Research Article

Daubechies Wavelet Scaling Function Approach to Solve Volterra’s Population Model

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Received 25 November 2021; Accepted 13 January 2022; Published 19 September 2022

Academic Editor: Marianna A. Shubov

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In this paper, we focus on a collocation approach based on Daubechies wavelet scaling functions for approximating the solution of Volterra’s model of population growth of a species with a closed system. We present that the integral and derivative terms, which appear in Volterra’s model of the population, will be computed exactly in dyadic points. Utilizing this collocation technique, Volterra’s population model reduces into a system of nonlinear algebraic equations. In addition, an error bound for our method will be explored. The numerical results demonstrate the applicability and accuracy of our method.

1. Introduction

Daubechies wavelet scaling functions have become a popular method for applications in signal analysis and approximation [1–6]. The Daubechies wavelet and their scaling functions have some features such as orthogonality, vanishing moment, and compact support, and applying them in numerical analysis makes sense. The main issue with Daubechies scaling functions is the lack of closed form, and their value is not known except in dyadic points. Despite this issue, because of their refinement relation, we are able to compute their derivation and integral in dyadic intervals accurately [7]. Adding this feature to their previously mentioned remarkable properties makes the result obtained with them reliable and more accurate than other methods.

Volterra’s population model for population growth of a species within a closed system is given [8, 9].

\[
\frac{dp}{dt} = ap - bp^2 - cp \int_0^t p(x)dx, \quad p(0) = p_0,
\]

where positive constants \(a, b\) and \(c\) are the birthday rate, the crowding coefficient, and the toxicity coefficient, respectively. \(p_0\) is the initial population and \(p(t)\) denotes the population time \(t\). The additional integral term indicates the accumulated toxicity of the species. Dimensionless variables are introduced as follows by applying the scale and time population:

\[
t = \frac{tc}{b},
\]

\[
u = \frac{pb}{a},
\]

to obtain the nondimensional problem,

\[
\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx, \quad u(0) = u_0,
\]

where \(u(t)\) is the scaled population of identical individuals at the time \(t\). The trivial solution for equation (3) is \(u(t) = 0\), and the nontrivial analytical solution given in [10] shows that \(u(t) > 0\) for all \(t\) if \(u_0 > 0\).

Several numerical and analytical approximations for solving equation (3) have been reported such as the successive approximation method [8], singular perturbation method [9], Euler and Runge Kutta methods [10]. In [11], a domain decomposition method and the sinc-Galerkin method were compared, and in [12], the series solution method, decomposition method, and Pade approximation were compared. The spectral methods were also applied to solve equation (3) in [13–15]. In [16], the rational Chebyshev methods...
functions and the Hermite function collocation approach are used to solve equation (3). Equation (3) in [16] is converted to a non-linear ODE. In most of these methods (see [11, 17]), for computing the integral in (3), quadrature approximation methods are used, but in our method, we compute it exactly.

This paper is organized as follows: in Section 2, multi-resolution analysis and scaling functions of the Daubechies wavelet are introduced. In Section 3, computations of the integral and the derivation of the scaling functions of the Daubechies wavelet are illustrated. In Section 4, the application of these scaling functions to construct a numerical method for solving (2) is extended, and error analysis is investigated. Finally, in Section 5, the numerical results and the conclusion are presented.

2. Multiresolution Analysis and Daubechies Wavelets

Multiresolution analysis is applied to represent a given function \( f \) into different scale components [4, 5]. In order to do this, the basis function \( \phi \) that can be scaled to give multiple resolutions of the original function is introduced. In order to achieve representation of functions in the Hilbert space \( L^2(\mathbb{R}) \) at different levels of resolution, a nested sequence of closed subspaces \( V_j \) was sought as shown in the following equation:

\[
V_j \subset V_{j+1}, \quad j \in \mathbb{Z},
\]

with the following properties [2, 5]:

1. \( \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \)
2. \( \cap_{j \in \mathbb{Z}} V_j = \{0\} \)
3. \( f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}, \quad j \in \mathbb{Z} \)
4. \( f(x) \in V_0 \Leftrightarrow f(x-k) \in V_k, \quad k \in \mathbb{Z} \)
5. There exists \( \phi \in V_0 \) in which the set \( \{\phi(x-k) : k \in \mathbb{Z}\} \) is an orthonormal basis of \( V_0 \)

Then, these properties \( \psi_{j,k}(x) = 2^{-j/2}\phi(2^jx-k) \) form an orthonormal basis for \( V_0 \). For each \( j \), since \( V_j \) is a proper subspace of \( V_{j+1} \), there is some space in \( V_{j+1} \) called \( W_j \), which when combined with \( V_j \) gives us \( V_{j+1} \). This space \( W_j \) is called the wavelet subspace and is orthogonal complementary to \( V_j \) in \( V_{j+1} \). In other words,

\[
V_j \cap W_j = \{0\}, \quad j \in \mathbb{Z},
\]

\[
V_j \perp W_j, \quad j \in \mathbb{Z},
\]

and so

\[
V_{j+1} = V_j \oplus W_j,
\]

where \( \oplus \) represents a direct sum. It follows, then, we have

\[
L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j.
\]

Now, let us introduce a wavelet function \( \psi(x) \) that \( \psi(x-k) \) forms an orthonormal basis of the subspace \( W_0 \), i.e., \( W_0 = \text{span}\{\psi(x-k) : k \in \mathbb{Z}\} \). Then, we obtain

\[
\psi_{j,k}(x) = 2^{j/2}\psi(2^jx-k), \quad j, k \in \mathbb{Z},
\]

which is an orthonormal basis for \( W_j \). Furthermore, since \( W_0 \) and \( V_0 \) are contained in the space \( V_j \), we can express the scaling function \( \phi(x) \) and the wavelet function \( \psi(x) \) in terms of the scaling function at the next higher scale as shown in the following equations:

\[
\phi(x) = \sum_{k \in \mathbb{Z}} h_k \phi(2x-k),
\]

\[
\psi(x) = \sum_{k \in \mathbb{Z}} g_k \phi(2x-k).
\]

Equation (9) is referred to as the dilation equation, and the parameters \( h_k \) are known as filter coefficients. Equation (10) is referred to as the wavelet equation, and the parameters \( g_k \) are known as wavelet parameters.

2.1. Daubechies Scaling Coefficients. In order to determine scaling coefficients, Daubechies imposed some useful conditions as follows:

1. A scaling function must be able to exactly represent polynomials of order smaller than \( p \).
2. \( \int_0^{2^{-p-1}} \phi(x)dx = 1 \).
3. Orthogonality of a scaling function to its integer translates and orthogonality of a scaling function to the Daubechies wavelet are shown in the following equations:

\[
\int_0^{2^{-p-1}} \phi(x-k)\phi(x-l)dx = \delta_{k,l}, \quad l, k \in \mathbb{Z},
\]

\[
\int_0^{2^{-p-1}} \phi(x)\psi(x-l)dx = 0, \quad l \in \mathbb{Z}.
\]

These conditions conclude the following results

\[
\sum_{k=0}^{2^{-p}-1} (-1)^k k^m h_k = 0, \quad m = 0, 1, \ldots, p-1, \quad p \leq 15
\]

\[
\sum_{k=0}^{2^{-p}-1} h_k = 2,
\]

\[
\sum_{k=0}^{2^{-p}-1} h_k h_{2m+k} = 2\delta_{0,m}, \quad k = 0, 1, \ldots, p-1,
\]

\[
h_k = 0, \quad \forall k \notin \{0, 1, \ldots, 2p-1\}.
\]

Imposing equations (12-15) to filter coefficients enables us to compute \( \{h_k\}_{k=0}^{2^{p-1}} \), so dilation equation (9) for the Daubechies scaling function will be as follows:

\[
\phi(x) = \sum_{k=0}^{2^{p-1}} h_k \phi(2x-k).
\]

In [18], the exact values of \( h_k \) are evaluated. In [19], it is proved that for a fixed \( p \), there exists only one linear
independent scaling function $\phi$ which satisfies the dilation equation with these filter coefficients. The support of $\phi$ lies in $[0, 2p - 1]$, and if we denote $D = 2p$, then the parameter $D$ is called the wavelet genus or Daubechies’ number $[1, 6]$. The relation between $h_k$ and $g_k$ is shown in the following equation [5]:

$$g_k = (-1)^k h_{D-k-1}.$$  (17)

In Table 1, filter coefficients of the Daubechies scaling function with $p$ vanishing moments have been shown.

2.2. Computing Daubechies Scaling Functions and Their Derivative at Integers and Dyadic Points. Generally, scaling functions of the Daubechies wavelet do not have a closed form. Instead, they have to be generated recursively from dilation equation (10). The scaling function $\phi$ has support for the interval $[0, D - 1]$. Putting $x = 0, \ldots, D - 1$ in (10) yields a homogeneous linear system of equations. For the Daubechies scaling functions, we have $\phi(0) = 0$ and $\phi(D - 1) = 0$, and so, we will discard $\phi(D - 1)$ in our computations, but for reasons which are explained below, we keep $\phi(0)$. In a matrix form, it becomes

$$
\begin{bmatrix}
    h_0 & 0 & 0 & \ldots & 0 & 0 \\
    h_1 & h_0 & 0 & \ldots & 0 & 0 \\
    h_2 & h_1 & h_0 & \ldots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & h_{D-3} & h_{D-4} \\
    0 & 0 & 0 & \ldots & h_{D-1} & h_{D-2}
\end{bmatrix}
\begin{bmatrix}
    \phi(0) \\
    \phi(1) \\
    \phi(2) \\
    \vdots \\
    \phi(D - 3) \\
    \phi(D - 2)
\end{bmatrix}
= 
\begin{bmatrix}
    \phi(0) \\
    \phi(1) \\
    \phi(2) \\
    \vdots \\
    \phi(D - 3) \\
    \phi(D - 2)
\end{bmatrix}.
$$

$$M \phi = \phi.$$  (18)

Thus, the vector of integer values of the scaling function $\phi$ is the eigenvector of $M$ corresponding to the eigenvalue 1. As in all eigenvalue problems, the solution to the following system:

$$(M - I) \phi = 0,$$  (19)

for obtaining $\phi$, the solution is not unique. In order to obtain a unique solution, the normalizing condition is as follows:

$$\sum_{k=0}^{D-1} \phi(k) = 1,$$  (20)

which will be imposed on (19) [4]. So, by considering $\phi(0) = \phi(D - 1) = 0$, from (20), we have

$$\sum_{k=1}^{D-2} \phi(k) = 1.$$  (21)

Therefore, from (20) and (21), we obtain the following equation:

$$M_{\text{new}} \phi_{\text{new}} = e_1,$$  (22)

where $M_{\text{new}}$ is

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The values of the scaling function at the integers are given by the solution to equation (22). Now, we can use (16) to obtain the scaling function at all midpoints between integers in the interval, and the value of the Daubechies scaling function in all dyadic points will be determined. The complete explanation of this procedure is named the cascade algorithm and is introduced in [6, 17].

We applied the same procedure as the cascade algorithm for evaluating the scaling function’s derivatives at dyadic points. Differentiating equation (16) $m$ times yields

$$\phi^{(m)}(x) = 2^m \sum_{k=0}^{N-1} h_k \phi^{(m)}(2x - k),$$

and by substituting all integer values of the interval $[0, D - 1]$, equation (24) gives a linear homogeneous system. The coefficient matrix of this homogeneous system is singular. Thus, a normalization condition is required, in order to determine a unique solution. The following important additional property of the Daubechies scaling function $\phi$ can be used [20]:

$$\sum_k k^m \phi(x - k) = x^m + \sum_{k=1}^{m} (-1)^k \frac{m!}{(m-k)!k!} x^{m-k} \int_{-\infty}^{\infty} \phi(z) z^k dz,$$

where $m$ is a positive integer number. By differentiating $m$ times, the previous equation yields

$$\sum_k k^m \phi^{(m)}(x - k) = m!.$$  

Using this normalizing equation (26) and equation (24) for $x = 1, 2, \ldots, D - 1$ yields an inhomogeneous system, which can be solved, and derivatives can be evaluated at the integer values of $x$. Similar to the cascade algorithm, using the dilation equation once again, the values of $\phi^{(m)}(x)$ at $x = i2^{l-n}$ for $i = 1, 3, 5, \ldots, (D - 1)2^n - 1$ can be determined.

3. The Integrals of the Daubechies Scaling Function on Dyadic Intervals

As previously mentioned, the support of the Daubechies scaling function with $p$ vanishing moments is $[0, D - 1]$ where $D = 2p$. In this section, first, the integral of the Daubechies scaling function on integer intervals is computed exactly. Then, we use a recursion relation, in order to obtain the integral of the Daubechies scaling function, on dyadic intervals, i.e., on the intervals $[k2^{-n}, (k + 1)2^{-n}]$, in arbitrary resolution $j$ will be computed [3].

3.1. The Integrals of the Daubechies Scaling Function on Integer Intervals

We define $A_{[a,b]}$, which is as follows:

$$A_{[a,b]} = \int_{a}^{b} \phi(x) dx,$$

and using dilation equation (16) to the scaling function $\phi$, then the following result is obtained:

$$A_{[0,j]} = \frac{1}{2} \sum_{k=0}^{D-1} h_k A_{[0,2^{j-k}]} \phi(2x - k) dx, \quad i > 0$$

$$= \frac{1}{2} \sum_{k=0}^{D-1} h_k \int_{-\infty}^{\infty} \phi(t) dt$$

$$= \frac{1}{2} \sum_{k=0}^{D-1} h_k A_{[0,2^{j-k}]}$$

in other words,

$$A_{[0,j]} = \frac{1}{2} \sum_{k=0}^{D-1} h_k A_{[0,2^{j-k}]}.$$  

From equation (17), it is clear that

$$A_{[0,j]} = 1, \quad i \geq D - 1,$$

$$A_{[0,j]} = 0, \quad i \leq 0.$$  

Equations (29)–(31) can be written as a sparse linear system of equations $C_{ij} = b_i$. The entries of matrices $C$, $x$ and $b$ are given as

$$C_{i,j} = \begin{cases} cl - h_i, & \text{for } i = j, \\ -h_{2j-1}, & \text{for } i \neq j \end{cases}$$

$$b_i = \begin{cases} cl_0, & i \leq p - 1, \\ \sum_{k=0}^{2^{(j-p)}+1} h_k, & i \geq p. \end{cases}$$

$$x_i = A_{[0,j]}, \quad 1 \leq i \leq D - 2.$$  

It is obvious that the matrix of coefficients in these linear systems is sparse and also diagonally dominant, so is nonsingular. For example, the linear system of equations $C = 4$ when $D = 4$ and $D = 6$ is given as follows:
In this section, we propose our method to approximate the solution of equation (3). Let \( U \) be the approximate solution of \( u \) as

\[
R(t) = \sum_{k=k_1}^{k_2} a_{jk} \left[ \kappa \phi_{jk}(t) - \phi_{jk}(t) \sum_{k=k_1}^{k_2} a_{jk} \phi_{jk}(t) + \phi_{jk}(t) \sum_{k=k_1}^{k_2} a_{jk} \phi_{jk}(t) \right].
\]

As we know, Volterra’s population model is defined on the interval \([0, +\infty)\) and decays rapidly [8]. The support of Daubechies scaling functions with \( N \) vanishing moment is \([0, D-1]\), where \( D = 2N \), so we let \( k_1 \) and \( k_2 \) be such that, in which the last point in support of \( \phi_{jk} \), is smaller than 0 and the first point in support of \( \phi_{jk} \) is bigger than \( D-1 \); therefore, we have

\[
k_1 = 2 - D, \\
k_2 = 2^j (D - 1) - 1.
\]

By integrating equation (37), we can get the following result:

\[
\int_0^t U_j(x)dx = \sum_{k=k_1}^{k_2} a_{jk} \int_0^t \phi_{jk}(x)dx.
\]

Therefore, suppose that

\[
\int_0^t U(x)dx = \sum_{k=k_1}^{k_2} a_{jk} H_{jk}(t),
\]

where

\[
H_{jk}(t) = 2^{j/2} \int_0^t \phi(2^j x - k)dx,
\]

\[
H_{jk}(t) = 2^{-j/2} A_{[-k,2^j t-k]}.
\]

In order to approximate \( u(t) \) in equation (3) by differentiating equation (37), we have

\[
U_j'(x) = \sum_{k=k_1}^{k_2} a_{jk} \phi_{jk}'(x).
\]

By substituting equations (37)–(42), in equation (3), we construct the residual as

\[
R(t) = \kappa \frac{dU_j(t)}{dt} - U_j(t) + U_j^2(t) + U_j(t) \int_0^t U_j(x)dx,
\]

\[
U_j(0) = u_0,
\]

and by substituting equation (40) in equation (43), we have
In the above equation for obtaining the coefficients $a_{j,k}$, let

$$R(t_j) = 0, \quad \text{where } j = 1, 2, \ldots, k_2 - k_1,$$

and also, the initial condition is

$$\sum_{k=k_1}^{k_2} a_{j,k} \phi_{j,k}(0) = u_0,$$

in which the collocation points $t_j$ are dyadic points on $[0, D-1]$. The equations, (45) and (46) are a system of nonlinear $k_2 - k_1 + 1$ equation, which can be solved by Newton’s iteration method.

### 5. Error Analysis

Let $\mathcal{N}$ be the nonlinear operator, then we have

$$E_j = \|\mathcal{N}(U_j)\| = \|\mathcal{N}(U_j) - \mathcal{N}(u)\| = \|\kappa(u' - U'_j) + (u - U_j) + (u^2 - U^2_j) + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right)\|.$$  

Therefore, we have

$$E_j \leq \|\kappa(u' - U'_j)\| + \|u - U_j\| + \|u^2 - U^2_j\| + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right),$$  

and by using the bound of errors as $\|u(t) - U_j(t)\| \leq C_1 2^{-j}$ and $\|u(t) - U'_j(t)\| \leq C_2 2^{-j}$, which are given in [21] and also $C_1$ and $C_2$ are constants, we have

$$E_j \leq \kappa C_1 2^{-j} + C_1 2^{-j} + 2C_1 C_3 2^{-j} + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right).$$

Now, let $U_j(t) = u(t) + \varepsilon(t)$, then the above inequality can be written as

$$E_j \leq (C_1 + \kappa C_2 + 2C_3 C_1) 2^{-j} + \left(\varepsilon(t) \int_0^t \varepsilon(x)dx\right) + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right).$$

and as we know $\|\varepsilon(t)\| \leq C_1 2^{-j}$, $t \leq (L - 1)$ and if we let $C = C_1 + \kappa C_2 + 2C_3 C_1$, then, we have

$$\mathcal{N}(u) = \kappa u'(t) - u(t) + u^2(t) + u(t) \int_0^t u(x)dx,$$

and it is obvious that $\mathcal{N}(u) = 0$ when $u(t)$ is the exact solution of (47). Let $U_j(t) = \sum_{k=k_1}^{k_2} a_{j,k} \phi_{j,k}$, where $j$ is the resolution of the Daubechies wavelet, then the error of our method is

$$E_j = \|\mathcal{N}(U_j)\|.$$  

Let $U_j(t)$ be the approximate solution of (3) given by (37), and the error of our method is given in (48). Then, the error is

$$E_j = O(2^{-j}).$$

**Proof.** From equation (47) and $\mathcal{N}(u) = 0$, we have

$$E_j = \|\mathcal{N}(U_j)\| = \|\kappa(u' - U'_j) + (u - U_j) + (u^2 - U^2_j) + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right)\|.$$  

$$E_j \leq C_1 2^{-j} + (L - 1) C_3^2 2^{-2j} + 2C_1 C_3 (L - 1) 2^{-j},$$

and by using the bound of errors as $\|u(t) - U_j(t)\| \leq C_1 2^{-j}$ and $\|u(t) - U'_j(t)\| \leq C_2 2^{-j}$, which are given in [21] and also $C_1$ and $C_2$ are constants, we have

$$E_j \leq \kappa C_1 2^{-j} + C_1 2^{-j} + 2C_1 C_3 2^{-j} + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right).$$

Now, let $U_j(t) = u(t) + \varepsilon(t)$, then the above inequality can be written as

$$E_j \leq (C_1 + \kappa C_2 + 2C_3 C_1) 2^{-j} + \left(\varepsilon(t) \int_0^t \varepsilon(x)dx\right) + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right).$$

and as we know $\|\varepsilon(t)\| \leq C_1 2^{-j}$, $t \leq (L - 1)$ and if we let $C = C_1 + \kappa C_2 + 2C_3 C_1$, then, we have

$$E_j = \|\mathcal{N}(U_j)\| = \|\kappa(u' - U'_j) + (u - U_j) + (u^2 - U^2_j) + \left(u(t) \int_0^t u(x)dx - U_j(t) \int_0^t U_j(x)dx\right)\|.$$  

This theorem indicates that the error of our method is bounded, and when $j$ increases, then the error of our method tends to zero, in the other words, $E_j = O(2^{-j})$, and the numerical results in Table 3 are consistent with this upper bound.

### 6. Numerical Results

In this section, we use the Daubechies scaling function with 4 vanishing moments, and the computational results are summarized in Table 3. In Figure 1, the results showed for $\kappa = 0.02, 0.1, 0.2$ and 0.5. This figure shows the rapid rise along the logistic curve followed by the slow exponential decay after reaching the maximum point, and when $t$ increases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases. All computational efforts in this work have been carried out using Maple software in 30 decimal digits.
7. Conclusion

In this paper, we have constructed an efficient and accurate numerical method based on Daubechies wavelet scaling functions for the solution of Volterra’s population model. Our method was used in a direct way without using linearization, perturbation, or restrictive assumption. The refinement equation in the Daubechies wavelet scaling function has enabled us exactly to compute the integral and the derivative of unknown functions in 2. Essential data have been saved, and this fact strongly has reduced time-consuming computations and gives us the advantage of requiring a shorter time to run the procedure in more accurate resolutions. We have shown that our method is simple. The only error that affected the results was computations in the Newton method for solving a system of nonlinear equations. In the error analysis, we showed that when finer resolutions applied for big $j$, the error decreases, and numerical results satisfy this result. The stability and convergence of Daubechies wavelet scaling functions make this approach very attractive for computation at the dyadic points. Considering this exact computation and helpful characteristics of Daubechies scaling functions, promising results were obtained.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


