427050283
2	1.0243269442591967
3	0.1957669613478087
4	−0.3426567153829353
5	−0.0456011318835469
6	0.1097026586421339
7	−0.0088268001083583
8	−0.0177918701019542
9	0.0047174279390679
	$p = 6$
0	0.1577424320027466
1	0.6995038140774233
2	1.0622637598801890
3	0.4458313229311702
4	−0.3199865989409983
5	−0.1835180641065938
6	0.1378880929785304
7	0.0389232097078970
8	−0.0446637483054601
9	0.0007832511506546
10	0.0067560623615907
11	−0.0015235338263795
	$p = 7$
0	0.1100994307456160
1	0.5607912836254882
2	1.0311484916361415
3	0.6643724822110735
4	−0.2035138224626306
5	−0.3168350112806179
6	0.1008464650093839
7	0.1140034451597351
8	−0.0537824525896852
9	−0.0234399415642046
10	0.0177497923793598
11	0.0006075149954022
12	−0.0025479047181871
13	0.0005002268531225

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ h_2 & h_1 - 1 & h_0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & h_{D-3} - 1 & h_{D-4} \\ 0 & 0 & 0 & \dots & h_{D-1} & h_{D-2} - 1 \end{bmatrix},$$
$$\phi_{\text{new}} = \begin{bmatrix} \phi(0) \\ \phi(1) \\ \dots \\ \phi(D-3) \\ \phi(D-2) \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \\ 0 \end{bmatrix}. \tag{23}$$

The values of the scaling function at the integers are given by the solution to equation (22). Now, we can use (16) to obtain the scaling function at all midpoints between integers in the interval, and the value of the Daubechies scaling function in all dyadic points will be determined. The complete explanation of this procedure is named the cascade algorithm and is introduced in [6, 17].

We applied the same procedure as the cascade algorithm for evaluating the scaling function's derivatives at dyadic points. Differentiating equation (16) m times yields

$$\phi^{(m)}(x) = 2^m \sum_{k=0}^{N-1} h_k \phi^{(m)}(2x - k), \tag{24}$$

and by substituting all integer values of the interval $[0, D-1]$, equation (24) gives a linear homogeneous system. The coefficient matrix of this homogeneous system is singular. Thus, a normalization condition is required, in order to determine a unique solution. The following important additional property of the Daubechies scaling function ϕ can be used [20]:

$$\sum_k k^m \phi(x-k) = x^m + \sum_{k=1}^{m} (-1)^k \frac{m!}{(m-k)!k!} x^{m-k} \cdot \int_{-\infty}^{\infty} \phi(z) z^k \mathrm{d}z, \tag{25}$$

where m is a positive integer number. By differentiating m times, the previous equation yields

$$\sum_k k^m \phi^{(m)}(x-k) = m!. \tag{26}$$

Using this normalizing equation (26) and equation (24) for $x = 1, 2, \dots, D-1$ yields an inhomogeneous system, which can be solved, and derivatives can be evaluated at the integer values of x. Similar to the cascade algorithm, using the dilation equation once again, the values of $\phi^m(x)$ at $x = i2^{(-n)}$ for $i = 1, 3, 5, \dots, (D-1)2^n - 1$ can be determined.

3. The Integrals of the Daubechies Scaling Function on Dyadic Intervals

As previously mentioned, the support of the Daubechies scaling function with p vanishing moments is $[0, D-1]$ where $D = 2p$. In this section, first, the integral of the Daubechies scaling function on integer intervals is computed exactly. Then, we use a recursion relation, in order to obtain the integral of the Daubechies scaling function, on dyadic intervals, i.e, on the intervals $[k2^{-n}, (k+1)2^{-n}]$, in arbitrary resolution j will be computed [3].

3.1. The Integrals of the Daubechies Scaling Function on Integer Intervals. We define $A_{[a,b]}$, which is as follows:

$$A_{[a,b]} = \int_a^b \phi(x)\mathrm{d}x, \tag{27}$$

and using dilation equation (16) to the scaling function ϕ, then the following result is obtained:

$$\begin{aligned} A_{[0,i]} &= \int_0^i \sum_{k=0}^{D-1} h_k \phi(2x-k)\mathrm{d}x, \quad i > 0 \\ &= \frac{1}{2} \sum_{k=0}^{D-1} h_k \int_{-k}^{2i-k} \phi(t)\mathrm{d}t \\ &= \frac{1}{2} \sum_{k=0}^{D-1} h_k A_{[0,2i-k]}, \end{aligned} \tag{28}$$

in other words,

$$A_{[0,i]} = \frac{1}{2} \sum_{k=0}^{D-1} h_k A_{[0,2i-k]}. \tag{29}$$

From equation (17), it is clear that

$$A_{[0,i]} = 1, \quad i \geq D-1, \tag{30}$$

$$A_{[0,i]} = 0, \quad i \leq 0. \tag{31}$$

Equations (29)–(31) can be written as a sparse linear system of equations $C\mathbf{x} = \mathbf{b}$. The entries of matrices C, $\mathbf{x}$ and $\mathbf{b}$ are given as

$$\begin{aligned} C_{i,j} &= \begin{cases} cl2 - h_i, & \text{for \$}i = j\text{\$}, \\ -h_{2i-j}, & \text{for \$}i \neq j\text{\$}, \end{cases} \\ b_i &= \begin{cases} cl0, & i \leq p-1, \\ \sum_{k=0}^{2(i-p)+1} h_k, & i \geq p, \end{cases} \\ \mathbf{x}_i &= \mathbf{A}_{[0,i]}, \quad 1 \leq i \leq D-2. \end{aligned} \tag{32}$$

It is obvious that the matrix of coefficients in these linear systems is sparse and also diagonally dominant, so is nonsingular. For example, the linear system of equations C when $D = 4$ and $D = 6$ is given as follows:

$$D = 4$$

$$\begin{bmatrix} cc2 - h_1 & -h_0 \\ -h_3 & 2 - h_2 \end{bmatrix} \begin{bmatrix} A_{[0,1]} \\ A_{[0,2]} \end{bmatrix} = \begin{bmatrix} 0 \\ h_0 + h_1 \end{bmatrix}. \quad (33)$$

$$D = 4$$

$$\begin{bmatrix} cccc2 - h_1 & -h_0 & 0 & 0 \\ -h_3 & 2 - h_2 & -h_1 & -h_0 \\ -h_5 & -h_4 & 2 - h_3 & -h_2 \\ 0 & 0 & -h_5 & 2 - h_4 \end{bmatrix} \begin{bmatrix} A_{[0,1]} \\ A_{[0,2]} \\ A_{[0,3]} \\ A_{[0,4]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h_0 + h_1 \\ \sum_{i=0}^{3} h_i \end{bmatrix}. \quad (34)$$

In this case, $D = 4$, the answers are

$$A_{[0,1]} = 0.8496793685588857,$$
$$A_{[0,2]} = 1.016346035225553. \quad (35)$$

3.2. Integrals of the Daubechies Scaling Function on Dyadic Intervals. In the previous subsection, the values of $A_{[a,b]}$, when a and b are integers, have been computed. Now, by using dilation equation (10), a recursion relation for $j \in \mathbb{N}$ and $0 \le l \le (D-1)2^j - 1$ will be introduced as

$$\begin{aligned} A_{[l2^{-j},(l+1)2^{-j}]} &= \int_{l2^{-j}}^{(l+1)2^{-j}} \phi(x)\mathrm{d}x \\ &= \int_{l2^{-j}}^{(l+1)2^{-j}} \sum_{k=0}^{D-1} h_k \phi(2x-k)\mathrm{d}x \\ &= \sum_{k=0}^{D-1} \frac{1}{2} h_k \int_{l2^{1-j}-k}^{(l+1)2^{1-j}-k} \phi(t)\mathrm{d}t \\ &= \frac{1}{2} \sum_{k=0}^{D-1} h_k A_{[l2^{1-j}-k,(l+1)2^{1-j}-k]}. \end{aligned} \quad (36)$$

By using this recursion relation, the values of $A_{[a,b]}$, when a and b are dyadic points, are computed. In Table 2, the values of $A_{[a,b]}$ of the Daubechies scaling function, when $D = 4$ and a and b are dyadic points, are given.

4. Solution of Volterra's Population Model by the Scaling Function of the Daubechies Wavelet

In this section, we propose our method to approximate the solution of equation (3). Let U be the approximate solution of u as

$$u(x) \cong U_j(x) = \sum_{k=k_1}^{k_2} a_{j,k}\phi_{j,k}(x). \quad (37)$$

As we know, Volterra's population model is defined on the interval $[0, +\infty)$ and decays rapidly [8]. The support of Daubechies scaling functions with N vanishing moment is $[0, D-1]$, where $D = 2N$, so we let k_1 and k_2 be such that, in which the last point in support of ϕ_{j,k_1} is smaller than 0 and the first point in support of ϕ_{j,k_2} is bigger than $D-1$; therefore, we have

$$k_1 = 2 - D,$$
$$k_2 = 2^j(D-1) - 1. \quad (38)$$

By integrating equation (37), we can get the following result:

$$\int_0^t U_j(x)\mathrm{d}x = \sum_{k=k_1}^{k_2} a_{j,k} \int_0^t \phi_{j,k}(x)\mathrm{d}x. \quad (39)$$

Therefore, suppose that

$$\int_0^t U(x)\mathrm{d}x = \sum_{k=k_1}^{k_2} a_{j,k} H_{j,k}(t), \quad (40)$$

where

$$H_{j,k}(t) = 2^{j/2} \int_0^t \phi\left(2^j x - k\right)\mathrm{d}x,$$
$$H_{j,k}(t) = 2^{-(j/2)} A_{[-k,2^j t-k]}. \quad (41)$$

In order to approximate $u\prime(t)$ in equation (3) by differentiating equation (37), we have

$$U_j'(x) = \sum_{k=k_1}^{k_2} a_{j,k}\phi_{j,k}'(x). \quad (42)$$

By substituting equations (37)–(42), in equation (3), we construct the residual as

$$R(t) = \kappa \frac{\mathrm{d}U_j(t)}{\mathrm{d}t} - U_j(t) + U_j^2(t) + U_j(t) \int_0^t U_j(x)\mathrm{d}x,$$
$$U_j(0) = u_0, \quad (43)$$

and by substituting equation (40) in equation (43), we have

$$R(t) = \sum_{k=k_1}^{k_2} a_{j,k} \left[\kappa\phi_{j,k}'(t) - \phi_{j,k}(t) + \phi_{j,k}(t) \sum_{k=k_1}^{k_2} a_{j,k}\phi_{j,k}(t) + \phi_{j,k}(t) \sum_{k=k_1}^{k_2} a_{j,k} H_{j,k}(t) \right]. \quad (44)$$

Table 2: The values $A_{[a,b]}$ for $D = 4$ and $j = 2$.

$A_{[(0)(1/4)]} = 9.9095205 \times 10^{-2}$	$A_{[(1)(5/4)]} = 1.7456594 \times 10^{-1}$	$A_{[(2)(9/4)]} = -2.3661145 \times 10^{-2}$
$A_{[(1/4)(2/4)]} = 1.9107569 \times 10^{-1}$	$A_{[(5/4)(6/4)]} = 5.3104960 \times 10^{-2}$	$A_{[(9/4)(10/4)]} = 5.8193442 \times 10^{-3}$
$A_{[(2/4)(3/4)]} = 2.4938899 \times 10^{-1}$	$A_{[(6/4)(7/4)]} = -1.0216272 \times 10^{-3}$	$A_{[(10/4)(11/4)]} = 1.6326381 \times 10^{-3}$
$A_{[(3/4)(1)]} = 3.1011948 \times 10^{-1}$	$A_{[(7/4)(2)]} = -5.9982607 \times 10^{-2}$	$A_{[(11/4)(3)]} = -1.3687209 \times 10^{-4}$

In the above equation for obtaining the coefficients $a_{j,k}$, let

$$R(t_j) = 0, \quad \text{where } j = 1, 2 \ldots, k_2 - k_1, \tag{45}$$

and also, the initial condition is

$$\sum_{k=k_1}^{k_2} a_{j,k}\phi_{j,k}(0) = u_0, \tag{46}$$

in which the collocation points t_j are dyadic points on $[0, D-1]$. The equations. (45) and (46) are a system of nonlinear $k_2 - k_1 + 1$ equation, which can be solved by Newton's iteration method.

5. Error Analysis

Let $\mathcal{N}$ be the nonlinear operator, then we have

$$\mathcal{N}(u) = \kappa u'(t) - u(t) + u^2(t) + u(t)\int_0^t u(x)\mathrm{d}x, \tag{47}$$

and it is obvious that $\mathcal{N}(u) = 0$ when $u(t)$ is the exact solution of (47). Let $U_j(t) = \sum_{k=k_1}^{k_2} a_{j,k}\phi_{j,k}$, where j is the resolution of the Daubechies wavelet, then the error of our method is

$$E_j = \left\|\mathcal{N}(U_j)\right\|. \tag{48}$$

Let $U_j(t)$ be the approximate solution of (3) given by (37), and the error of our method is given in (48). Then, the error is

$$E_j = O\left(2^{-j}\right). \tag{49}$$

Proof. From equation (47) and $\mathcal{N}(u) = 0$, we have

$$\begin{aligned} E_j &= \left\|\mathcal{N}(U_j)\right\| \\ &= \left\|\mathcal{N}(U_j) - \mathcal{N}(u)\right\| \\ &= \left\|\kappa\left(u\prime - U_j'\right) + \left(u - U_j\right) + \left(u^2 - U_j^2\right) + \left(u(t)\int_0^t u(x)\mathrm{d}x - U_j(t)\int_0^t U_j(x)\mathrm{d}x\right)\right\|. \end{aligned} \tag{50}$$

Therefore, we have

$$\begin{aligned} E_j \le{}& \left\|\kappa\left(u' - U_j'\right)\right\| + \left\|u - U_j\right\| + \left\|u - U_j\right\|\left\|u + U_j\right\| \\ &+ \left\|u(t)\int_0^t u(x)\mathrm{d}x - U_j(t)\int_0^t U_j(x)\mathrm{d}x\right\|, \end{aligned} \tag{51}$$

and by using the bound of errors as $\|u(t) - U_j(t)\| \le C_1 2^{-j}$ and $\|u\prime(t) - U_j'(t)\| \le C_2 2^{-j}$, which are given in [21] and also C_1 and C_2 are constants, we have

$$\begin{aligned} E_j \le{}& \kappa C_2 2^{-j} + C_1 2^{-j} + 2C_3C_1 2^{-j} \\ &+ \left\|u(t)\int_0^t u(x)\mathrm{d}x - U_{N,j}(t)\int_0^t U_j(x)\mathrm{d}x\right\|. \end{aligned} \tag{52}$$

Now, let $U_j(t) = u(t) + \varepsilon(t)$, then the above inequality can be written as

$$\begin{aligned} E_j \le{}& (C_1 + \kappa C_2 + 2C_3C_1)2^{-j} + \left\|\varepsilon(t)\int_0^t \varepsilon(x)\mathrm{d}x\right\| \\ &+ \left\|u(t)\int_0^t \varepsilon(x)\mathrm{d}x\right\| + \left\|\varepsilon(t)\int_0^t u(x)\mathrm{d}x\right\|, \end{aligned} \tag{53}$$

and as we know $\|\varepsilon(t)\| \le C_1 2^{-j}$, $t \le (L-1)$ and if we let $C = C_1 + \kappa C_2 + 2C_3C_1$, then, we have

$$E_j \le C2^{-j} + \left((L-1)C_1^2\right)2^{-2j} + 2C_3C_1(L-1)2^{-j}, \tag{54}$$

hence

$$E_j = O\left(2^{-j}\right). \tag{55}$$

This theorem indicates that the error of our method is bounded, and when j increases, then the error of our method tends to zero, in the other words, $E_j = O(2^{-j})$, and the numerical results in Table 3 are consistent with this upper bound. □

6. Numerical Results

In this section, we use the Daubechies scaling function with 4 vanishing moments, and the computational results are summarized in Table 3. In Figure 1, the results showed for $\kappa = 0.02, 0.1, 0.2$ and 0.5. This figure shows the rapid rise along the logistic curve followed by the slow exponential decay after reaching the maximum point, and when t increases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases. All computational efforts in this work have been carried out using Maple software in 30 decimal digits.

Table 3: A comparison between our method at the dyadic point with the exact value.

κ	Exact $u_{\max}$	Dyadic $u_{\max}$ in our method		
		$j = 6$	$j = 7$	$j = 8$
0.02	0.92342717	0.95427522	0.92252698	0.92332478
0.1	0.76974149	0.76921312	0.76974563	0.76974145
0.2	0.65905038	0.65907693	0.65905164	0.65905037
0.5	0.48519030	0.48519131	0.48519030	0.48519030

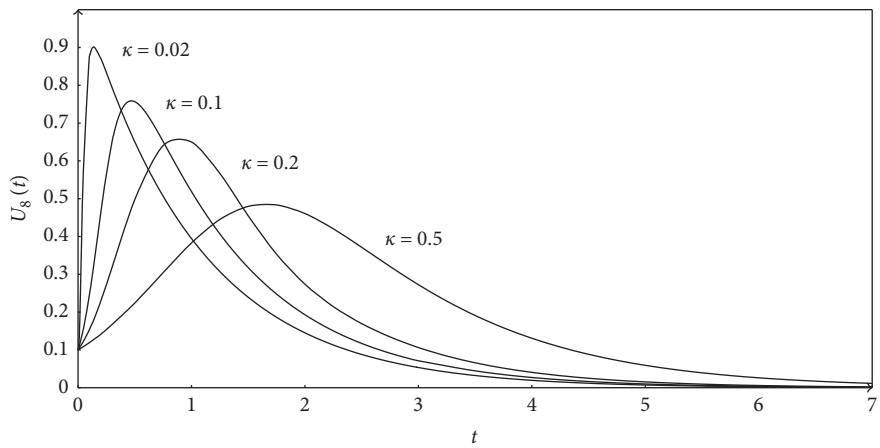

Figure 1: The result of Daubechies wavelet scaling functions for $\kappa = 0.02, 0.1, 0.2, 0.5$.

7. Conclusion

In this paper, we have constructed an efficient and accurate numerical method based on Daubechies wavelet scaling functions for the solution of Volterra's population model. Our method was used in a direct way without using linearization, perturbation, or restrictive assumption. The refinement equation in the Daubechies wavelet scaling function has enabled us exactly to compute the integral and the derivative of unknown functions in 2. Essential data have been saved, and this fact strongly has reduced time-consuming computations and gives us the advantage of requiring a shorter time to run the procedure in more accurate resolutions. We have shown that our method is simple. The only error that affected the results was computations in the Newton method for solving a system of nonlinear equations. In the error analysis, we showed that when finer resolutions applied for big j, the error decreases, and numerical results satisfy this result. The stability and convergence of Daubechies wavelet scaling functions make this approach very attractive for computation at the dyadic points. Considering this exact computation and helpful characteristics of Daubechies scaling functions, promising results were obtained.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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