Research Article

Multiplicity Results for Weak Solutions of a Semilinear Dirichlet Elliptic Problem with a Parametric Nonlinearity

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This paper deals with the existence of weak solutions to a Dirichlet problem for a semilinear elliptic equation involving the difference of two main nonlinearities functions that depends on a real parameter \( \lambda \). According to the values of \( \lambda \), we give both nonexistence and multiplicity results by using variational methods. In particular, we first exhibit a critical positive value such that the problem admits at least a nontrivial non-negative weak solution if and only if \( \lambda \) is greater than or equal to this critical value.

Furthermore, for \( \lambda \) greater than a second critical positive value, we show the existence of two independent nontrivial non-negative weak solutions to the problem.

1. Introduction

In the last years, most works studied the existence, nonexistence, and multiplicity of nontrivial weak solutions of a semilinear Dirichlet problem of the form as follows:

\[
\begin{aligned}
-\Delta u &= f_\lambda(x, u), & x &\in \Omega, \\
u &= 0, & x &\in \partial \Omega,
\end{aligned}
\] (1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( \lambda \) is a real parameter, and \( f_\lambda: \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a nonlinear function taking different forms. According to the values of \( \lambda \), Ambrosetti et al. studied in [1], the existence and multiplicity of non-negative weak solutions of the problem (1) when \( f_\lambda(x, u) = \lambda u^q + u^p \) with \( 0 < q < 1 < p \). For example, by using variational method, they show the existence of infinitely many solutions of the problem as follows:

\[
\begin{aligned}
-\Delta u &= \lambda |u|^{q-1}u + |u|^{p-1}u, & x &\in \Omega, \\
u &= 0, & x &\in \partial \Omega,
\end{aligned}
\] (2)

for \( \lambda > 0 \) and small. Later, Alama and Tarantello in [2] studied the semilinear Dirichlet problem (1) by searching non-negative solutions with

\[
f_\lambda(x, u) = \lambda u + \omega(x)u^{q-1} - h(x)u^{r-1},
\] (3)

where \( \lambda \in \mathbb{R} \), \( \Omega \subset \mathbb{R}^N (N \geq 3) \) is a bounded domain with smooth boundary, \( \omega \) and \( h \) are suitable functions, and \( 1 < q < r \). In this case also, the authors show the influence of values of \( \lambda \) on the existence and multiplicity of weak solutions of the problem. These different studies on nonexistence, existence, and multiplicity results for nontrivial weak solutions depending on a parameter for a Dirichlet problem for a semilinear elliptic equation were extensively investigated in the literature (see, for e.g., [3–6] and the references therein). Similar results, depending on a real parameter, are obtained in the case of quasilinear elliptic equations in bounded domains or in entire space \( \mathbb{R}^N \). For example, we can mention the papers [7–9], which are devoted to the unbounded case. In [7], the authors deal with the nonexistence and existence of nontrivial weak solutions of the quasilinear problem:
\begin{equation}
\begin{cases}
-\text{div}(a(x)\nabla u)|^{p-2}\nabla u) + |u|^{p-2}u = \lambda g(x)|u|^{r-2}u \quad \text{in } \Omega,
\end{cases}
\end{equation}
\begin{equation}
\begin{cases}
a(x)\nabla u|^{p-2}\partial u + b(x)|\nabla u|^{p-2}u = 0, \text{ on } \partial \Omega,
\end{cases}
\end{equation}

where \( \Omega \) is a smooth exterior domain in \( \mathbb{R}^N \), \( v \) is the unit vector of the outward normal on \( \partial \Omega \), \( \lambda \) is a real parameter. By taking inspiration on the method developed in \([8, 10]\), we use variational arguments to study the existence and the multiplicity of nontrivial weak solutions of the following Dirichlet problem for a semilinear elliptic equation:

\begin{equation}
\begin{cases}
-\Delta u + u = \lambda a(|u|)u - b(|u|)u \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{equation}

where \( \Omega \subset \mathbb{R}^N \), \( N \geq 3 \), is a bounded domain with smooth boundary, \( a, b \) are suitable non-negative functions, and \( \lambda \) is a real parameter. By taking inspiration on the method developed in \([8, 10]\), we use variational arguments to study the existence and the multiplicity of nontrivial weak solutions of the following Dirichlet problem for a semilinear elliptic equation:

\begin{enumerate}
\item[(\( \mathcal{A}_1 \))] \( a \) is a function continuous on \( \mathbb{R}_+ \) and of \( C^1((0, +\infty), \mathbb{R}_+) \) such that
\begin{equation}
a(t) > 0, \quad (a(t))' > 0 \quad \text{for all } t > 0, \tag{8}
\end{equation}
\begin{equation}
a(t) = 0 \iff t = 0, \tag{9}
\end{equation}
\begin{equation}
\lim_{t \to +\infty} a(t) = +\infty. \tag{10}
\end{equation}

Let us set
\begin{equation}
\lambda \mathcal{A} = \int_0^t a(s)\,ds \quad \text{for } t \geq 0.
\end{equation}

\item[(\( \mathcal{A}_2 \))] There exist \( p, q \in \mathbb{R}_+ \) such that
\begin{equation}
p \leq q < 2^* = 2N/(N-2)
\end{equation}
and
\begin{equation}
p - 2 \leq \frac{a'(t)t}{a(t)} \leq q - 2, \quad \text{for all } t > 0.
\end{equation}

Let us set
\begin{equation}
\lambda \mathcal{B} = \int_0^t b(s)\,ds \quad \text{for } t \geq 0.
\end{equation}

\item[(\( \mathcal{B}_2 \))] There exists \( r \in \mathbb{R}_+ \) such that \( q < r \) and
\begin{equation}
q - 2 \leq \frac{b'(t)t}{b(t)} \leq r - 2, \quad \text{for all } t > 0.
\end{equation}
\end{enumerate}

Some examples of functions \( a \) and \( b \) in \( (\varepsilon_i) \) satisfying the previous assumptions \( (\mathcal{A}_1), (\mathcal{A}_2),(\mathcal{B}_1), \) and \( (\mathcal{B}_2) \):

1. For \( \varepsilon \geq 0 \), we can have \( a(u) = u^{p-2} \) and \( b(u) = u^{r-2} \), in \((\varepsilon_1)\) with \( 2 < p < r < N < 2^* \).
2. Another example of functions \( a \) and \( b \) is the following:

\begin{align*}
a(u) &= \begin{cases} 
    u^{\alpha-2} & \text{for } u > 0, \\
    0 & \text{for } u = 0,
\end{cases} \\
b(u) &= \begin{cases} 
    u^{\beta-2} & \text{for } u > 0, \\
    0 & \text{for } u = 0,
\end{cases}
\end{align*}

with \( 3 < \alpha < \alpha + 1 < \beta < N < 2^* \).

Theorem 1. Two theorems: equation studied in [7, 8] or [10].

In this case, for all \( t > 0 \), we have

\[
\begin{align*}
\alpha - 3 & \leq \frac{a'(t)t}{a(t)} \leq \alpha - 2, \\
\alpha - 2 & < \beta - 3 \leq \frac{b'(t)t}{b(t)} \leq \beta - 2.
\end{align*}
\]

(19)

(3) Let \( \sigma \) fixed in \((1, +\infty)\). For all \( u \geq 0 \),

\[
\begin{align*}
a(u) &= \frac{u^{\alpha - 2}}{\ln(\sigma + u)}, \\
b(u) &= \frac{u^{\beta - 2}}{\ln(\sigma + u)},
\end{align*}
\]

with \( 2 + \frac{\sigma}{\sigma + t_0} < \alpha < \frac{\sigma}{\sigma + t_0} < \beta < N < 2^* \),

where \( t_0 \) is the unique solution of the equation \( \sigma \ln(\sigma + t) - t = 0 \) on \((0, +\infty)\). Hence, for all \( t > 0 \), we have

\[
\begin{align*}
\alpha - 2 - \frac{\sigma}{\sigma + t_0} & \leq \frac{a'(t)t}{a(t)} \leq \alpha - 2, \\
\alpha - 2 - \frac{\sigma}{\sigma + t_0} & \leq \frac{b'(t)t}{b(t)} \leq \beta - 2.
\end{align*}
\]

(21)

Thus, it is clear that the functions \( a \) and \( b \) of the Dirichlet problem \((\varepsilon_i)\) of our present work generalize the functions \(|u|^{\sigma-2}\) and \(|u|^{\sigma-2} (2 < q < r)\) which appear in the main equation studied in [7, 8] or [10].

The main goal of this paper is the proof of the following two theorems:

**Theorem 1.** By the fulfillment of assumptions \((\mathcal{A}_1)\), \((\mathcal{A}_2)(\mathcal{B}_1)\), and \((\mathcal{B}_2)\), there exists a critical value \( \lambda > 0 \) such that the Dirichlet problem \((\varepsilon_i)\) admits at least a nontrivial non-negative weak solution if and only if \( \lambda \geq \hat{\lambda} \).

**Theorem 2.** Suppose that the assumptions \((\mathcal{A}_1), (\mathcal{A}_2)(\mathcal{B}_1),\) and \((\mathcal{B}_2)\) are fulfilled. Then, there exists a critical value \( \lambda \) satisfying \( \lambda \geq \hat{\lambda} \) such that for all \( \lambda \geq \hat{\lambda} \), the Dirichlet problem \((\varepsilon_i)\) admits at least two nontrivial non-negative weak solutions.

In Section 2, we talk about Orlicz spaces, which we will use in our work. In Section 3, we give different imbeddings between the working spaces of this paper and prove the non-existence of a nontrivial weak solution when \( \lambda \) in \((\varepsilon_i)\) is less than a positive number. The conditions for existence of weak solutions of \((\varepsilon_i)\) are established in Section 4. Section 5 has devoted to prove Theorem 1, and Section 6 deals with the proof of Theorem 2.

### 2. Notions on Orlicz Spaces (See Chapter 8 in [11])

**Definition 1.** (definition of a \(N\)-function). Let \( \psi \) be a real-valued function defined on \([0, \infty)\) and having the following properties:

(a) \( \psi(0) = 0 \), \( \psi(t) > 0 \) if \( t > 0 \), \( \lim_{t \to \infty} = +\infty \)

(b) \( \psi \) is nondecreasing, that is, \( s > t \) implies \( \psi(s) \geq \psi(t) \)

(c) \( \psi \) is right continuous, that is, if \( t \geq 0 \), then \( \lim_{s \to t^+} \psi(s) = \psi(t) \)

Then, the real-valued function \( \Psi \) defined on \([0, +\infty)\) by

\[
\Psi(t) = \int_0^t \psi(s) ds,
\]

is called an \(N\)-function.

Any such \(N\)-function \( \Psi \) has the following properties:

(i) \( \psi \) is continuous on \([0, \infty)\)

(ii) \( \psi \) is strictly increasing

(iii) \( \psi \) is convex

(iv) \( \lim_{t \to 0^+} \psi(t)/t = 0 \) and \( \lim_{t \to +\infty} \psi(t)/t = +\infty \)

(v) The function \( t \mapsto \psi(t) \) is strictly increasing on \((0, \infty)\)

For any \(N\)-function \( \psi = \psi(t) \) and an open set \( \Omega \subset \mathbb{R}^N \), the Orlicz space \( L_\psi(\Omega) \) is defined. When \( \Psi \) satisfies \( \Delta_2 \)-condition, i.e.,

\[
\psi(2t) \leq k \psi(t), \quad \text{for all } t \geq 0,
\]

for some constant \( k > 0 \), then

\[
L_\psi(\Omega) = \left\{ u: \Omega \to \mathbb{R} \text{ measurable: } \int_\Omega \psi(|u(x)|) dx < \infty \right\}.
\]

Endowed with the norm

\[
\|u\|_\psi = \inf \left\{ k > 0: \int_\Omega \psi \left( \frac{|u(x)|}{k} \right) dx \leq 1 \right\},
\]

which is called the Luxembourg norm, the Orlicz space \( L_\psi(\Omega) \) is a Banach space. It is known that if \( \int_\Omega \psi(|u(x)|/k_0) dx = 1 \), then \( \|u\|_\psi = k_0 \) with \( k_0 > 0 \).

The complement of \( \Psi \) is given by the Legendre transformation as follows:

\[
\bar{\Psi}(s) = \max_{t \geq 0} (st - \Psi(t)) \text{for } s \geq 0.
\]

We say that \( \Psi \) and \( \bar{\Psi} \) are complementary \(N\)-functions of each other.

For all \( t > 0 \), we have the inequality \( \bar{\Psi}(\Psi(t)/t) \leq \Psi(t) \).

From Young’s inequality

\[
st \leq \Psi(t) + \bar{\Psi}(s),
\]

a generalized version of Hölder’s inequality is obtained as follows:
3. Preliminaries and Nonexistence of Nontrivial Solution for $\lambda$ Small

By condition (a') and the definition of $A$ and by condition (R1) and the definition of $B$, the functions $A$ and $B$ are $N$-functions, with $\psi(s) = sa(s)$ or $\psi(s) = sb(s)$, respectively.

**Lemma 1.** The $N$-function $A$ satisfies the $\Delta_2$-condition.

**Proof.** In fact, by (12),
\[
\text{for all } t > 0, \quad \rho \leq \frac{a(t)^2}{A(t)} \leq q.
\]
(29)

Thus, by Theorem 4.1 in [12], it follows that $A$ satisfies the $\Delta_2$-condition. \(\square\)

**Lemma 2.** The $N$-function $B$ satisfies the $\Delta_2$-condition.

**Proof.** In fact, by (17)
\[
\text{for all } t > 0, \quad q \leq \frac{b(t)^2}{B(t)} \leq r,
\]
(30)

and $B$ satisfies the $\Delta_2$-condition by Theorem 4.1 in [12].

Let us consider the Sobolev space $H_0^1(\Omega)$, which is the completion of $C_0^\infty(\Omega)$ with the norm
\[
\|u\|_{H_0^1(\Omega)} = \left( \int_\Omega |\nabla u|^2 \right)^{1/2} = \|\nabla u\|_2.
\]
(31)

Let $X$ denote the completion of $C_0^\infty(\Omega)$ with respect to the norm
\[
\|u\|_X = \left( \|u\|_{H_0^1(\Omega)}^2 + \|u\|_A^2 \right)^{1/2},
\]
(32)

where
\[
\|u\|_A = \inf \left\{ \lambda > 0 : \int_\Omega B\left( \frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\},
\]
(33)

is the Luxembourg norm in the Orlicz space $L_A(\Omega)$. The space $X$ is the space in which we will find our nontrivial weak solutions. \(\square\)

**Lemma 3.** The embeddings $X \rightarrow H_0^1(\Omega) \rightarrow L^2(\Omega)$ are continuous with $\|u\|_{H_0^1(\Omega)} \leq \|u\|_X$, for all $u \in X$ and $\|u\|_A \leq C_2\|u\|_{H_0^1(\Omega)}$, for all $u \in H_0^1(\Omega)$.

Moreover, $H_0^1(\Omega) \rightarrow L^p(\Omega)$ and $X \rightarrow L^p(\Omega)$ are compact for all $p$ such that $1 \leq p < 2^*$.

**Proof.** The first imbedding that i.e., $X \rightarrow H_0^1(\Omega)$ is followed from the definition of the norm in $X$. The second imbedding is followed from Talenti’s work in [13]; $C_2$, is the Talenti constant. The imbedding compact $H_0^1(\Omega) \rightarrow L^p(\Omega)$ for $p \in [1, 2^*)$ is obtained by Rellich’s Theorem. It follows that the mapping $X \rightarrow L^p(\Omega)$ is compact for $p \in [1, 2^*)$. \(\square\)

**Lemma 4.** Let $\zeta_0(t) = \min(t^p, t^q)$, $\zeta_1(t) = \max(t^q, t^p)$, $\zeta_2(t) = \min(t^q, t^p)$, and $\zeta_3(t) = \max(t^q, t^p)$ for all $t \geq 0$. Then,
\[
\zeta_0(\rho)A(t) \leq A(\rho t) \leq \zeta_1(\rho)A(t) \quad \text{for } \rho \geq 1, \quad (34)
\]
\[
\zeta_2(\rho)B(t) \leq B(\rho t) \leq \zeta_3(\rho)B(t) \quad \text{for } \rho \geq 1, \quad (35)
\]
\[
\zeta_2(\|u\|_B) \leq B(\|u\|_B) \leq \zeta_3(\|u\|_B) \quad \text{for } u \in L_B(\Omega). \quad (37)
\]

**Proof.** The proof is given in [14] (see Lemma 2.1 of [14]). In fact by integrating the inequalities (29) and (30) respectively, we get inequalities (34) and (35). From (34) and (35) and the definition of Luxembourg norm we get respectively (35) and (37). \(\square\)

**Lemma 5.** The space $L_B(\Omega)$ is imbedded continuously in $L_A(\Omega)$ with $\|u\|_A \leq H\|u\|_B$, where
\[
H = 2k\omega,
\]
(38)

with
\[
k = \max\{1, (A(1)/B(1))^{1/q}\} \quad \text{and} \quad \omega = \max\{1, (A(1)/B(1))^{1/q}(1/2\text{mes}(\Omega))\}.
\]

**Proof.** This proof is based on the proof of Theorem 8.12 in [11]. Fix $k$, with $k \geq 1$, then
\[
\text{for all } t \geq 1, \quad \frac{A(t)}{B(kt)} \leq \frac{A(1)}{k\omega B(1)}, \quad (39)
\]
by (34) and (36). Thus, by taking $k = \max\{1, (A(1)/B(1))^{1/q}\}$, we have
\[
A(t) \leq B(kt), \quad \text{for all } t \geq 1.
\]
(40)

By the proof of (Theorem 8.12 [11]),
\[
\text{for all } t \geq A^{-1}\left(\frac{1}{2\text{mes}(\Omega)}\right), \quad A(t) \leq \omega B(kt),
\]
(41)

where $\omega = \max\left\{1, \frac{A(1)}{B(kA^{-1}(1/2\text{mes}(\Omega)))}\right\}$.

Let $u \in L_B(\Omega)$ such that $u \neq 0$. Let us set $\Omega'(u) = \{x \in \Omega : \|u(x)/2\|_B \leq A^{-1}(1/2\text{mes}(\Omega))\}$ and $\Omega''(u) = \Omega - \Omega'(u)$.
\[
\int \Omega \left( \frac{|u(x)|}{2\omega (A(1))^{1/2} ||u||_{L^2}} \right) dx = \int \Omega' (u) \left( \frac{|u(x)|}{2\omega (A(1))^{1/2} ||u||_{L^2}} \right) dx + \int \Omega^* (u) \left( \frac{|u(x)|}{2\omega (A(1))^{1/2} ||u||_{L^2}} \right) dx
\]
\[
\leq \frac{1}{2 \text{mes} (\Omega)} \int \Omega' (u) dx + \omega \int \Omega^* (u) \left( \frac{|u(x)|^{2'}}{2\omega ||u||_{L^2}} \right) dx \leq \frac{1}{2} + \frac{1}{2} \int \Omega \left( \frac{|u(x)|^{2'}}{||u||_{L^2}^2} \right) dx \leq 1.
\]

It follows that \( ||u||_{L^2} \leq 2\omega (A(1))^{1/2} ||u||_{L^2} \) and consequently \( L^2 (\Omega) \) is imbedded continuously in \( L^2 (\Omega) \).

**Lemma 6.** The \( N \) - function \( A \) satisfies the following:
\[
\lim_{t \to +\infty} \frac{A(kt)}{t^2} = 0 \text{ for all } k \geq 0.
\]

**Proof.** The result is followed from property (34). In fact, by (34), for all \( k \geq 0 \), we have
\[
0 \leq \frac{A(kt)}{t^2} \leq \frac{\zeta (t)}{t^2} = t^{2-2^*} A(k).
\]

**Lemma 7.** The imbedding \( L^2 (\Omega) \to L^2 (\Omega) \) is continuous with
\[
\int \Omega \left( \frac{|u(x)|}{2\omega (A(1))^{1/2} ||u||_{L^2}} \right) dx = \int \Omega' (u) \left( \frac{|u(x)|}{2\omega (A(1))^{1/2} ||u||_{L^2}} \right) dx + \int \Omega^* (u) \left( \frac{|u(x)|}{2\omega (A(1))^{1/2} ||u||_{L^2}} \right) dx
\]
\[
\leq \frac{1}{2 \text{mes} (\Omega)} \int \Omega' (u) dx + \omega \int \Omega^* (u) \left( \frac{|u(x)|^{2'}}{2\omega ||u||_{L^2}} \right) dx \leq \frac{1}{2} + \frac{1}{2} \int \Omega \left( \frac{|u(x)|^{2'}}{||u||_{L^2}^2} \right) dx \leq 1.
\]

It follows that \( ||u||_{L^2} \leq 2\omega (A(1))^{1/2} ||u||_{L^2} \) and consequently \( L^2 (\Omega) \) is imbedded continuously in \( L^2 (\Omega) \).

**Lemma 8.** The imbedding \( H^1_0 (\Omega) \to L^2 (\Omega) \) is continuous with
\[
||u||_{L^2} \leq C_A ||u||_{H^1_0} \text{ for all } u \in H^1_0 (\Omega),
\]
where \( C_A = C_{2^*} C_{A^*} \).

Moreover, the imbeddings \( H^1_0 (\Omega) \to L^2 (\Omega) \) and \( X \to L^2 (\Omega) \) are compacts.

**Proof.** The continuity of the imbedding \( H^1_0 (\Omega) \to L^2 (\Omega) \) is followed from Lemmas 3 and 7. By Lemma 6 and Theorem 8.36 in [11], it follows that the imbedding \( H^1_0 (\Omega) \to L^2 (\Omega) \) is compact. It follows also that \( X \to L^2 (\Omega) \) is compact because \( X \to H^1_0 (\Omega) \) is continuous by Lemma 3.

**Lemma 9.** If \( u \) is an element of \( X \setminus \{0\} \) and \( \lambda \) is a real such that
\[
\int \Omega |u|^2 dx + \int \Omega |u|^2 dx + \int \Omega b(|u|)u^2 dx = \lambda \int \Omega a(|u|)u^2 dx,
\]
then \( \lambda > 0 \) and there exists two positive constants \( k_1 \) and \( k_2 \) independent of \( u \) such that
\[
k_1 f_1 (\lambda) \leq ||u||_{L^2} \leq k_2 f_2 (\lambda),
\]
where \( f_1 \) and \( f_2 \) are two functions of \( \lambda \).

**Proof.** Let us take \( u \in X \setminus \{0\} \) and \( \lambda \in \mathbb{R} \) such that (49) hold. Thus, we have \( 0 < \||u||_{H^1_0}^2 \|\leq \lambda \int \Omega a(|u|)u^2 dx \). As \( a \) is a non-negative function in \( (0, +\infty) \), it follows that \( \lambda > 0 \). Let us show the second part of the Lemma. Firstly, by using respectively (29), (35), (48), and (49), we get
\[
||u||_{L^2} \leq \lambda C_A C_{2^*} \lambda \leq k_2 f_2 (\lambda),
\]
Thus, this last inequality yields
\[
||u||_{L^2} \geq k_1 f_1 (\lambda) \leq \lambda C_A C_{2^*} \lambda \leq k_2 f_2 (\lambda).
\]

Secondly, by (29), (30), (48), and (49) we have
\[
\|u\|_A^2 \leq C^2_A \|u\|_{H^2(\Omega)}^2 \leq C^2_A \left( \lambda \int_{\Omega} a(|u|) u^2 \, dx - \int_{\Omega} b(|u|) u^2 \, dx \right) \leq C^2_A \lambda A(|u(x)|) - B(|u(x)|) \, dx.
\] (53)

Let us set \( \Omega_u = \{ x \in \Omega : |u(x)| \leq 1 \} \) and \( \Omega_u' = \Omega - \Omega_u \).
Since \( \lambda > 0 \), (34) and (36) imply

\[
\begin{aligned}
\lambda A(|u(x)|) - B(|u(x)|) &\leq \begin{cases} 
\lambda A(1)|u(x)|^p - B(1)|u(x)|^r, & \text{for } x \in \Omega_u, \\
\lambda A(1)|u(x)|^q - B(1)|u(x)|^r, & \text{for } x \in \Omega_u'.
\end{cases}
\end{aligned}
\] (54)

where \( a = (B(1))^{(s/r)}|u(x)|^{s}, \quad b = \lambda A(1)(B(1))^{-s/r} \), \( \alpha = (r/s) > 1 \), and \( \beta = (r/r - s) > 1 \). Inequality (52) yields

\[
\begin{aligned}
\lambda A(1)|u(x)|^s - B(1)|u(x)|^r &\leq B(1)|u(x)|^r \left( \frac{s}{r} - 1 \right) + \frac{r - s}{r} \left( \frac{\lambda A(1)}{(B(1))^{(s/r)}} \right)^{(r/(r-s))}, \\
&\leq \lambda^{(r/(r-s))} \left( \frac{s}{r} - 1 \right) + \frac{r - s}{r} \left( \frac{\lambda A(1)}{(B(1))^{(s/r)}} \right)^{(r/(r-s))},
\end{aligned}
\] (56)

being \( s < r \). Hence, (54) yields

\[
\begin{aligned}
\lambda A(|u(x)|) - B(|u(x)|) &\leq \begin{cases} 
\lambda^{(r/(r-p))} \frac{r - p}{r} \left( \frac{A(1)}{(B(1))^{(s/r)}} \right)^{(r/(r-p))}, & \text{for } x \in \Omega_u, \\
\lambda^{(r/(r-q))} \frac{r - q}{r} \left( \frac{A(1)}{(B(1))^{(s/r)}} \right)^{(r/(r-q))}, & \text{for } x \in \Omega_u'.
\end{cases}
\end{aligned}
\] (58)

As \( (r - p/r) > (r - q/r) \), it follows, from (58), that

\[
\begin{aligned}
\lambda A(|u(x)|) - B(|u(x)|) &\leq \frac{r - p}{r} \min \left\{ \lambda^{(r/(r-p))} \left( \frac{A(1)}{(B(1))^{(s/r)}} \right)^{(r/(r-p))}, \lambda^{(r/(r-q))} \left( \frac{A(1)}{(B(1))^{(s/r)}} \right)^{(r/(r-q))} \right\}, \text{ for all } x \in \Omega.
\end{aligned}
\] (59)
Thus, by (53) and (59), we get

\[ \|u\|_A \leq C_A \left( \frac{q(r-p)}{r} \text{mes}(\Omega) \right)^{\frac{1}{2}} \max \left\{ \lambda^{(r/2(r-p))} \left( \frac{A(1)}{(B(1))^{(p/r)}} \right)^{(r/2(r-p))}, \Lambda^{(r/2(r-q))} \left( \frac{A(1)}{(B(1))^{(q/r)}} \right)^{(r/2(r-q))} \right\}. \]  

(60)

Finally, the last inequality and (52) yield the result (50).

**Definition 2.** An element \( u \) of \( X \) is a weak solution of \( (\varepsilon_i) \) if

\[
\int_{\Omega} \nabla u v dx + \int_{\Omega} uv dx = \lambda \int_{\Omega} a(|u|)uv dx - \int_{\Omega} b(|u|)uv dx \quad \text{for all } v \in X.
\]

(61)

\[
\lambda_0 = \min \left\{ \left( \frac{\eta}{\kappa^\varepsilon} \right)^{(2(r-p)(p-2)/p(r-2))}, \left( \frac{\eta}{\kappa^\varepsilon} \right)^{(2(r-q)(q-2)/q(r-2))} \right\} > 0,
\]

(63)

with

\[
\eta = C_A^{-\frac{1}{2}} \left( \frac{q(r-p)}{r} \text{mes}(\Omega) \right)^{-(1/2)}, \\
\kappa = (qC_A)^{(1/(p-2))}, \\
\xi = (qC_A)^{(1/(q-2))}, \\
\theta = \left( \frac{A(1)}{(B(1))^{(p/r)}} \right)^{(r/2(r-p))}, \\
\varrho = \left( \frac{A(1)}{(B(1))^{(q/r)}} \right)^{(r/2(r-q))}.
\]

(64)

**Proof.** If \( (\varepsilon_i) \) admits a nontrivial weak solution \( u \in X \), then equality (49) is satisfied and \( \lambda > 0 \) by Lemma 9. Let us now show that \( \lambda \geq \lambda_0 \). By inequalities (50) and (60) of the proof of Lemma 9, we get the result.

We claim that the set \( E = \{ \lambda > 0 \} : (\varepsilon_i) \) admits only the trivial solution for all \( \mu < \lambda \) is not empty and bounded above. Indeed, by Lemma 9, for all \( \lambda < \lambda_0 \), \( (\varepsilon_i) \) admit only trivial solution. Thus, \( \lambda_0 \in E \). Now suppose that for all \( M > 0 \), there exists \( \lambda_M \) such that \( \lambda_M > M \). Therefore, there exists a sequence \( (\lambda_n)_{n \in \mathbb{N}} \) of elements of \( E \) such that \( 0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \). For all \( n \in \mathbb{N} \), for all \( \mu \) such that \( 0 < \mu < \lambda_n \), \( (\varepsilon_i) \) admits only trivial solution. By hypothesis, the sequence \( (\lambda_n)_{n \in \mathbb{N}} \) tends to \( +\infty \). Hence, for all \( \mu \in (0, +\infty) \), \( (\varepsilon_i) \) admits only trivial solution. This contradicts the Lemma 9.

Let us define

\[
\bar{\lambda} = \sup \{ \lambda : (\varepsilon_i) \text{ admits only the trivial solution for all } \mu < \lambda \}.
\]

(65)

It is clear that \( \bar{\lambda} \geq \lambda_0 > 0 \). In Section 5, we will prove that \( \bar{\lambda} \) is the required critical value of the Theorem 1.

### 4. Basic Results for Existence of Nontrivial Solution

The results in the previous section require us to work from now on with \( \lambda > 0 \).

**Lemma 11.** The energy functional \( \Phi_\lambda \) in coercive on \( X \).

**Proof.** Let \( u \in X \).
\[ \Phi_{\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\Omega} |u|^2 \, dx - \lambda \int_{\Omega} A(|u|) \, dx + \int_{\Omega} B(|u|) \, dx, \]
\[ \geq \frac{1}{2} \|u\|_{H^1_0(\Omega)}^2 - \frac{\lambda}{2} \int_{\Omega} A(|u|) \, dx - \frac{1}{2} \int_{\Omega} B(|u|) \, dx \]  \tag{66}

By using Young’s inequality, as we use it in (55), we have
\[ \lambda A(|u(x)|) \leq \frac{1}{2} B(|u(x)|) \leq 2^{(q/(r-q))} r \frac{\lambda}{r} c_1, \text{ for all } x \in \Omega, \]
\[ \lambda \triangleq \max\left\{ \lambda^{(r/(r-p))}(A(1)/(B(1)))^{(p/r)} r, \lambda^{(r/(q-r))}(A(1)/(B(1)))^{(p/r)} (r-q)^{(r-q)} \right\}. \]

Proof. Let \( Y = H^1_0(\Omega) \times L^p(\Omega) \), which we endow with the norm \( \|u\|_Y = \|u\|_{H^1_0(\Omega)} + \|u\|_{L^p} \). Since \( H^1_0(\Omega) \) and \( L^p(\Omega) \) are reflexive Banach spaces, then \((Y, \|\cdot\|_Y)\) is a separable and reflexive Banach space by Theorem 1.23 in [11]. Let us consider the operator \( \Gamma : (X, \|\cdot\|_X) \rightarrow (Y, \|\cdot\|_Y) \) defined by \( \Gamma(u) = (u, u) \). \( \Gamma \) is well defined, linear, and isometric. Therefore, \( \Gamma(X) \) is a closed subspace of \( Y \) and so \( \Gamma(X) \) is separable and reflexive by Theorem 1.22 in [11]. Consequently, \((X, \|\cdot\|_X)\) is a separable reflexive Banach space, being isomorphic to a separable, reflexive space. Finally, we conclude that \((X, \|\cdot\|_X)\) is a separable reflexive Banach space because reflexivity and separability are preserved under equivalent norms. \( \square \)

**Lemma 12.** Let \( \overline{A} \) and \( \overline{B} \) respectively be the complements of \( N \)-functions \( A \) and \( B \). Then, we have
\[ \frac{p}{q-1} \leq \frac{A'(s)}{A(s)} \leq \frac{q}{p-1} \quad \text{for } s > 0, \]
\[ \frac{r}{r-1} \leq \frac{B'(s)}{B(s)} \leq \frac{q}{p-1} \quad \text{for } s > 0, \]
where \( \overline{A}'(s) \) and \( \overline{B}'(s) \) are respectively the derivatives of \( \overline{A}(s) \) and \( \overline{B}(s) \).

**Proof.** The results are obtained by using (29) and (30) and the proof of the point (2.7) of Lemma 2.5 in [14]. \( \square \)

**Proposition 1.** The spaces \( L_A(\Omega) \) and \( L_B(\Omega) \) are reflexive and separable Banach spaces.

**Proof.** The \( N \)-functions \( A \) and \( B \) satisfy \( \Delta_2 \)-condition respectively by Lemmas 1 and 2. Moreover, the inequalities (69) and (70) of Lemma 12 imply that the \( N \)-functions \( \overline{A} \) and \( \overline{B} \) satisfy the \( \Delta_2 \)-condition. Thus, by (Theorem 8.20 and Remark 8.22 in [11]), the spaces \( L_A(\Omega) \) and \( L_B(\Omega) \) are reflexive and separable Banach spaces. \( \square \)

**Proposition 2.** \((X, \|\cdot\|_X)\) is a separable reflexive Banach space.

**Proof.** The convexity of functional \( \Phi_{\Delta}(u) = (1/2) \int_{\Omega} |\nabla u|^2 \, dx \) is followed from the convexity of function \( r \mapsto r^2 \) in \( \mathbb{R} \).

Let us show the convexity of \( \Phi_{\Delta}(u) \) on \( X \). Let \( (u_n) \) be a sequence of which elements are in \( X \) and let \( u \in X \) such that \( u_n \rightharpoonup u \) in \( X \). Let \( (u_n)_k \) be an arbitrary subsequence of \( (u_n) \). The subsequence \( u_n \rightharpoonup u \) in \( X \) and hence, by Lemma 3, \( \|u_n - u\| \rightarrow 0 \) in \( (L^2(\Omega))^N \). By Theorem 4.9 in
there exists a subsequence \((u_{n_j})_j\) of \(u_n\) and a function 
\(\psi \in L^1(\Omega)\) such that \(\nabla u_{n_j} \rightarrow \nabla u\) a.e in \(\Omega\) as \(j 
rightarrow \infty\) and 
\(\|\nabla u_{n_j}\| \leq \psi \) a.e in \(\Omega\) for all \(j \in \mathbb{N}\). So \(\|\nabla u_{n_j}\| \rightarrow \|\nabla u\|\) a.e in \(\Omega\) as \(j 
rightarrow \infty\) and \(\|\nabla u_{n_j}\| \leq \psi^2 \in L^1(\Omega)\) a.e in \(\Omega\) for all \(j \in \mathbb{N}\).

The dominated convergence theorem implies that 
\(\|\nabla u_{n_j}\|^2 \rightarrow \|\nabla u\|^2\) in \(L^1(\Omega)\) as \(j 
rightarrow \infty\). The subsequence \((u_{n_j})_j\) being arbitrary, we deduce that 
\(\|\nabla u_{n_j}\|^2 \rightarrow \|\nabla u\|^2\) in \(L^1(\Omega)\) as \(n 
rightarrow \infty\). Then, we get the continuity of \(\Phi^\prime_\delta\) on \(X\)
and also, \(\Phi^\prime_\delta\) is sequentially weakly lower semicontinuous in \(X\) by Corollary 3.9 in [15]. Moreover, \(\Phi^\prime_\delta\) is Gateaux-differentiable in \(X\) and for all \(u, \varphi \in X\), we have

\[
\langle \Phi^\prime_\delta(u), \varphi \rangle = \int_{\Omega} \nabla u \varphi \, dx. \quad (71)
\]

Let us finish the proof by showing that \(\Phi^\prime_\delta\) is continuous. Let \((u_n)_n\) be a sequence of functions in \(X\) and \(u \in X\) such that 
\(u_n \rightarrow u\) in \(X\) when \(n \rightarrow \infty\). By a simple calculus we have

\[
\|\Phi^\prime_\delta(u_n) - \Phi^\prime_\delta(u)\|_X \leq \|u_n - u\|_H^\prime(u) \leq \|u_n - u\|_X. \quad (72)
\]

This inequality yields the continuity of \(\Phi^\prime_\delta\).

**Lemma 15.** The functional \(\Phi_2(u) = (1/2) \int_{\Omega} |u|^2 \, dx\) is convex, of class \(C^1\) and sequentially weakly lower semicontinuous in \(X\). Furthermore, if \((u_n)_n\) is a sequence of elements of \(X\) and \(u\) belongs to \(X\) such that \(u_n\) converge weakly to \(u\) in \(X\), then 
\(\Phi^\prime_2(u_n) \rightarrow \Phi^\prime_2(u)\) in \(X^*\).

**Proof.** The convexity of the functional \(\Phi_2\) is obvious. The continuity of \(\Phi_2\) in \(X\) follows from Lemma 3. Thus, \(\Phi_2\) is sequentially weakly lower semicontinuous in \(X\) by Corollary 3.9 in [15]. To complete the proof of the theorem, it suffices to show the last part of the theorem. Therefore, let us take \((u_n)_n \subset X\) and \(u \in X\) such that \(u_n \rightarrow u\) weakly in \(X\). Since \(X \rightarrow L^2(\Omega)\) is compact by lemma 3, we have 
\(\|\Phi^\prime_2(u_n) - \Phi^\prime_2(u)\|_X \leq \|u_n - u\|_2\) and then \(\Phi^\prime_2(u_n) \rightarrow \Phi^\prime_2(u)\) in \(X^*\).

**Proposition 3.** (i) If \(u_n\) converge strongly to \(u\) in \(L^2(\Omega)\), then 
\(a(|u_n|)u_n \rightarrow a(|u|)u\) in \(L^2(\Omega)\). (ii) If \(u_n\) converge strongly to \(u\) in \(L^2(\Omega)\), then 
\(b(|u_n|)u_n \rightarrow b(|u|)u\) in \(L^2(\Omega)\).

**Proof.** (i) Suppose that \(u_n\) converge strongly to \(u\) in \(L^2(\Omega)\). Let \((v_n)_k\) be a fixed subsequence of the sequence \(n \rightarrow a(|u_n|)u_n\). For all \(k\), we have \(v_n = a(|u_n|)u_n\), where \((v_n)_k\) is a subsequence of the sequence \((u_n)_n\), which converge to \(u\) in \(L^2(\Omega)\). Thus, 
\(\int_{\Omega} A(a(|u_n|)u_n - a(|u|)u) \, dx \rightarrow 0\) as \(k \rightarrow \infty\). It follows that, there exists a subsequence \((u_{n_k})_k\) of \((u_n)_n\) and a positive function 
\(\psi \in L^1(\Omega)\) such that 
\(A(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \rightarrow 0\) a.e in \(\Omega\) and 
\(0 \leq A(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \leq \psi(x)\) a.e in \(\Omega\), for all \(j \in \mathbb{N}\). As \(A\) is continuous and strictly increasing in \(\mathbb{R}_+\), we have 
\(u_{n_k} \rightarrow u\) a.e in \(\Omega\). Hence, 
\(v_n \rightarrow v = a(|u|)u\) a.e in \(\Omega\) and 
\(A(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \rightarrow 0\) a.e in \(\Omega\). On other hand, there exists \(h \in L^1(\Omega)\) such that 
\(A(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \leq h(x)\) a.e in \(\Omega\). Indeed, by the fact that 
\(A(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \leq \psi(x)\) a.e in \(\Omega\) and the function \(t \rightarrow a(t)\) is continuous and increases strictly in \(\mathbb{R}_+\), we have

\[
\tilde{A}(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \leq \tilde{A}(a(|u|)u) = A(a(|u|)u) + \tilde{A}(a(|u|)u) \leq A(a(|u|)u) + |a(|u|)u|, \quad \text{for } x \in \Omega a.e. \quad (73)
\]

By (13) and since \(\tilde{A}\) satisfy \(\Delta_2\)-condition, we have

\[
\tilde{A}(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \leq \delta \tilde{A}(A^{-1}(\psi(x)) + |u|), \quad \text{for } x \in \Omega a.e, \quad (74)
\]

where \(\delta\) and \(\delta'\) are constants. Thus, \(h(x) = \delta A^{-1}(\psi(x), |u(x)|)\). In consequence, it yields that 
\(\tilde{A}(v_{n_k} - v) = A(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \rightarrow 0\) in \(L^1(\Omega)\), as \(j \rightarrow \infty\). Due to arbitrariness of \((v_{n_k})_k\), we deduce that 
\(\tilde{A}(v_{n_k} - v) = A(a(|u_{n_k}|)u_{n_k} - a(|u|)u) \rightarrow 0\) in \(L^1(\Omega)\), as \(n \rightarrow \infty\). Hence, \(a(|u_{n_k}|)u_{n_k} \rightarrow a(|u|)u\) in \(L^2(\Omega)\).

(ii) The proof is analogous to (i).

**Lemma 16.** The functional \(\Phi_\delta(u) = \int_{\Omega} A(|u|) \, dx\) is convex, of class \(C^1\) and sequentially weakly lower semicontinuous in \(X\).
where \( \varphi_x(t) = A(k|t| - kA(|t|)), \quad \psi_x(t) = 2A(|C_x|t) \), for all \( t \in \mathbb{R} \) with \( (1/C_x) = \varepsilon(k - 1) \).

Let \((u_n)_n\) be a sequence of \( X \) and \( u \) an element of \( X \) such that \( u_n \to u \) in \( X \). \( u \) belongs to \( L_B(\Omega) \) by Lemma 8 and we have \( \int_\Omega A(|u|)dx < \infty \). \( u_n \to u \) in \( X \) implies \( u_n \to u \) in \( H_B^1(\Omega) \). So, \( u_n \to u \) a.e in \( \Omega \). By (35), properties of function \( \zeta_1 \) and Lemma 8, we have

\[
0 \leq \int_\Omega \varphi_x(u_n - u)dx = \int_\Omega A(|u_n - u|) - kA(|u_n - u|)dx \\
\leq \zeta_1(k\|u_n - u\|_X) \leq k^2\zeta_1(\|u_n - u\|_X). \\
(76)
\]

As \( u_n \to u \) in \( X \), there exists \( M > 0 \) such that for all integer \( n \),

\[
0 \leq \int_\Omega \varphi_x(u_n - u)dx \leq k^2M. \\
(77)
\]

Since \( u_n \to u \) in \( X \), by Lemma 8 and Proposition 3, it follows that \( \|a(|u_n|)u_n - a(|u|)u\|_A \to 0 \) and consequently, \( \Phi'_A(u_n) \to \Phi'_A(u) \) in \( X' \). In particular, this shows that \( \Phi_A \) is Class \( C^1 \) in \( X \) and the proof of the Lemma is finished. \( \square \)

**Lemma 17.** The functional \( \Phi_B(u) = \int_\Omega B(|u|)dx \) is convex, of class \( C^1 \) and sequentially weakly lower semicontinuous in \( X \). Moreover, if \((u_n)_n\) is a sequence of elements in \( X \) and \( u \in X \) such that \( u_n \to u \) in \( X \) as \( n \to \infty \), then \( \Phi_B(u_n) \to \Phi_B(u) \) in \( X' \).

**Proof.** The convexity of \( \Phi_B \) follows from the convexity of the positive function \( \varphi_B(t) = B(k|t| - kB(|t|)) \) defined on \( \mathbb{R} \). The proof of the convexity of \( \Phi_B \) in \( X \) is analogous to the proof of the convexity of \( \Phi_A \) in \( X \) in the previous Lemma. In fact, by Lemma 3 in [16], for all \( k > 1 \), for \( \varepsilon > 0 \), such that \( 0 < \varepsilon < 1/(k) \), we have

\[
|B(x + y)| - B(|x|) \leq \varepsilon \varphi_{-1}(x + \psi_{-1}(y)), \quad \text{for all } x, y \in \mathbb{R},
\]

where \( \varphi_{-1}(t) = B(k|t|) - kB(|t|), \quad \psi_{-1}(t) = 2B(|C_x|t) \), for all \( t \in \mathbb{R} \) with \( (1/C_x) = \varepsilon(k - 1) \).

Let \((u_n)_n \subset X \) and \( u \in X \) such that \( u_n \to u \) in \( X \). By the definition of the space \( X \), we have \( X \to L_B(\Omega) \),

\[
\Phi_B'(u_n) = \sup_{\|\varphi\|_{X'} = 1} \int_\Omega [b(|u_n|)u_n - b(|u|)u] \varphi dx \\
\leq 2 \sup_{\|\varphi\|_{X'} = 1} [b(|u_n|)u_n - b(|u|)u] \|\varphi\|_{X'}.
\]

Since \( u \) belongs to \( L_B(\Omega) \), \( \Phi_B(|u|) \in L^1(\Omega) \) for all \( \theta \in \mathbb{R} \) and thus,

\[
0 \leq \int_\Omega \psi_x(u)dx < \infty, \text{ for all } \varepsilon > 0.
\]

Hence, by Theorem 2 in [16], it follows that \( \Phi_B(u_n) \to \Phi_B(u) \) as \( n \to \infty \). This assures the continuity of the functional \( \Phi_B \). It then follows that \( \Phi_B \) is sequentially weakly lower semicontinuous in \( X \) by Corollary 3.9 in [15]. Furthermore, \( \Phi_B \) is Gateaux-differentiable in \( X \) and for all \( u, \varphi \in X \),

\[
\langle \Phi_B'(u), \varphi \rangle = \int_\Omega a(|u(x)|)|u(x)|\varphi(x)dx.
\]

Let \((u_n)_n \subset X \) and \( u \in X \) such that \( u_n \to u \) in \( X \) as \( n \to \infty \). Let us show that \( \Phi_B'(u_n) \to \Phi_B'(u) \) in \( X' \). By Hölder’s inequality, we have

\[
\Phi_B'(u_n) = \sup_{\|\varphi\|_{X'} = 1} \int_\Omega [b(|u_n|)u_n - b(|u|)u] \varphi dx \\
\leq 2 \sup_{\|\varphi\|_{X'} = 1} [b(|u_n|)u_n - b(|u|)u] \|\varphi\|_{X'}.
\]

Since \( u_n \to u \) in \( X \), by Lemma 8 and Proposition 3, it follows that \( \|a(|u_n|)u_n - a(|u|)u\|_A \to 0 \) and consequently, \( \Phi_B'(u_n) \to \Phi_B'(u) \) in \( X' \). In particular, this shows that \( \Phi_B \) is Class \( C^1 \) in \( X \) and the proof of the Lemma is finished. \( \square \)

**Lemma 17.** The functional \( \Phi_B(u) = \int_\Omega B(|u|)dx \) is convex, of class \( C^1 \) and sequentially weakly lower semicontinuous in \( X \). Moreover, if \((u_n)_n \) is a sequence of elements in \( X \) and \( u \in X \) such that \( u_n \to u \) in \( X \) as \( n \to \infty \), then \( \Phi_B(u_n) \to \Phi_B(u) \) in \( X' \).

Proof. The convexity of \( \Phi_B \) follows from the convexity of the positive function \( \varphi_B(t) = B(k|t| - kB(|t|)) \) defined on \( \mathbb{R} \). The proof of the convexity of \( \Phi_B \) in \( X \) is analogous to the proof of the convexity of \( \Phi_A \) in \( X \) in the previous Lemma. In fact, by Lemma 3 in [16], for all \( k > 1 \), for \( \varepsilon > 0 \), such that \( 0 < \varepsilon < 1/(k) \), we have

\[
|B(x + y)| - B(|x|) \leq \varepsilon \varphi_{-1}(x + \psi_{-1}(y)), \quad \text{for all } x, y \in \mathbb{R},
\]

where \( \varphi_{-1}(t) = B(k|t|) - kB(|t|), \quad \psi_{-1}(t) = 2B(|C_x|t) \), for all \( t \in \mathbb{R} \) with \( (1/C_x) = \varepsilon(k - 1) \).

Let \((u_n)_n \subset X \) and \( u \in X \) such that \( u_n \to u \) in \( X \). By the definition of the space \( X \), we have \( X \to L_B(\Omega) \),

\[
\Phi_B'(u_n) = \sup_{\|\varphi\|_{X'} = 1} \int_\Omega [b(|u_n|)u_n - b(|u|)u] \varphi dx \\
\leq 2 \sup_{\|\varphi\|_{X'} = 1} [b(|u_n|)u_n - b(|u|)u] \|\varphi\|_{X'}.
\]

Since \( u \) belongs to \( L_B(\Omega) \), \( \Phi_B(|\theta u|) \in L^1(\Omega) \) for all \( \theta \in \mathbb{R} \) and thus,

\[
0 \leq \int_\Omega \psi_x(u)dx < \infty, \text{ for all } \varepsilon > 0.
\]

Hence, by Theorem 2 in [16], it follows that \( \Phi_B(u_n) \to \Phi_B(u) \) as \( n \to \infty \). This assures the continuity of the functional \( \Phi_B \). It then follows that \( \Phi_B \) is sequentially weakly lower semicontinuous in \( X \) by Corollary 3.9 in [15]. Furthermore, \( \Phi_B \) is Gateaux-differentiable in \( X \) and for all \( u, \varphi \in X \),

\[
\langle \Phi_B'(u), \varphi \rangle = \int_\Omega a(|u(x)|)|u(x)|\varphi(x)dx.
\]

Let \((u_n)_n \subset X \) and \( u \in X \) such that \( u_n \to u \) in \( X \) as \( n \to \infty \). Let us show that \( \Phi_B(u_n) \to \Phi_B(u) \) in \( X' \). By Hölder’s inequality, we have

\[
\Phi_B'(u_n) = \sup_{\|\varphi\|_{X'} = 1} \int_\Omega [b(|u_n|)u_n - b(|u|)u] \varphi dx \\
\leq 2 \sup_{\|\varphi\|_{X'} = 1} [b(|u_n|)u_n - b(|u|)u] \|\varphi\|_{X'}.
\]
According to Proposition 3 (ii), it follows that $\|b(\|u_n\|)u_n - b(\|u\|)u\|_B \to 0$ and consequently, $\Phi'_B(u) \to \Phi'_B(u)$ in $X'$. Thus, $\Phi_B$ is class $C^1$ in $X$.

Let us show the last part of this Lemma. Let $(u_n) \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in $X$. We shall prove $\Phi_B(u_n) \to \Phi_B(u)$ in $X'$ as $n \to \infty$. For any integer $n$, let us set $v_n = b(\|u_n\|)u_n$ and consider an arbitrary subsequence $(v_{n_k})$ of the sequence $(v_n)$. We have

$$
\left\| \left[ b(\|u_{n_k}\|)u_{n_k} - b(\|u\|)u \right] \right\| \leq \left( b(\|u_{n_k}\|) + b(\|u\|) \right) |\varphi|,
$$

$$
\leq rB(1)|\varphi| \max \left\{ \left( u_{n_k} \right)^{g^{-1}}, \left| u_{n_k} \right|^{g^{-1}} \right\} + \max \left( |u|^{g^{-1}}, |u|^{g^{-1}} \right) \text{a.e in } \Omega,
$$

$$
\leq rB(1)|\varphi| \max \left( |u|^{g^{-1}}, |u|^{g^{-1}} \right) \text{a.e in } \Omega.
$$

Thus, $h = rB(1)|\varphi| \max \left( |u|^{g^{-1}}, |u|^{g^{-1}} \right) \in L^1(\Omega)$. Hence,

$$
\int_{\Omega} \left[ v_{n_k} - b(\|u\|)u \right] \varphi \, dx = \int_{\Omega} \left[ b(\|u_{n_k}\|)u_{n_k} - b(\|u\|)u \right] \varphi \, dx \to 0.
$$

In conclusion, $\Phi'_B(u_n) \rightharpoonup \Phi'_B(u)$ in $X'$, as $n \to \infty$, since $(v_{n_k})$ is an arbitrary subsequence of the sequence $(v_n)$.

For any $(x, u) \in \Omega \times \mathbb{R}$, let us set

$$
f(x, u) = \lambda a(\|u\|)u - b(\|u\|)u,
$$

$$
F(x, u) = \int_0^u f(x, v(x)) \, dx
$$

(88)

$$
= \lambda A(\|u\|) - B(\|u\|).
$$

\[\square\]

$$
|\mathcal{F}_u(\nu)| \leq \lambda \int_{\Omega} a(\|u\|)|\varphi| \, dx + \int_{\Omega} b(\|u\|)|\varphi| \, dx \leq 2\lambda \|a(\|u\|)\|_A \|\varphi\|_A + 2\|b(\|u\|)\|_B \|\varphi\|_X.
$$

(90)

Thus, we conclude that $\mathcal{F}_u$ is continuous on $X$. Then, it follows that, $v_n \rightharpoonup v$ in $X$ yields $\mathcal{F}_u(v_n) \to \mathcal{F}_u(v)$.

\[\square\]

**Lemma 19.** The functional $\Phi_\lambda$ is of class $C^1$ and sequentially weakly lower semicontinuous in $X$, that is if $u_n \rightharpoonup u$ in $X$, then $v_n = b(\|u_n\|)u_n$, where $(u_n)_k$ is a subsequence of $(u_n)$. Then, $u_n \rightharpoonup u$ weakly in $X$ when $k \to \infty$. By Lemma 3, it follows that $u_n \to u$ in $L^2(\Omega)$. Thus, there exists a subsequence $(u_{n_k})_k$ of $(u_n)_k$ such that $u_{n_k} \to u$ a.e in $\Omega$ and $|u_{n_k}| \leq g$ a.e in $\Omega$, where $g \in L^1(\Omega)$. In consequence, for all $\varphi \in X$, $v_{n_k} \varphi \to b(\|u\|)u \varphi$ a.e in $\Omega$ and there exists $h \in L^1(\Omega)$ such that $|b(\|u_{n_k}\|)u_{n_k} \varphi - b(\|u\|)u \varphi| \leq h$ a.e in $\Omega$. In fact, by (30) and (36) we have

$$
\int_{\Omega} \left( v_{n_k} - b(\|u\|)u \right) \varphi \, dx = \int_{\Omega} \left[ b(\|u_{n_k}\|)u_{n_k} - b(\|u\|)u \right] \varphi \, dx \to 0.
$$

\[\square\]

**Lemma 18.** For any fixed $u \in X$, the functional $\mathcal{F}_u : X \to \mathbb{R}$, $v \mapsto \int_{\Omega} f(x, u(x))v(x) \, dx$ is in $X'$ ($X$ is the Banach dual space of $X$). In particular, if $v_n \rightharpoonup v$ in $X$, then $\mathcal{F}_u(v_n) \to \mathcal{F}_u(v)$.

**Proof.** Let us take $u \in X$. It is easy to show that $\mathcal{F}_u$ is linear. Moreover, for all $v \in X$, by using Hölder inequality and imbedding properties of $X$ in $L_A(\Omega)$ and $X \times \mathbb{R}$, we have

$$
\Phi'_A(u) \leq \liminf_{n \to \infty} \Phi'_A(u_n).
$$

\[\square\]

**Lemma 19.** The functional $\Phi_\lambda$ is of class $C^1$ and sequentially weakly lower semicontinuous in $X$, that is if $u_n \rightharpoonup u$ in $X$, then

$$
\Phi_\lambda(u) \leq \liminf_{n \to \infty} \Phi_\lambda(u_n).
$$

**Proof.** Lemmas 14–17 imply $\Phi_\lambda \in C^1(X)$. Let $(u_n)_n \in X$ and $u \in X$ such that $u_n \rightharpoonup u$ in $X$. The definition of $\Phi_\lambda$ and (89) allow us to write
\[\Phi_\lambda (u) - \Phi_\lambda (u_n) = \frac{1}{2} (\|u\|_{H^1_0(\Omega)}^2 - \|u_n\|_{H^1_0(\Omega)}^2) + \frac{1}{2} \left(\|u\|_2^2 - \|u_n\|_2^2\right) + \int_\Omega \left[ F(x, u_n(x)) - F(x, u(x)) \right]. \]  

(92)

Since \(u_n \rightharpoonup u\) in \(X\), Lemmas 14 and 15 yield respectively that

\[\text{lim sup}_{n \to \infty} [\Phi_\lambda (u) - \Phi_\lambda (u_n)] \leq \text{lim sup}_{n \to \infty} \int_\Omega [ F(x, u_n(x)) - F(x, u(x))] .\]  

(95)

By (88) and (89), for all \(s \in [0,1]\),

\[F_u(x, u + s(u_n - u)) = \frac{\partial F}{\partial u} (x, u + s(u_n - u)) \]
\[= f(x, u + s(u_n - u)) \]
\[= f(x, u) + (u_n - u) \int_0^s f_u(x, u + t(u_n - u))dt .\]  

(96)

where \(f_u(x, u) = (\partial f/\partial u)(x, u) = \lambda |u|a'(|u|) + a(|u|) - |u|b'(|u|) + b(|u|))\), for all \(u \in X\).

Multiplying (96) by \((u_n - u)\) and integrating the result over \([0,1]\) with respect to \(s\), we obtain

\[F(x, u_n) - F(x, u) = f(x, u)(u_n - u) + (u_n - u)\int_0^1 \left(\int_0^s f_u(x, u + t(u_n - u))dt \right)ds.\]  

(97)

According to the assumptions \((\mathcal{A}_2)\) and \((\mathcal{B}_2)\), we have for all \(u \in X\), \(f_u(x, u) \leq (q - 1) [\lambda A(|u|) - B(|u|)]|u|^{-2}\).  

(98)

By (24), for all \(u \in X\setminus\{0\}\), we have

\[|\lambda A(|u(x)|) - B(|u(x)|)|u|^{-2} \leq \begin{cases} \frac{\lambda A(1)|u(x)|^{p-2} - B(1)|u(x)|^{r-2}}{\lambda A(1)|u(x)|^{r-2} - B(1)|u(x)|^{r-2}}, & \text{for } x \in \Omega_u, \\ \frac{\lambda A(1)|u(x)|^{p-2}}{\lambda A(1)|u(x)|^{r-2}}, & \text{for } x \in \Omega_u. \end{cases}\]  

(100)

Let \(s = p - 2\) or \(s = q - 2\). By applying Young’s inequality \(ab \leq (a^\alpha/\alpha) + (b^\beta/\beta)\) for \(a, b > 0\) and \(\alpha > 1\) and \(\beta > 1\), we have

\[\lambda A(1)|u(x)|^s \leq \frac{s}{r-2} B(1)|u(x)|^{r-2} + \frac{r-2-s}{r-2} \left(\frac{\lambda A(1)}{(B(1))^{(r/(r-2))}}\right)^{(r-2)/(r-2-s)} ,\]  

(101)
where we take \( a = (B(1))^{\alpha/(r-2)}|u(x)|^\alpha \), \( b = \lambda A(1) (B(1))^{-(\alpha/(r-2))} \), \( \alpha = ((r-2)/s) > 1 \), and \( \beta = ((r-2)/(r-2-s)) > 1 \). Inequality (101) yields

\[
\lambda A(1)|u(x)|^\alpha - B(1)|u(x)|^{\alpha-2} \leq \frac{r-2-s}{r-2} \left( \frac{\lambda A(1)}{(B(1))^{\alpha/(r-2)}} \right)^{(r-2)/(r-2-s)},
\]

being \( s < r - 2 \). Hence, (100) yields

\[
[\lambda A(|u(x)|) - B(|u(x)|)]|u(x)|^{\alpha-2} \leq \begin{cases} 
\lambda^{(r/2)/(r-p)} \frac{r - p}{r - 2} \left( \frac{A(1)}{(B(1))^{1/(p-2)/(r-2)}} \right)^{(r-2)/(r-p)}, & \text{for } x \in \Omega_u, \\
\lambda^{(r/2)/(r-q)} \frac{r - q}{r - 2} \left( \frac{A(1)}{(B(1))^{1/(q-2)/(r-2)}} \right)^{(r-2)/(r-q)}, & \text{for } x \in \Omega_u'.
\end{cases}
\]

As \( r > (r - p)/(r - 2)(r - q/r - 2) \), it follows, from (104), that

\[
[\lambda A(|u(x)|) - B(|u(x)|)]|u(x)|^{\alpha-2} \leq \frac{r - p}{r - 2} c_1',
\]

for all \( x \in \Omega \),

\[
c_1' = \max \left\{ \lambda^{(r-2)/(r-p)} \left( \frac{A(1)}{(B(1))^{1/(p-2)/(r-2)}} \right)^{(r-2)/(r-p)}, \lambda^{(r-2)/(r-q)} \left( \frac{A(1)}{(B(1))^{1/(q-2)/(r-2)}} \right)^{(r-2)/(r-q)} \right\}.
\]

From (104) and (105), we obtain

\[
\text{for all } u \in X, \ f_u(x, u) \leq \frac{q(1)(r - p)}{r - 2} c_1',
\]

Consequently, (97) yields

\[
\int_{\Omega} [F(x, u_n) - F(x, u)] dx \leq \int_{\Omega} f(x, u)(u_n - u) dx + \frac{q(1)(r - p)}{2} c_1'/2 (r - 2) \int_{\Omega} (u_n - u)^2 dx
\]

\[
\leq \mathcal{F}_u(u_n - u) + \frac{q(1)(r - p)}{2} c_1'/2 (r - 2) \|u_n - u\|_{\mathbf{L}^2}.
\]

Since \( u_n \to u \) in \( X \), Lemmas 3 and 18 yield respectively \( \mathcal{F}_u(u_n - u) \to 0 \) and \( \|u_n - u\|_{\mathbf{L}^2} \to 0 \). Thus, \( \limsup_{n \to \infty} \int_{\Omega} [F(x, u_n(x)) - F(x, u(x))] dx \leq 0 \) and then, by (95), we get the claim (91). □

5. Existence of Weak Solutions of (\( \varepsilon_1 \)) for Large Values of \( \lambda \)

Let
\[
\overline{\lambda} = \inf_{\Phi_{\lambda}(u) \in \lambda} \left\{ \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + |u|^2) + \int_{\Omega} B(|u(x)|)dx \right\} \text{with } \Phi_{\lambda}(u)
\]
\[
= \int_{\Omega} A(|u(x)|)dx.
\]

(109)

Thus,
\[
\overline{\lambda} \geq \frac{1}{2C_A} + \zeta_2 \left( \frac{1}{H} \right) > 0.
\]

(112)

**Lemma 20.** For all \( \lambda > \overline{\lambda} \), the functional \( \Phi_{\lambda} \) admits in \( X \) a global nontrivial non-negative minimizer \( m \) with negative energy, that is \( \Phi_{\lambda}(m) < 0 \).

**Proof.** For each \( \lambda > 0 \), Lemmas 11 and 19 and Corollary 3.23 in [15] assure to \( \Phi_{\lambda} \) the existence of a global minimizer \( m \in X \), that is
\[
\Phi_{\lambda}(m) = \inf_{u \in X} \Phi_{\lambda}(u).
\]

(113)

So, \( m \) is a solution of \((\epsilon_1)\). Let us prove that \( m \) is nontrivial as soon as \( \lambda > \overline{\lambda} \). For that we shall just prove that \( \inf_{u \in X} \Phi_{\lambda}(u) < 0 \). Let \( \lambda > \overline{\lambda} \). By the definition of \( \overline{\lambda} \), there exists a function \( \varphi \in X \), with \( \int_{\Omega} A(\varphi(x))dx = 1 \), such that
\[
\lambda \int_{\Omega} A(\varphi(x))dx = \lambda > \frac{1}{2} \int_{\Omega} (|\nabla \varphi|^2 + |\varphi(x)|^2)
\]
\[
+ \int_{\Omega} B(|\varphi(x)|)dx \geq \overline{\lambda}.
\]

Hence,
\[
\frac{1}{2} \int_{\Omega} (|\nabla \varphi|^2 + |\varphi(x)|^2) - \lambda \int_{\Omega} A(\varphi(x))dx + \int_{\Omega} B(|\varphi(x)|)dx < 0
\]

(115)

and then
\[
\Phi_{\lambda}(\varphi) < 0.
\]

(116)

We remark that \( \overline{\lambda} > 0 \). Indeed, by the fact that \( \Phi_{\lambda}(u) = 1 \Rightarrow \|u\|_A = 1 \), (48) implies that

\[\text{for all } u \in X \text{ with } \Phi_{\lambda}(u) = 1.\]

(110)

By the property (37), Lemma 5 and the monotony of function \( \zeta_2 \), we have

\[\text{for all } u \in X \text{ with } \Phi_{\lambda}(u) = 1.\]

(111)

\[\Phi_{\lambda}(\varphi) < 0.\]

(117)

Therefore,
\[
\Phi_{\lambda}(m) = \inf_{u \in X} \Phi_{\lambda}(u) \leq \Phi_{\lambda}(\varphi) < 0.
\]

(118)

Hence, it follows that, for \( \lambda > \overline{\lambda} \), equation \((\epsilon_1)\) has a nontrivial weak solution \( m \in X \) with \( \Phi_{\lambda}(m) < 0 \). To complete the proof, we can assume \( m \geq 0 \text{ a.e in } \Omega \) because \( |m| \in X \) and \( \Phi_{\lambda}(m) = \Phi_{\lambda}(|m|) \).

Let us define
\[
\overline{\lambda} = \inf \{ \lambda > 0 : (\epsilon_1) \text{has an non trivial weak solution} \}.
\]

(119)

By Lemma 20, it is clear that this definition is meaningful. Furthermore, it follows easily that \( \overline{\lambda} > \overline{\lambda} \). \( \square \)

**Theorem 3.** For any \( \lambda > \overline{\lambda} \), the problem \((\epsilon_1)\) admits a nontrivial non-negative weak solution \( u_1 \in X \).

**Proof.** Fix \( \lambda > \overline{\lambda} \). By definition of \( \overline{\lambda} \), there exists \( h \in (\overline{\lambda}, \lambda) \) such that \( \Phi_h \) has a nontrivial critical point \( u_h \in X \). Without loss of generality, we assume that \( u_h \geq 0 \text{ a.e in } \Omega \), since \( |u_h| \) is also a solution of \((\epsilon_h)\). We can easily see that \( u_h \) is a subsolution for \((\epsilon_1)\). Let us consider the following minimization problem:
\[
\inf_{u \in \mathcal{C}} \Phi_{\lambda}(u) \text{ where } \mathcal{C} = \{ v \in X : \text{forall } \Omega, v(x) \geq u_h(x) \}.
\]

(119)

We remark easily that \( \mathcal{C} \) is a closed and convex set. Thus, \( \mathcal{C} \) is weakly closed. Moreover, being \( \Phi_{\lambda} \) is coercive in \( X \) by Lemma 11, it follows that it is coercive in \( \mathcal{C} \). Finally, \( \Phi_{\lambda} \) is sequentially weakly lower semicontinuous in \( X \) and so in \( \mathcal{C} \).
Hence, Corollary 3.23 of [15] implies that there exists $u_\lambda \in C^0(\Omega)$ such that
\[ \Phi_\lambda (u_\lambda) = \inf_{v \in C^0(\Omega)} \Phi_\lambda (v). \] (120)

Now, we shall prove that $u_\lambda$ is a solution of $(\epsilon \lambda)$. Let $\theta \in C^0_0(\Omega)$ and $\epsilon > 0$. Let us set $\theta_\epsilon = \max\{0, u_h - u_\lambda - \epsilon \theta\}$ and $\omega_\epsilon = u_\lambda + \epsilon \theta + \theta_\epsilon$. It is clear that $\omega_\epsilon \in C^0(\Omega)$ and
\[ 0 \leq \langle \Phi_\lambda'(u_\lambda), \omega_\epsilon - u_\lambda \rangle = \epsilon \langle \Phi_\lambda'(u_\lambda), \theta \rangle + \langle \Phi_\lambda'(u_\lambda), \theta_\epsilon \rangle. \] (121)

Hence
\[ \langle \Phi_\lambda'(u_\lambda), \theta \rangle \geq - \frac{1}{\epsilon} \langle \Phi_\lambda'(u_\lambda), \theta_\epsilon \rangle. \] (122)

\[ \langle \Phi_\lambda'(u_\lambda), \theta \rangle \leq \langle \Phi_\lambda'(u_\lambda) - \Phi_\lambda'(u_h), \theta \rangle \\
\leq \int_{\Omega} (\nabla u_\lambda - \nabla u_h) : \nabla (u_h - u_\lambda - \epsilon \theta) dx + \int_{\Omega} (u_h - u_\lambda) (u_h - u_\lambda - \epsilon \theta) dx \\
- \int_{\Omega} [f(u_\lambda) - f(u_h)] \cdot (u_h - u_\lambda - \epsilon \theta) dx \\
\leq \int_{\Omega} (\nabla u_\lambda - \nabla u_h) : \nabla (-\epsilon \theta) dx + \int_{\Omega} (u_h - u_\lambda) (u_h - u_\lambda - \epsilon \theta) dx \\
- \int_{\Omega} [f(u_\lambda) - f(u_h)] \cdot (u_h - u_\lambda - \epsilon \theta) dx. \] (125)

(i) For $\Omega_\epsilon = \emptyset$, it is clear that $\langle \Phi_\lambda'(u_\lambda), \theta \rangle \leq 0$. Hence by (122), we have $\langle \Phi_\lambda'(u_\lambda), \theta \rangle \geq 0$.

\[ \langle \Phi_\lambda'(u_\lambda), \theta \rangle \leq \epsilon \left( \int_{\Omega} (\nabla u_\lambda - \nabla u_h) : \nabla (\theta) dx + \int_{\Omega} |(u_h - u_\lambda)| |\theta| dx + \int_{\Omega} |(u_h - u_\lambda)| |\theta| dx \right) \leq \epsilon \int_{\Omega} \psi(x) dx, \] (126)

where $\psi(x) = (\nabla u_h - \nabla u_\lambda) : \nabla \theta + |(u_h - u_\lambda)| + |f(u_\lambda) - f(u_h)| |\theta|$. We claim that $\psi \in L^1(\text{supp} \theta)$. Indeed, $\nabla u_h$ and $\nabla u_\lambda$ are in $[L^2(\Omega)]^N$ and $u_h$ and $u_\lambda$ are in $L^2_{\text{loc}}(\Omega)$ since $X \subset H^1_0(\Omega)$. Moreover, by (A1) and (B1), the functions $t \to a(t)$ and $t \to b(t)$ are continuous in $\mathbb{R}_+$. Thus, the functions $a(|u_\lambda|)u_\lambda$, $a(|u_h|)u_h$, $b(|u_\lambda|)u_\lambda$, and $b(|u_h|)u_h$ are continuous in $\Omega$. Consequently, $a(|u_\lambda|)u_\lambda$, $a(|u_h|)u_h$, $b(|u_\lambda|)u_\lambda$, and $b(|u_h|)u_h$ belong to $L^1_{\text{loc}}(\Omega)$. Finally, $|f(u_\lambda) - f(u_h)| \in L^1_{\text{loc}}(\Omega)$ since $|f(u_\lambda) - f(u_h)| \leq \lambda (a(|u_\lambda|)u_\lambda + a(|u_h|)u_h) + b(|u_\lambda|)u_\lambda + b(|u_h|)u_h)$. Therefore, the claim is obtained. Thus, since $|\Omega_\epsilon| \to 0$ as $\epsilon \to 0^+$. Then, (126) implies that $\langle \Phi_\lambda'(u_\lambda), \theta \rangle \leq \epsilon \to 0^+$. So, by (122), it follows that $\langle \Phi_\lambda'(u_\lambda), \theta \rangle \geq 0$.

We deduce that, $\langle \Phi_\lambda'(u_\lambda), \theta \rangle \geq 0$ for all $\theta \in C^0_0(\Omega)$, that is $\Phi_\lambda'(u_\lambda), \theta \rangle = 0$ for all $\theta \in C^0_0(\Omega)$. Thus, $u_\lambda$ is a weak solution of $(\epsilon \lambda)$ in $X$ because $X$ is the completion of $C^0_0(\Omega)$ with respect to the norm $\| \cdot \|_X$. Finally, $u_\lambda$ is nontrivial and non-negative, since $u_\lambda > u_h$.

Lemma 21. $\bar{\lambda} = \lambda$.

Proof. This proof is the same to that one did in the step 5 of the proof of (Theorem 1.1 in [7]). In fact, by Theorem 3, we have $\lambda \geq \bar{\lambda}$. Indeed, for all $\lambda$ such that $(\epsilon \lambda)$ a weak nontrivial
solution, we have $\lambda > \tilde{\lambda}$ and thus, $\tilde{\lambda} \geq \lambda$. Let us show that $\tilde{\lambda} \leq \lambda$. Suppose that $\lambda > \tilde{\lambda}$. For $\lambda < \tilde{\lambda}$, the problem $(\varepsilon_\lambda)$ cannot admit a nontrivial solution $u \in X$ since this would contradict the minimality of $\lambda$. Thus, for all $\lambda \in [\tilde{\lambda}, \lambda)$ the unique solution of $(\varepsilon_\lambda)$ is $u \equiv 0$. This assertion is still again impossible because it would contradict the maximality of $\lambda$. Hence, $\lambda = \tilde{\lambda}$. □

**Theorem 4.** The problem $(\varepsilon_\lambda)$ admits a nontrivial non-negative weak solution in $X$.

Proof. Let $(\lambda_n)_{n \in \mathbb{N}^*}$ be a strictly decreasing sequence converging to $\tilde{\lambda}$ and $u_n \in X$ be a nontrivial non-negative weak solution of $(\varepsilon_{\lambda_n})$ for all $n \in \mathbb{N}^*$. For all $n$, we have

$$\int_{\Omega} \nabla u_n \nabla v dx = -\int_{\Omega} u_n v dx + \lambda_n \int_{\Omega} a(|u_n|)u_n v dx - \int_{\Omega} b(|u_n|)u_n v dx,$$

for all $v \in X$. \hspace{1cm} (128)

$$C_{\lambda_n} = \max \left\{ \lambda_n^{1/p}(A(1)/(B(1))^{1/p}), \lambda_n^{1/q/2}(r-p)/(r-q), \lambda_n^{1/q/2}(1/(r-q)) \right\},$$

for all $n \in \mathbb{N}^*$. \hspace{1cm} (131)

Thus, by monotony of sequence $(\lambda_n)_{n \in \mathbb{N}^*}$, we get

$$\|u_n\|_{H^1_0(\Omega)} + qC_{\lambda_n}(\|u_n\|_B) \leq K_A q C_{\lambda_n}, \quad \text{for all } n \in \mathbb{N}^*. \hspace{1cm} (132)$$

By (29), (30), (49), and Lemma 4 we have

$$\|u\|_{H^1_0(\Omega)} + qC_{\lambda_1}(\|u\|_B) \leq qC_{\lambda_1}(\|u\|_\lambda). \hspace{1cm} (129)$$

Furthermore, by inequality (60) and the monotony of $\zeta_1$ there exists a constant $K_A$ such that

$$\|u_n\|_{H^1_0(\Omega)} + qC_{\lambda_n}(\|u_n\|_B) \leq K_A q C_{\lambda_n}, \quad \text{for all } n \in \mathbb{N}^*, \hspace{1cm} (130)$$

where

$$u_n \rightharpoonup u \text{ in } X, u_n \rightharpoonup u \text{ in } L^A(\Omega), u_n \rightarrow u \text{ a.e. in } \Omega, \nabla u_n \rightharpoonup \nabla u \text{ in } [L^2(\Omega)]^N, \hspace{1cm} (133)$$

as $n \rightarrow \infty$. By passing to the limit in (128) as $n \rightarrow \infty$, we get by (134)–(136)

$$\int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} a(|u|)uv dx - \int_{\Omega} b(|u|)uv dx, \hspace{1cm} (134)$$

as $n \rightarrow \infty$, since $\nabla u_n \rightharpoonup \nabla u$ in $[L^2(\Omega)]^N$ by (133). Since $u_n \rightarrow u$ in $X$, Lemma 15 yields in particular that for all $v \in X$, $u_n \rightharpoonup u$ in $\Omega$, \hspace{1cm} (135)

as $n \rightarrow \infty$. Moreover, Lemmas 16 and 17 imply that for all $v \in X$

$$\int_{\Omega} a(|u_n|)u_n v dx \rightarrow \int_{\Omega} a(|u|)uv dx,$$

$$\int_{\Omega} b(|u_n|)u_n v dx \rightarrow \int_{\Omega} b(|u|)uv dx,$$

as $n \rightarrow \infty$. Hence, $u$ is a weak non-negative solution of $(\varepsilon_\lambda)$. It remains to show that the solution $u$ is nontrivial. Since $u_n \rightarrow u$ in $X$ by (133), Lemma 8 yields in particular that $\|u\|_A = \lim \|u_n\|_A$. Moreover, (52) applied to each $u_n \equiv 0$, implies that

$$\|u\|_A \geq \min \left\{ \left( qC_A \right)^{1/(2-\rho)} \lambda_n^{1/(2-\rho)}, qC_A^{1/(2-\rho)} \lambda_n^{1/(2-\rho)} \right\}. \hspace{1cm} (138)$$

Thus, by passing to the limit as $n \rightarrow \infty$, we get

$$u \rightharpoonup u \text{ in } X, u \rightharpoonup u \text{ in } L^A(\Omega), u \rightarrow u \text{ a.e. in } \Omega, \nabla u \rightharpoonup \nabla u \text{ in } [L^2(\Omega)]^N, \hspace{1cm} (133)$$

as $n \rightarrow \infty$. By passing to the limit in (128) as $n \rightarrow \infty$, we get by (134)–(136)

$$\int_{\Omega} \nabla u \nabla v dx + \int_{\Omega} a(|u|)uv dx - \int_{\Omega} b(|u|)uv dx,$$
\[ \|u\|_A \geq \min \left\{ \left( qC^A_\lambda \right)^{1/(2-p)} (\lambda_1) \left( \frac{1}{2} - \lambda \right) \left( p \right)^{1/(2-q)} \right\}, \]
\[ (139) \]

since \( \lambda_n \searrow \lambda \) and \( \lambda > 0 \). Consequently, \( u \) is nontrivial and non-negative by (133).

Proof of Theorem 1, Section 3, Lemma 20, and Theorems 3 and 4 show the existence of \( \lambda > 0 \) such that for all \( \lambda \geq \lambda \), \( (\epsilon_1) \) admits at least a nontrivial non-negative weak solution in \( X \).

6. Second Solution for Large Values of the Parameter

By variational methods, we prove, in this section, that the Dirichlet problem \( (\epsilon_1) \) admits at least two nontrivial weak solutions if \( \lambda \) is sufficiently large.

**Lemma 22.** For any \( m \in X \setminus \{0\} \) and \( \lambda > 0 \) there exist \( \rho \in (0, \|m\|_{\tilde{H}^1_0(\Omega)}) \) and \( \alpha = \alpha(\rho) > 0 \) such that \( \Phi_\lambda(u) \geq \alpha \) for all \( u \in X \), with \( \|u\|_{\tilde{H}^1_0(\Omega)} = \|m\|_{\tilde{H}^1_0(\Omega)} \).

**Proof.** Let \( m \in X \setminus \{0\} \) and \( \lambda > 0 \). Let \( u \) be in \( X \). By (35) and (48) and monotony of the function \( \zeta_1 \),

\[ \Phi_\lambda(u) \geq \frac{1}{2} \|u\|_{\tilde{H}^1_0(\Omega)}^2 - \lambda \int_\Omega A(|u|)dx \geq \frac{1}{2} \|u\|_{\tilde{H}^1_0(\Omega)}^2 - \lambda \zeta_1 \left( \|u\|_A \right) \geq \frac{1}{2} \|u\|_{\tilde{H}^1_0(\Omega)}^2 - \lambda \zeta_1 \left( C_\lambda \|u\|_{\tilde{H}^1_0(\Omega)} \right). \]
\[ (140) \]

So, to find the result of the Lemma, it is enough to take \( \rho \) such that

\[ \% \ 0 < \rho < \min \left\{ \left( 2\lambda C^A_\lambda \right)^{1/(2-p)}, \left( 2\lambda C^A_\lambda \right)^{1/(2-q)}, \|m\|_{\tilde{H}^1_0(\Omega)} \right\}, \]
\[ (141) \]

\[ \alpha = \alpha(\rho) = \left[ \frac{1}{2} - \lambda \max \left\{ C^A_\lambda \rho^{p-2}, C^A_\lambda \rho^{q-2} \right\} \right] \rho^2 > 0. \]
\[ (142) \]

In Lemma 20, we have shown that for all \( \lambda > \lambda \), the problem \( (\epsilon_1) \) admits a nontrivial non-negative weak solution \( m \in X \) which is above all a global minimizer of \( \Phi_\lambda \) in \( X \), with \( \Phi_\lambda(m) < 0 \). In this section, we are looking for a second nontrivial weak solution of \( (\epsilon_1) \) when \( \lambda > \lambda \).

**Lemma 23.** There exists \( \tilde{k} > 0 \) such that

for all \( (x, y) \in \mathbb{R}^2 \), \((b(|x|)x - b(|y|)y)(x - y) \geq \tilde{k}B(|x - y|)\) \( \geq 0 \).
\[ (143) \]

**Proof.**

(i) If \( x = y \), the inequality is verified for all \( \tilde{k} > 0 \).

(ii) If \( x \neq 0 \) and \( y = 0 \) (or \( x = 0 \) and \( y \neq 0 \)), the inequality is obtained by (30) and \( \tilde{k} > q > 0 \).

(iii) Suppose now that \( x \neq y \) with \( x \neq 0 \) and \( x \neq 0 \). By hypothesis \( (\epsilon_1) \), the function \( t \mapsto b(|t|)t \) is strictly increased in \([0; +\infty)\) and consequently it is strictly increased in \(( -\infty; 0)\) because it is an odd function. Thus, for \( x \neq y \) with \( x \neq 0 \) and \( x \neq 0 \), \((b(|x|)x - b(|y|)y)(x - y) > 0 \) and

\[ (144) \]

Theorem 5 (see Theorem A.3 in [10]). Let \( (X, \| \cdot \|) \) and \( (X, \| \cdot \|_E) \) be two Banach spaces such that \( X \longrightarrow E \). Let \( \Phi: X \longrightarrow \mathbb{R} \) be a \( C^1 \) functional with \( \Phi(0) = 0 \). Suppose that there exist \( \rho > 0 \) and \( \alpha > 0 \) and \( m \in X \) such that \( \|m\|_E > \rho \), \( \Phi(m) < \alpha \), and \( \Phi(u) \geq \alpha \) for all \( u \in X \) with \( \|u\|_E = \|m\|_E \). Then, there exists a sequence \( \{u_n\} \in X \) such that for all \( n \)

\[ c \leq \Phi(u_n) \leq c + \frac{1}{n^2} \|\Phi'(u_n)\|_\infty \leq \frac{2}{n}, \]
\[ (146) \]

where

\[ c = \inf_{y \in \Gamma} \text{max}_{t \in [0,1]} \Phi(y(t)), \Gamma = \{ y \in C([0,1]; X): y(0) = 0, y(1) = m \}, \]
\[ (147) \]

**Proof.** One can find the proof of this theorem in the section (Appendix A in [10]).

Proof of Theorem 2. Let \( \lambda \) be a strictly positive fixed number such that \( \lambda > \lambda \). Then, let \( m \in X \) be the global minimizer of \( \Phi_\lambda \) given by Lemma 20. Thanks to Lemma 22 and the fact that \( \Phi_\lambda(m) < 0 \), the assumptions of Theorem 5
which is a variant of Ekeland’s variational principle are fulfilled for the energy functional \( \Phi_{\lambda} \) of \((\varepsilon_\lambda)\). Hence, there exists a sequence \((u_n) \subset X\) such that

\[
\Phi_{\lambda}(u_n) \rightarrow c, \\
\|\Phi'(u_n)\|_X \rightarrow 0,
\]

where

\[
c = \inf_{y \in \Gamma} \max_{t \in [0,1]} \Phi(y(t)),
\]

\[
\Gamma = \{ y \in C([0,1]; X): \gamma(0) = 0, \gamma(1) = m \}.
\]

In the sequel, we shall prove that the sequence \((u_n)\) strongly converges to some \(u \in X\) and that \(u\) is a second nontrivial non-negative weak solution of \((\varepsilon_\lambda)\). \(\square\)

**Step 1.** By Lemma 11 (coercivity of \(\Phi_{\lambda}\) on \(X\)), the sequence \((u_n)\) is bounded in \(X\) and consequently, \((u_n)\) is bounded in \(H^1_0(\Omega)\). For the sequel, let us use the argument of the proof of Theorem 4. Hence, by Propositions 1 and 2 and Lemma 8 we can extract from the sequence \((u_n)\) a subsequence still relabeled \((u_n)\) and satisfying

\[
\begin{align*}
&u_n \rightharpoonup u \text{ in } X, \\
&u_n \rightarrow u \text{ in } L_A(\Omega), \\
&u_n \rightarrow u \text{ in } L_B(\Omega), \\
&u_n \rightharpoonup u \text{ a.e. in } \Omega, \\
&\nabla u_n \rightharpoonup \nabla u \text{ in } \left[ L^2(\Omega) \right]^N,
\end{align*}
\]

for some \(u \in X\). It remains to show that \(u\) is a nontrivial solution of \((\varepsilon_\lambda)\), with \(u \neq m\). We have

\[
\langle \Phi_{\lambda}'(u_n), v \rangle = \int_\Omega \nabla u_n \nabla v \ dx - \int_\Omega [-u_n + \lambda a(|u_n|)u_n - b(|u_n|)u_n] v \ dx, \text{ for all } n, \text{ for all } v \in X.
\]

By the same argument used in the proof Theorem 4, we have, for all \(v \in X\),

\[
\begin{align*}
&\int_\Omega \nabla u_n \nabla v \ dx \rightarrow \int_\Omega \nabla u \nabla v \ dx, \\
&\int_\Omega u_n v \ dx \rightarrow \int_\Omega u v \ dx, \\
&\int_\Omega a(|u_n|)u_n v \ dx \rightarrow \int_\Omega a(|u|)u v \ dx,
\end{align*}
\]

as \(n \rightarrow \infty\). Hence, passing to the limit as \(n \rightarrow \infty\) in (151), using and the fact that \(\langle \Phi_{\lambda}'(u_n), v \rangle \rightarrow 0\) as \(n \rightarrow \infty\) for all \(v \in X\), we get

\[
\int_\Omega \nabla u_n \nabla v \ dx + \int_\Omega u v \ dx = \lambda \int_\Omega a(|u|)u v \ dx
\]

\[- \int_\Omega b(|u|)u v \ dx, \text{ for all } v \in X,
\]

so \(u\) is a weak solution of \((\varepsilon_\lambda)\).

**Step 2.** We claim that

\[
0 \leq \int_\Omega (a(|u_n|)u_n - a(|u|)u)(u_n - u) \ dx \leq 2\|a(|u_n|)u_n - a(|u|)u\|_A \|u_n - u\|_A \rightarrow 0,
\]

as \(n \rightarrow \infty\). By hypothesis \((\phi_{\lambda})\), the function \(t \rightarrow a(|t|)t\) is strictly increased in \([0; +\infty)\) and consequently, it is strictly increased in \((-\infty; +\infty)\) because it is an odd function. Thus, we have

\[
\mathcal{J}_a(n) \geq 0.
\]

By (150), \(u_n \rightarrow u\) in \(L_A(\Omega)\). Thus, \(a(|u_n|)u_n \rightharpoonup a(|u|)u\) in \(L^\infty(\Omega)\) by Proposition 3. Therefore, by Hölder inequality, we get
as $n \to \infty$. This gives the proof of the claim.

Step 3. In this step, we show that $\|u_n - u\|_X \to 0$ as $n \to \infty$. Let us set

$$\mathcal{J}_\lambda(n) = \int_\Omega \left( b\left( |u_n| \right) u_n - b(|u|) u (u_n - u) \right) dx. \quad (157)$$

as $n \to \infty$. Furthermore, we have

$$\|u_n - u\|^2_{H^1_0(\Omega)} \to 0, \quad (160)$$

as $n \to \infty$. By (143) and (159), it follows that

$$\int_\Omega B\left( |u_n - u| \right) dx \leq \frac{1}{K} \mathcal{J}_\lambda(n) \to 0, \quad (161)$$

as $n \to \infty$. Thus, by section 8.13 in [11],

$$\|u_n - u\|_B \to 0 \text{ as } n \to \infty, \quad (162)$$

since the $N$-function $B$ satisfies the $\Delta_2$-condition. Finally,

$$\|u_n - u\|_X \to 0 \text{ as } n \to \infty, \quad (163)$$

by (160) and (162).

Step 4. Since $u_n \to u$ in $X$ and $\Phi_\lambda(u) \in C^1(X)$, we get $\Phi_\lambda(u) = c = \lim_{n \to \infty} \Phi_\lambda(u_n)$. So, $u$ is a second independent nontrivial weak solution of $(\varepsilon_\lambda)$, with $\Phi_\lambda(u) = c > 0 > \Phi_\lambda(m)$.

We can assume $u \geq 0$ a.e. in $\Omega$, since $|u|$ is also a solution of $(\varepsilon_\lambda)$ due to $\Phi_\lambda(|u|) = \Phi_\lambda(u)$.

This concludes the proof.

7. Conclusion

In this paper we studied the nonexistence, the existence, and multiplicity results for nontrivial weak solutions of the semilinear elliptic Dirichlet problem with $(\varepsilon_\lambda)$ involving a positive parameter $\lambda$. Our main results are obtained in the Theorems 1 and 2 by using variational methods.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References


