Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2022, Article ID 6011860, 19 pages
https://doi.org/10.1155/2022/6011860

Research Article

Multiplicity Results for Weak Solutions of a Semilinear Dirichlet Elliptic Problem with a Parametric Nonlinearity

Ayékotan Messan Joseph Tchalla and Kokou Tcharie

Département de Mathématiques, Université de Lomé, 01 B.P 1515, Lomé 1, Togo

Correspondence should be addressed to Ayékotan Messan Joseph Tchalla; atchalla@univ-lome.tg

Received 24 August 2022; Accepted 22 October 2022; Published 17 November 2022

Academic Editor: Nawab Hussain

This paper deals with the existence of weak solutions to a Dirichlet problem for a semilinear elliptic equation involving the difference of two main nonlinearities functions that depends on a real parameter λ. According to the values of λ, we give both nonexistence and multiplicity results by using variational methods. In particular, we first exhibit a critical positive value such that the problem admits at least a nontrivial non-negative weak solution if and only if λ is greater than or equal to this critical value. Furthermore, for λ greater than a second critical positive value, we show the existence of two independent nontrivial non-negative weak solutions to the problem.

1. Introduction

In the last years, most works studied the existence, nonexistence, and multiplicity of nontrivial weak solutions of a semilinear Dirichlet problem of the form as follows:

$$\begin{cases} -\Delta u = f_\lambda(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, λ is a real parameter, and $f_\lambda\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear function taking different forms. According to the values of λ, Ambrosetti et al. studied in [1], the existence and multiplicity of non-negative weak solutions of the problem (1) when $f_\lambda(x,u) = \lambda u^q + u^p$ with $0 < q < 1 < p$. For example, by using variational method, they show the existence of infinitely many solutions of the problem as follows:

$$\begin{cases} -\Delta u = \lambda |u|^{q-1}u + |u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

for $\lambda > 0$ and small. Later, Alama and Tarantello in [2] studied the semilinear Dirichlet problem (1) by searching non-negative solutions with

$$f_\lambda(x,u) = \lambda u + \omega(x)u^{q-1} - h(x)u^{r-1}, \tag{3}$$

where $\lambda \in \mathbb{R}, \Omega \subset \mathbb{R}^N\,(N \geq 3)$ is a bounded domain with smooth boundary, ω and h are suitable functions, and $1 < q < r$. In this case also, the authors show the influence of values of λ on the existence and multiplicity of weak solutions of the problem. These different studies on nonexistence, existence, and multiplicity results for nontrivial weak solutions depending on a parameter for a Dirichlet problem for a semilinear elliptic equation were extensively investigated in the literature (see, for e.g., [3–6] and the references therein). Similar results, depending on a real parameter, are obtained in the case of quasilinear elliptic equations in bounded domains or in entire space $\mathbb{R}^N$. For example, we can mention the papers [7–9], which are devoted to the unbounded case. In [7], the authors deal with the nonexistence and existence of nontrivial weak solutions of the quasilinear problem:

$$\begin{cases} -\mathrm{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) + |u|^{q-2}u = \lambda g(x)|u|^{r-2}u \text{ in } \Omega, \\ a(x)|\nabla u|^{p-2}\partial_\nu u + b(x)|\nabla u|^{p-2}u = 0, \text{ on } \partial\Omega, \end{cases} \quad (4)$$

where Ω is a smooth exterior domain in $\mathbb{R}^N$, ν is the unit vector of the outward normal on $\partial\Omega$, $1 < p < N$, $(p < r < q < p^* = (Np/(N-p)))$ or $p < q < r < p^*$, and a, b, and g are suitable functions. They showed in different cases that the existence of weak solutions of the problem depends on the values of λ relative to the value of some critical value.

In [8], Pucci and Radulescu studied the following problem in whole space:

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = \lambda|u|^{q-2}u - h(x)|u|^{r-2}u \text{ in } \mathbb{R}^N, \\ u \geq 0, \text{ in } \mathbb{R}^N, \end{cases} \quad (5)$$

where $h > 0$ satisfies

$$0 < \int_{\mathbb{R}^N} h(x)^{(q/(q-r))} < \infty, \quad (6)$$

$\lambda > 0$ is a parameter and $2 \leq p < q < \min\{r, p^*\}$ with $p^* = (Np/(N-p))$ if $N > p$ and $p^* = \infty$ if $N \leq p$. They obtained that the nonexistence and multiplicity of nontrivial weak solutions of this quasilinear elliptic equation are corresponding to the smallness and the largeness of λ, respectively. In [10], Autuori and Pucci extended the results in [8] by solving a more general quasilinear elliptic equation with the same variational method. Motivated by these previous results, we are concerned in this paper with the existence, nonexistence, and multiplicity of nontrivial weak solutions of the following Dirichlet problem for a semilinear elliptic equation:

$$\begin{cases} -\Delta u + u = \lambda a(|u|)u - b(|u|)u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} (\varepsilon_\lambda), \quad (7)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with smooth boundary, a, b are suitable non-negative functions, and λ is a real parameter. By taking inspiration on the method developed in [8, 10], we use variational arguments to study the existence and the multiplicity of nontrivial weak solutions of problem (ε_λ) according to the values of the parameter λ. To obtain our results in this work, we require in (ε_λ) the following assumptions:

$(\mathscr{A}_1)a$ is a function continuous on $\mathbb{R}_+$ and of $C^1((0,+\infty),\mathbb{R}_+)$ such that

$$a(t) > 0, \quad (a(t)t)' > 0 \text{ for all } t > 0;, \quad (8)$$

$$a(t) = 0 \Longleftrightarrow t = 0, \quad (9)$$

$$\lim_{t \longrightarrow +\infty} a(t) = +\infty. \quad (10)$$

Let us set

$$A(t) = \int_0^t sa(s)\mathrm{d}s \text{ for } t \geq 0. \quad (11)$$

$(\mathscr{A}_2)$ There exist $p, q \in]2, N[$ such that $p \leq q < 2^* = 2N/(N-2)$ and

$$p - 2 \leq \frac{a'(t)t}{a(t)} \leq q - 2, \quad \text{for all } t > 0. \quad (12)$$

$(\mathscr{B}_1)b$ is a function continuous on $\mathbb{R}_+$ and of $C^1((0,+\infty),\mathbb{R}_+)$ such that

$$b(t) > 0, (b(t)t)' > 0 \text{ for all } t > 0, \quad (13)$$

$$b(t) = 0 \Longleftrightarrow t = 0, \quad (14)$$

$$\lim_{t \longrightarrow +\infty} b(t) = +\infty. \quad (15)$$

Let us set

$$B(t) = \int_0^t sb(s)\mathrm{d}s \text{ for } t \geq 0. \quad (16)$$

$(\mathscr{B}_2)$ There exists $r \in]2, N[$ such that $q < r$ and

$$q - 2 < \frac{b'(t)t}{b(t)} \leq r - 2, \quad \text{for all } t > 0. \quad (17)$$

Some examples of functions a and b in (ε_λ)satisfying the previous assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_2)(\mathscr{B}_1)$, and $(\mathscr{B}_2)$:

(1) For $u \geq 0$, we can have $a(u) = u^{p-2}$ and $b(u) = u^{r-2}$ in (ε_λ) with $2 < p < r < N < 2^*$.

(2) Another example of functions a and b is the following:

$$a(u) = \begin{cases} \dfrac{u^{\alpha-2}}{\ln(1+u)}, & \text{for } u > 0, \\ 0, & \text{for } u = 0, \end{cases}$$

$$b(u) = \begin{cases} \dfrac{u^{\beta-2}}{\ln(1+u)}, & \text{for } u > 0, \\ 0, & \text{for } u = 0, \end{cases} \text{ with } 3 < \alpha < \alpha + 1 < \beta < N < 2^*. \quad (18)$$

In this case, for all t > 0, we have

$$\alpha - 3 \le \frac{a'(t)t}{a(t)} \le \alpha - 2,$$
$$\alpha - 2 < \beta - 3 \le \frac{b'(t)t}{b(t)} \le \beta - 2. \tag{19}$$

(3) Let σ fixed in $(1, +\infty)$. For all $u \ge 0$,

$$a(u) = \frac{u^{\alpha-2}}{\ln(\sigma + u)},$$
$$b(u) = \frac{u^{\beta-2}}{\ln(\sigma + u)}, \tag{20}$$
$$\text{with } 2 + \frac{\sigma}{\sigma + t_0} < \alpha < \alpha + \frac{\sigma}{\sigma + t_0} < \beta < N < 2^*,$$

where t_0 is the unique solution of the equation $\sigma \ln(\sigma + t) - t = 0$ on $(0, +\infty)$. Hence, for all $t > 0$, we have

$$\alpha - 2 - \frac{\sigma}{\sigma + t_0} \le \frac{a'(t)t}{a(t)} \le \alpha - 2,$$
$$\alpha - 2 < \beta - 2 - \frac{\sigma}{\sigma + t_0} \le \frac{b'(t)t}{b(t)} \le \beta - 2. \tag{21}$$

Thus, it is clear that the functions a and b of the Dirichlet problem (ε_λ) of our present work generalize the functions $|u|^{q-2}$ and $|u|^{r-2}$ $(2 < q < r)$ which appear in the main equation studied in [7, 8] or [10].

The main goal of this paper is the proof of the following two theorems:

Theorem 1. *By the fulfillment of assumptions* $(\mathscr{A}_1)$, $(\mathscr{A}_2)$ $(\mathscr{B}_1)$, *and* $(\mathscr{B}_2)$, *there exists a critical value* $\widehat{\lambda} > 0$ *such that the Dirichlet problem* (ε_λ) *admits at least a nontrivial non-negative weak solution if and only if* $\lambda \ge \widehat{\lambda}$.

Theorem 2. *Suppose that the assumptions* $(\mathscr{A}_1)$, $(\mathscr{A}_2)$ $(\mathscr{B}_1)$, *and* $(\mathscr{B}_2)$ *are fulfilled. Then, there exists a critical value* $\overline{\lambda}$ *satisfying* $\overline{\lambda} \ge \widehat{\lambda}$ *such that for all* $\lambda > \overline{\lambda}$, *the Dirichlet problem* (ε_λ) *admits at least two nontrivial non-negative weak solutions.*

In Section 2, we talk about Orlicz spaces, which we will use in our work. In Section 3, we give different imbeddings between the working spaces of this paper and prove the nonexistence of a nontrivial weak solution when λ in (ε_λ) is least than a positive number. The conditions for existence of weak solutions of (ε_λ) are established in Section 4. Section 5 has devoted to prove Theorem 1, and Section 6 deals with the proof of Theorem 2.

2. Notions on Orlicz Spaces (See Chapter 8 in [11])

Definition 1. (definition of a N-function). Let ψ be a real-valued function defined on $[0, \infty)$ and having the following properties:

(a) $\psi(0) = 0$, $\psi(t) > 0$ if $t > 0$, $\lim_{t \longrightarrow +\infty} = +\infty$
(b) ψ is nondecreasing, that is, $s > t$ implies $\psi(s) \ge \psi(t)$
(c) ψ is right continuous, that is, if $t \ge 0$, then $\lim_{s \longrightarrow t^+} \psi(s) = \psi(t)$

Then, the real-valued function Ψ defined on $[0, +\infty)$ by

$$\Psi(t) = \int_0^t \psi(s) \mathrm{d}s, \tag{22}$$

is called an N-function.

Any such N-function Ψ has the following properties:

(i) Ψ is continuous on $[0, \infty)$
(ii) Ψ is strictly increasing
(iii) Ψ is convex
(iv) $\lim_{t \longrightarrow 0} (\Psi(t)/t) = 0$ and $\lim_{t \longrightarrow +\infty} (\Psi(t)/t) = +\infty$
(iv) The function $t \longmapsto (\Psi(t)/t)$ is strictly increasing on $(0, \infty)$

For any N-function $\Psi = \Psi(t)$ and an open set $\Omega \subset \mathbb{R}^N$, the Orlicz space $L_\Psi(\Omega)$ is defined. When Ψ satisfies Δ_2 -condition, i.e.,

$$\Psi(2t) \le k\Psi(t), \quad \text{for all } t \ge 0, \tag{23}$$

for some constant $k > 0$, then

$$L_\Psi(\Omega) = \left\{u: \Omega \longrightarrow \mathbb{R} \text{ measurable: } \int_\Omega \Psi(|u(x)|)\mathrm{d}x < \infty\right\}. \tag{24}$$

Endowed with the norm

$$\|u\|_\Psi = \inf\left\{k > 0: \int_\Omega \Psi\left(\frac{|u(x)|}{k}\right)\mathrm{d}x \le 1\right\}, \tag{25}$$

which is called the Luxembourg norm, the Orlicz space $L_\Psi(\Omega)$ is a Banach space. It is known that if $\int_\Omega \Psi(|u(x)|/k_0)\mathrm{d}x = 1$, then $\|u\|_\Psi = k_0$ with $k_0 > 0$.

The complement of Ψ is given by the Legendre transformation as follows:

$$\widetilde{\Psi}(s) = \max_{t \ge 0}(st - \Psi(t)) \text{for } s \ge 0. \tag{26}$$

We say that Ψ and $\widetilde{\Psi}$ are complementary N-functions of each other.

For all $t > 0$, we have the inequality $\widetilde{\Psi}(\Psi(t)/t) \le \Psi(t)$.

From Young's inequality

$$st \le \Psi(t) + \widetilde{\Psi}(s), \tag{27}$$

a generalized version of Hölder's inequality is obtained as follows:

$$\left|\int_{\Omega} u(x)v(x)dx\right| \le 2\|u\|_{\Psi}\|v\|_{\widetilde{\Psi}} \text{for } u \in L_{\Psi}(\Omega), v \in L_{\widetilde{\Psi}}(\Omega). \tag{28}$$

3. Preliminaries and Nonexistence of Nontrivial Solution for λ Small

By condition $(\mathscr{A}_1)$ and the definition of A and by condition $(\mathscr{B}_1)$ and the definition of B, the functions A and B are N-functions, with $\psi(s) = sa(s)$ or $\psi(s) = sb(s)$, respectively.

Lemma 1. *The N-function A satisfies the Δ_2-condition.*

Proof. In fact, by (12),

$$\text{for all } t > 0, \quad p \le \frac{a(t)t^2}{A(t)} \le q. \tag{29}$$

Thus, by Theorem 4.1 in [12], it follows that A satisfies the Δ_2-condition. □

Lemma 2. *The N-function B satisfies the Δ_2-condition.*

Proof. In fact, by (17)

$$\text{for all } t > 0, \quad q < \frac{b(t)t^2}{B(t)} \le r, \tag{30}$$

and B satisfies the Δ_2-condition by Theorem 4.1 in [12].

Let us consider the Sobolev space $H_0^1(\Omega)$, which is the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\|_{H_0^1(\Omega)} = \left(\int_{\Omega} |\nabla u|^2\right)^{(1/2)} = \|\nabla u\|_2. \tag{31}$$

Let X denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_X = \left(\|u\|^2_{H_0^1(\Omega)} + \|u\|_B^2\right)^{(1/2)}, \tag{32}$$

where

$$\|u\|_B = \text{in} f\left\{\lambda > 0: \int_{\Omega} B\left(\frac{|u(x)|}{\lambda}\right)dx \le 1\right\}, \tag{33}$$

is the Luxembourg norm in the Orlicz space $L_B(\Omega)$. The space X is the space in which we will find our nontrivial weak solutions. □

Lemma 3. *The embeddings $X \longrightarrow H_0^1(\Omega) \longrightarrow L^{2^*}(\Omega)$ are continuous with $\|u\|_{H_0^1(\Omega)} \le \|u\|_X$, for all $u \in X$ and $\|u\|_{2^*} \le C_{2^*}\|u\|_{H_0^1(\Omega)}$, for all $u \in H_0^1(\Omega)$.*

Moreover, $H_0^1(\Omega) \longrightarrow \longrightarrow L^\rho(\Omega)$ and $X \longrightarrow \longrightarrow L^\rho(\Omega)$ are compact for all ρ such that $1 \le \rho < 2^*$.

Proof. The first imbedding that i.e., $X \longrightarrow H_0^1(\Omega)$ is followed from the definition of the norm in X. The second imbedding is followed from Talenti's work in [13]; C_{2^*} is the Talenti constant. The imbedding compact $H_0^1(\Omega) \longrightarrow \longrightarrow L^\rho(\Omega)$ for $\rho \in [1, 2^*)$ is obtained by Rellich's Theorem. It follows that the mapping $X \longrightarrow \longrightarrow L^\rho(\Omega)$ is compact for $\rho \in [1, 2^*)$. □

Lemma 4. *Let $\zeta_0(t) = \min(t^p, t^q)$, $\zeta_1(t) = \max(t^p, t^q)$, $\zeta_2(t) = \min(t^q, t^r)$, and $\zeta_3(t) = \max(t^q, t^r)$ for all $t \ge 0$. Then,*

$$\zeta_0(\rho)A(t) \le A(\rho t) \le \zeta_1(\rho)A(t) \text{ for } \rho, \quad t \ge 0, \tag{34}$$

$$\zeta_0(\|u\|_A) \le \int_{\Omega} A(|u(x)|dx \le \zeta_1(\|u\|_A) \text{ for } u \in L_A(\Omega), \tag{35}$$

$$\zeta_2(\rho)B(t) \le B(\rho t) \le \zeta_3(\rho)B(t) \text{ for } \rho, \quad t \ge 0, \tag{36}$$

$$\zeta_2(\|u\|_B) \le \int_{\Omega} B(|u(x)|dx \le \zeta_3(\|u\|_B) \text{ for } u \in L_B(\Omega). \tag{37}$$

Proof. The proof is given in [14] (see Lemma 2.1 of [14]). In fact by integrating the inequalities (29) and (30) respectively, we get inequalities (34) and (35). From (34) and (35) and the definition of Luxembourg norm we get respectively (35) and (37). □

Lemma 5. *The space $L_B(\Omega)$ is imbedded continuously in $L_A(\Omega)$ with $\|u\|_A \le \mathscr{H}\|u\|_B$, where*

$$\mathscr{H} = 2k\omega, \tag{38}$$

with $k = \max\left\{1, (A(1)/B(1))^{(1/q)}\right\}$ *and* $\omega = \max\{1, (A(1)/B(kA^{-1}(1/2\, mes(\Omega))))\}$.

Proof. This proof is based on the proof of Theorem 8.12 in [11]. Fix k, with $k \ge 1$, then

$$\text{for all } t \ge 1, \frac{A(t)}{B(kt)} \le \frac{A(1)}{k^q B(1)}, \tag{39}$$

by (34) and (36). Thus, by taking $k = \max\left\{1, (A(1)/B(1))^{(1/q)}\right\}$, we have

$$A(t) \le B(kt), \quad \text{for all } t \ge 1. \tag{40}$$

By the proof of (Theorem 8.12 [11]),

$$\begin{gathered}\text{for all } t \ge A^{-1}\left(\frac{1}{2\text{mes}(\Omega)}\right), \quad A(t) \le \omega B(kt), \\ \text{where } \omega = \max\left\{1, \frac{A(1)}{B\left(kA^{-1}(1/2\text{mes}(\Omega))\right)}\right\}.\end{gathered} \tag{41}$$

Let $u \in L_B(\Omega)$ such that $u \ne 0$. Let us set $\Omega'(u) = \left\{x \in \Omega: (|u(x)|/2\omega k\|u\|_B) \le A^{-1}(1/2\text{mes}(\Omega))\right\}$ and $\Omega''(u) = \Omega - \Omega'(u)$. B being convex, we have

$$\int_{\Omega} A\left(\frac{|u(x)|}{2\omega k\|u\|_B}\right)dx = \int_{\Omega'(u)} A\left(\frac{|u(x)|}{2\omega k\|u\|_B}\right)dx + \int_{\Omega''(u)} A\left(\frac{|u(x)|}{2\omega k\|u\|_B}\right)dx \le \frac{1}{2\text{mes}(\Omega)}\int_{\Omega'(u)} dx$$
$$+\ \omega\int_{\Omega''(u)} B\left(\frac{|u(x)|}{2\omega\|u\|_B}\right)dx \le \frac{1}{2} + \frac{1}{2}\int_{\Omega} B\left(\frac{|u(x)|}{\|u\|_B}\right)dx \le 1. \tag{42}$$

It follows that $\|u\|_A \le 2\omega k\|u\|_B$ and consequently $L_B(\Omega)$ is imbedded continuously in $L_A(\Omega)$. □

Lemma 6. *The $N-functionA$ satisfies the following:*

$$\lim_{t\longrightarrow+\infty} \frac{A(kt)}{t^{2^*}} = 0 \text{ for all } k \ge 0. \tag{43}$$

Proof. The result is followed from property (34). In fact, by (34), for all $k \ge 0$, for all $t \ge 1$, we have

$$0 \le \frac{A(kt)}{t^{2^*}} \le \frac{\zeta_1(t)A(k)}{t^{2^*}} = t^{q-2^*}A(k). \tag{44}$$

□

Lemma 7. *The imbedding $L^{2^*}(\Omega) \longrightarrow L_A(\Omega)$ is continuous with*

$$\|u\|_A \le C_A^*\|u\|_{2^*}, \quad \text{for all } u \in L^{2^*}(\Omega), \tag{45}$$

where, $C_A^* = 2\max\{1;\ (A^{-1}(1/2mes(\Omega)))^{-2^*}\}(A(1))^{(1/2^*)}$.

Proof. The N-function A increases more slowly than $t \longmapsto t^{2^*}$ near infinity and by (34) it follows that for all $t \ge 1$, $A(t) \le A(1)t^{2^*}$. By the proof of (Theorem 8.12 [11]),

$$\text{for all } t \ge A^{-1}\left(\frac{1}{2\text{mes}(\Omega)}\right), \quad A(t) \le \omega A(1)t^{2^*},$$
$$\text{where } \omega = \max\left\{1, \left(A^{-1}\left(\frac{1}{2\text{mes}(\Omega)}\right)\right)^{-2^*}\right\}. \tag{46}$$

Let $u \in L_{2^*}(\Omega)$ such that $u \ne 0$. Let us set $\Omega'(u) = \{x \in \Omega\colon (|u(x)|/2\omega(A(1))^{(1/2^*)}\|u\|_{2^*}) \le A^{-1}(1/2\ \text{mes}(\Omega))\}$ and $\Omega''(u) = \Omega - \Omega'(u)$.

$$\int_{\Omega} A\left(\frac{|u(x)|}{2\omega(A(1))^{(1/2^*)}\|u\|_{2^*}}\right)dx = \int_{\Omega'(u)} A\left(\frac{|u(x)|}{2\omega(A(1))^{(1/2^*)}\|u\|_{2^*}}\right)dx + \int_{\Omega''(u)} A\left(\frac{|u(x)|}{2\omega(A(1))^{(1/2^*)}\|u\|_{2^*}}\right)dx$$
$$\le \frac{1}{2\text{mes}(\Omega)}\int_{\Omega'(u)} dx + \omega\int_{\Omega''(u)} \frac{|u(x)|^{2^*}}{(2\omega\|u\|_{2^*})^{2^*}}dx \le \frac{1}{2} + \frac{1}{2}\int_{\Omega} \frac{|u(x)|^{2^*}}{\|u\|_{2^*}^{2^*}}dx \le 1. \tag{47}$$

It follows that $\|u\|_A \le 2\omega(A(1))^{(1/2^*)}\|u\|_{2^*}$ and consequently $L_{2^*}(\Omega)$ is imbedded continuously in $L_A(\Omega)$. □

Lemma 8. *The imbedding $H_0^1(\Omega) \longrightarrow L_A(\Omega)$ is continuous with*

$$\|u\|_A \le C_A\|u\|_{H_0^1(\Omega)} \text{for all } u \in H_0^1(\Omega),$$
$$\text{where } C_A = C_{2^*}C_A^*. \tag{48}$$

Moreover, the imbeddings $H_0^1(\Omega) \longrightarrow\ \longrightarrow L_A(\Omega)$ and $X \longrightarrow\ \longrightarrow L_A(\Omega)$ are compacts.

Proof. The continuity of the imbedding $H_0^1(\Omega) \longrightarrow L_A(\Omega)$ is followed from Lemmas 3 and 7. By Lemma 6 and Theorem 8.36 in [11], it follows that the imbedding $H_0^1(\Omega) \longrightarrow L_A(\Omega)$ is compact. It follows also that $X \longrightarrow\ \longrightarrow L_A(\Omega)$ is compact because $X \longrightarrow H_0^1(\Omega)$ is continuous by Lemma 3. □

Lemma 9. *If u is an element of $X\setminus\{0\}$ and λ is a real such that*

$$\int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u|^2 dx + \int_{\Omega} b(|u|)u^2 dx = \lambda\int_{\Omega} a(|u|)u^2 dx, \tag{49}$$

then $\lambda > 0$ and there exists two positive constants k_1 and k_2 independent of u such that

$$k_1 f_1(\lambda) \le \|u\|_A \le k_2 f_2(\lambda), \tag{50}$$

where f_1 and f_2 are two functions of λ.

Proof. Let us take $u \in X\setminus\{0\}$ and $\lambda \in \mathbb{R}$ such that (49) hold. Thus, we have $0 < \|u\|_{H_0^1(\Omega)}^2 \le \lambda\int_{\Omega} a(|u|)u^2 dx$. As a is a non-negative function in $(0, +\infty)$, it follows that $\lambda > 0$. Let us show the second part of the Lemma. Firstly, by using respectively (29), (35), (48), and (49), we get

$$\|u\|_A^2 \le q\lambda C_A^2\zeta_1(\|u\|_A). \tag{51}$$

Thus, this last inequality yields

$$\|u\|_A \ge \min\left\{(qC_A^2)^{(1/(2-p))}\lambda^{(1/(2-p))}, (qC_A^2)^{(1/(2-p))}\lambda^{(1/(2-p))}\right\}. \tag{52}$$

Secondly, by (29), (30), (48), and (49) we have

$$\|u\|_A^2 \le C_A^2 \|u\|_{H_0^1(\Omega)}^2 \le C_A^2\left(\lambda\int_\Omega a(|u|)u^2\mathrm{d}x - \int_\Omega b(|u|)u^2\mathrm{d}x\right) \le C_A^2 q\int_\Omega \lambda A(|u(x)|) - B(|u(x)|)\mathrm{d}x. \tag{53}$$

Let us set $\Omega_u = \{x \in \Omega\colon |u(x)| \le 1\}$ and $\Omega_u' = \Omega - \Omega_u$. Since $\lambda > 0$, (34) and (36) imply

$$\lambda A(|u(x)|) - B(|u(x)|) \le \begin{cases} \lambda A(1)|u(x)|^p - B(1)|u(x)|^r, & \text{for } x \in \Omega_u, \\ \lambda A(1)|u(x)|^q - B(1)|u(x)|^r, & \text{for } x \in \Omega_u'. \end{cases} \tag{54}$$

Let $s = p$ or $s = q$. By applying Young's inequality $ab \le (a^\alpha/\alpha) + (b^\beta/\beta)$ for $a, b > 0$ and $\alpha > 1$ and $\beta > 1$, we have

$$\lambda A(1)|u(x)|^s \le \frac{s}{r}B(1)|u(x)|^r + \frac{r-s}{r}\left(\frac{\lambda A(1)}{(B(1))^{(s/r)}}\right)^{(r/(r-s))}, \tag{55}$$

where $a = (B(1))^{(s/r)}|u(x)|^s$, $b = \lambda A(1)(B(1))^{-(s/r)}$, $\alpha = (r/s) > 1$, and $\beta = (r/(r-s)) > 1$. Inequality (55) yields

$$\lambda A(1)|u(x)|^s - B(1)|u(x)|^r \le B(1)|u(x)|^r\left(\frac{s}{r} - 1\right) + \frac{r-s}{r}\left(\frac{\lambda A(1)}{(B(1))^{(s/r)}}\right)^{(r/(r-s))}, \tag{56}$$

$$\le \lambda^{(r/(r-s))}\frac{r-s}{r}\left(\frac{A(1)}{(B(1))^{(s/r)}}\right)^{(r/(r-s))}, \tag{57}$$

being $s < r$. Hence, (54) yields

$$\lambda A(|u(x)|) - B(|u(x)|) \le \begin{cases} \lambda^{(r/(r-p))}\dfrac{r-p}{r}\left(\dfrac{A(1)}{(B(1))^{(p/r)}}\right)^{(r/(r-p))}, & \text{for } x \in \Omega_u, \\ \lambda^{(r/(r-q))}\dfrac{r-q}{r}\left(\dfrac{A(1)}{(B(1))^{(q/r)}}\right)^{(r/(r-q))}, & \text{for } x \in \Omega_u'. \end{cases} \tag{58}$$

As $((r-p)/r) > ((r-q)/r)$, it follows, from (58), that

$$\lambda A(|u(x)|) - B(|u(x)|) \le \frac{r-p}{r}\max\left\{\lambda^{(r/(r-p))}\left(\frac{A(1)}{(B(1))^{(p/r)}}\right)^{(r/(r-p))}, \lambda^{(r/(r-q))}\left(\frac{A(1)}{(B(1))^{(q/r)}}\right)^{(r/(r-q))}\right\}, \quad \text{for all } x \in \Omega. \tag{59}$$

Thus, by (53) and (59), we get

$$\|u\|_A \le C_A \left\{ \frac{q(r-p)}{r} \operatorname{mes}(\Omega) \right\}^{(1/2)} \max \left\{ \lambda^{(r/2(r-p))} \left(\frac{A(1)}{(B(1))^{(p/r)}} \right)^{(r/2(r-p))}, \lambda^{(r/2(r-q))} \left(\frac{A(1)}{(B(1))^{(q/r)}} \right)^{(r/2(r-q))} \right\}. \quad (60)$$

Finally, the last inequality and (52) yield the result (50). □

Definition 2. An element u of X is a weak solution of (ε_λ) if

$$\begin{aligned} \int_\Omega \nabla u \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \lambda \int_\Omega a(|u|) uv \mathrm{d}x \\ - \int_\Omega b(|u|) uv \mathrm{d}x \text{ for all } v \in X. \end{aligned} \quad (61)$$

Consequently, the weak solutions of (ε_λ) are exactly the critical points of the energy functional $\Phi_\lambda \colon X \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} \Phi_\lambda(u) &= \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2} \int_\Omega |u|^2 \mathrm{d}x \\ &\quad - \lambda \int_\Omega A(|u|) \mathrm{d}x + \int_\Omega B(|u|) \mathrm{d}x. \end{aligned} \quad (62)$$

Lemma 10. *If (ε_λ) has a nontrivial weak solution $u \in X$, then $\lambda \ge \lambda_0$, where*

$$\lambda_0 = \min \left\{ \left(\frac{\eta}{\kappa \varpi} \right)^{2(r-p)(p-2)/(p(r-2))}, \left(\frac{\eta}{\kappa \varrho} \right)^{2(r-q)(p-2)/(rp-2q)}, \left(\frac{\eta}{\xi \varpi} \right)^{2(r-p)(q-2)/(rq-2p)}, \left(\frac{\eta}{\xi \varrho} \right)^{2(r-q)(q-2)/(q(r-2))} \right\} > 0, \quad (63)$$

with

$$\begin{aligned} \eta &= C_A^{-1} \left(\frac{q(r-p)}{r} \operatorname{mes}(\Omega) \right)^{-(1/2)}, \\ \kappa &= \left(q C_A^2 \right)^{(1/(p-2))}, \\ \xi &= \left(q C_A^2 \right)^{(1/(q-2))}, \\ \varpi &= \left(\frac{A(1)}{(B(1))^{(p/r)}} \right)^{(r/2(r-p))}, \\ \varrho &= \left(\frac{A(1)}{(B(1))^{(q/r)}} \right)^{(r/2(r-q))}. \end{aligned} \quad (64)$$

Proof. If (ε_λ) admits a nontrivial weak solution $u \in X$, then equality (49) is satisfied and $\lambda > 0$ by Lemma 9. Let us now show that $\lambda \ge \lambda_0$. By inequalities (50) and (60) of the proof of Lemma 9, we get the result.

We claim that the set $E = \{\lambda > 0 \colon (\varepsilon_\mu)$ admits only the trivial solution for all $\mu < \lambda\}$ is not empty and bounded above. Indeed, by Lemma 9, for all $\lambda < \lambda_0$, (ε_λ) admit only trivial solution. Thus, $\lambda_0 \in E$. Now suppose that for all $M > 0$, there exists λ_M such that $\lambda_M > M$. Therefore, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}^*}$ of elements of E such that $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ For all $n \in \mathbb{N}$, for all μ such that $0 < \mu < \lambda_n$, (ε_μ) admits only trivial solution. By hypothesis, the sequence $(\lambda_n)_{n \in \mathbb{N}}$ tends to $+\infty$. Hence, for all $\mu \in (0, +\infty)$, (ε_μ) admits only trivial solution. This contradicts the Lemma 9.

Let us define

$$\hat{\lambda} = \sup \left\{ \lambda > 0 \colon \left(\varepsilon_\mu \right) \text{admits only the trivial solution for all } \mu < \lambda \right\}. \quad (65)$$

It is clear that $\hat{\lambda} \ge \lambda_0 > 0$. In Section 5, we will prove that $\hat{\lambda}$ is the required critical value of the Theorem 1. □

4. Basic Results for Existence of Nontrivial Solution

The results in the previous section require us to work from now on with $\lambda > 0$.

Lemma 11. *The energy functional Φ_λ in coercive on X.*

Proof. Let $u \in X$.

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int_\Omega |u|^2 \mathrm{d}x - \lambda\int_\Omega A(|u|)\mathrm{d}x + \int_\Omega B(|u|)\mathrm{d}x,\\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \left[\lambda\int_\Omega A(|u|)\mathrm{d}x - \frac{1}{2}\int_\Omega B(|u|)\mathrm{d}x\right] + \frac{1}{2}\int_\Omega B(|u|)\mathrm{d}x.\end{aligned} \tag{66}$$

By using Young's inequality, as we use it in (55), we have

$$\lambda A(|u(x)|) - \frac{1}{2}B(|u(x)|) \leq 2^{(q/(r-q))}\frac{r-p}{r}c_\lambda, \text{for all } x \in \Omega, \tag{67}$$

where $c_\lambda = \max\{\lambda^{(r/(r-p))}(A(1)/(B(1))^{(p/r)})^{(r/(r-p))}, \lambda^{(r/(r-q))}(A(1)/(B(1))^{(q/r)})^{(r/(r-q))}\}$. Therefore, we have

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{1}{2}\int_\Omega B(|u|)\mathrm{d}x - 2^{(q/(r-q))}\frac{r-p}{r}c_\lambda \mathrm{mes}(\Omega),\\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{1}{2}\zeta_2(\|u\|_B) - 2^{(q/(r-q))}\frac{r-p}{r}c_\lambda \mathrm{mes}(\Omega),\\ &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} + \frac{1}{2}\left(\|u\|_B^2 - 1\right) - 2^{(q/(r-q))}\frac{r-p}{r}c_\lambda \mathrm{mes}(\Omega),\\ &\geq \frac{1}{2}\|u\|_X^2 - 2^{(q/(r-q))}\frac{r-p}{r}c_\lambda \mathrm{mes}(\Omega) - \frac{1}{2}.\end{aligned} \tag{68}$$

In conclusion, Φ_λ is coercive in X. □

Lemma 12. *Let $\widetilde{A}$ and $\widetilde{B}$ respectively be the complements of N-functions A and B. Then, we have*

$$\frac{p}{p-1} \leq \frac{\widetilde{A}'(s)s}{\widetilde{A}(s)} \leq \frac{q}{q-1} \text{for} s > 0, \tag{69}$$

$$\frac{q}{q-1} \leq \frac{\widetilde{B}'(s)s}{\widetilde{B}(s)} \leq \frac{r}{r-1} \text{for } s > 0, \tag{70}$$

where $\widetilde{A}'(s)$ and $\widetilde{B}'(s)$ are respectively the derivatives of $\widetilde{A}(s)$ and $\widetilde{B}(s)$.

Proof. The results are obtained by using (29) and (30) and the proof of the point (2.7) of Lemma 2.5 in [14]. □

Proposition 1. *The spaces $L_A(\Omega)$ and $L_B(\Omega)$ are reflexives and separables Banach spaces.*

Proof. The N-functions A and B satisfy Δ_2-condition respectively by Lemmas 1 and 2. Moreover, the inequalities (69) and (70) of Lemma 12 imply that the N-functions $\widetilde{A}$ and $\widetilde{B}$ satisfy the Δ_2-condition. Thus, by (Theorem 8.20 and Remark 8.22 in [11]), the spaces $L_A(\Omega)$ and $L_B(\Omega)$ are reflexives and separables Banach spaces. □

Proposition 2. $(X, \|.\|_X)$ *is a separable reflexive Banach space.*

Proof. Let $Y = H_0^1(\Omega) \times L_B(\Omega)$, which we endow with the norm $\|u\|_Y = \|u\|_{H_0^1(\Omega)} + \|u\|_B$. Since $H_0^1(\Omega)$ and $L_B(\Omega)$ are reflexives Banach spaces, then $(Y, \|.\|_Y)$ is a separable and reflexive Banach space by Theorem 1.23 in [11]. Let us consider the operator Γ: $(X, \|.\|_X) \longrightarrow (Y, \|.\|_Y)$ defined by $\Gamma(u) = (u, u)$. Γ is well defined, linear, and isometric. Therefore, $\Gamma(X)$ is closed subspace of Y and so $\Gamma(X)$ is separable and reflexive by Theorem 1.22 in [11]. Consequently, $(X, \|.\|_Y)$ is a separable reflexive Banach space, being isomorphic to a separable, reflexive space. Finally, we conclude that $(X, \|.\|_X)$ is a separable reflexive Banach space because reflexivity and separability are preserved under equivalent norms. □

Lemma 13. *Let (u_n) be a sequence in X such that $(\Phi_\lambda(u_n))_n$ is bounded. Then, (u_n) admits a weakly convergent subsequent in X*
.

Proof. The proof comes from the coercivity of Φ_λ in X and the reflexivity of the space X. □

Lemma 14. *The functional $\Phi_\Delta(u) = (1/2)\int_\Omega |\nabla u|^2 dx$ is convex, of class C^1 and is particularly sequentially weakly lower semicontinuous in X.*

Proof. The convexity of functional $\Phi_\Delta(u) = (1/2)\int_\Omega |\nabla u|^2 dx$ is followed from the convexity of function $t \longmapsto t^2$ in $\mathbb{R}$.

Let us show the continuity of $\Phi_\Delta(u)$ on X. Let $(u_n)_n$ be a sequence of which elements are in X and let $u \in X$ such that $u_n \longrightarrow u$ in X. Let $(u_{n_k})_k$ be an arbitrary subsequence of $(u_n)_n$. The subsequence $u_{n_k} \longrightarrow u$ in X and hence, by Lemma 3, $\nabla u_{n_k} \longrightarrow \nabla u$ in $(L^2(\Omega))^N$. By Theorem 4.9 in

[15], there exists a subsequence $(u_{n_{k_j}})_j$ of u_{n_k} and a function $\psi \in L^1(\Omega)$ such that $\nabla u_{n_{k_j}} \longrightarrow \nabla u$ a.e in Ω as $j \longrightarrow \infty$ and $|\nabla u_{n_{k_j}}| \le \psi$ a.e in Ω for all $j \in \mathbb{N}$. So $|\nabla u_{n_{k_j}}|^2 \longrightarrow |\nabla u|^2$ a.e in Ω as $j \longrightarrow \infty$ and $|\nabla u_{n_{k_j}}|^2 \le \psi^2 \in L^1(\Omega)$ a.e in Ω for all $j \in \mathbb{N}$. The dominated convergence theorem implies that $|\nabla u_{n_{k_j}}|^2 \longrightarrow |\nabla u|^2$ in $L^1(\Omega)$ as $j \longrightarrow \infty$. The subsequence $(u_{n_k})_k$ being arbitrary, we deduce that $|\nabla u_n|^2 \longrightarrow |\nabla u|^2$ in $L^1(\Omega)$ as $n \longrightarrow \infty$. Then, we get the continuity of Φ_Δ on X and also, Φ_Δ is sequentially weakly lower semicontinuous in X by Corollary 3.9 in [15]. Moreover, Φ_Δ is Gateaux-differentiable in X and for all $u, \varphi \in X$, we have

$$\langle \Phi_\Delta'(u), \varphi \rangle = \int_\Omega \nabla u \nabla \varphi dx. \tag{71}$$

Let us finish the proof by showing that Φ_Δ' is continuous. Let $(u_n)_n$ be a sequence of functions in X and $u \in X$ such that $u_n \longrightarrow u$ in X when $n \longrightarrow \infty$. By a simple calculus we have

$$\left\|\Phi_\Delta'(u_n) - \Phi_\Delta'(u)\right\|_{X'} \le \left\|u_n - u\right\|_{H_0^1(\Omega)} \le \left\|u_n - u\right\|_X. \tag{72}$$

This inequality yields the continuity of Φ_Δ'. ☐

Lemma 15. *The functional $\Phi_2(u) = (1/2)\int_\Omega |u|^2 dx$ is convex, of class C^1 and sequentially weakly lower semicontinuous in X. Furthermore, if (u_n) is a sequence of elements of X and u belongs to X such that u_n converge weakly to u in X, then $\Phi_2'(u_n) \longrightarrow \Phi_2'(u)$ in X'.*

Proof. The convexity of the functional Φ_2 is obvious. The continuity of Φ_2 in X follows from Lemma 3. Thus, Φ_2 is sequentially weakly lower semicontinuous in X by Corollary 3.9 in [15]. To complete the proof of the theorem, it suffices to show the last part of the theorem. Therefore, let us take $(u_n) \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ weakly in X. Since $X \longrightarrow \longrightarrow L^2(\Omega)$ is compact by lemma 3, we have $\|\Phi_2'(u_n) - \Phi_2'(u)\|_{X'} \le \|u_n - u\|_2$ and then $\Phi_2'(u_n) \longrightarrow \Phi_2'(u)$ in X'. ☐

Proposition 3. *(i) If u_n converge strongly to u in $L_A(\Omega)$, then $a(|u_n|)u_n$ converge to $a(|u|)u$ in $L_{\widetilde{A}}(\Omega)$. (ii) If u_n converge strongly to u in $L_B(\Omega)$, then $b(|u_n|)u_n$ converge to $b(|u|)u$ in $L_{\widetilde{B}}(\Omega)$.*

Proof

(i) Suppose that u_n converge strongly to u in $L_A(\Omega)$. Let $(v_{n_k})_k$ be a fixed subsequence of the sequence $n \longmapsto a(|u_n|)u_n$. For all k, we have $v_{n_k} = a(|u_{n_k}|)u_{n_k}$, where $(u_{n_k})_k$ is a subsequence of the sequence (u_n), which converge to u in $L_A(\Omega)$. Thus, $\int_\Omega A(|u_{n_k} - u|)dx \longrightarrow 0$, as $k \longrightarrow \infty$. It follows that, there exists a subsequence $(u_{n_{k_j}})_j$ of $(u_{n_k})_k$ and a positive function $\psi \in L^1(\Omega)$ such that $A(|u_{n_{k_j}} - u|) \longrightarrow 0$ a.e in Ω and $0 \le A(|u_{n_{k_j}} - u|) \le \psi(x)$ a.e in Ω, for all $j \in \mathbb{N}$. As A is continuous and strictly increasing in $\mathbb{R}_+$, we have $u_{n_{k_j}} \longrightarrow u a.e$ in Ω. Hence, $v_{n_{k_j}} \longrightarrow v = a(|u|)u a.e$ in Ω and $\widetilde{A}(|a(|u_{n_{k_j}}|)u_{n_{k_j}} - a(|u|)u|) \longrightarrow 0 a.e$ in Ω. On other hand, there exists $h \in L^1(\Omega)$ such that $\widetilde{A}(|a(|u_{n_{k_j}}|)u_{n_{k_j}} - a(|u|)u|) \le h(x) a.e$ in Ω. Indeed, by the fact that $A(|u_{n_{k_j}} - u|) < \psi(x) a.e$ in Ω and the function $t \longmapsto a(t)t$ is continuous and increases strictly in $\mathbb{R}_+$, we have

$$\widetilde{A}\left(\left|a\left(\left|u_{n_{k_j}}\right|\right)u_{n_{k_j}} - a(|u|)u\right|\right) \le \widetilde{A}\left[2a\left(A^{-1}(\psi(x)) + |u(x)|\right)\left(A^{-1}(\psi(x)) + |u(x)|\right)\right], \text{ for } x \in \Omega a.e. \tag{73}$$

By (13) and since $\widetilde{A}$ satisfy Δ_2-condition, we have

$$\begin{aligned}\widetilde{A}\left(\left|a\left(\left|u_{n_{k_j}}\right|\right)u_{n_{k_j}} - a(|u|)u\right|\right) &\le \delta \widetilde{A}\left[\frac{A\left(A^{-1}(\psi(x)) + |u(x)|\right)}{A^{-1}(\psi(x)) + |u(x)|}\right], \text{for } x \in \Omega a.e, \\ &\le \delta A\left(A^{-1}(\psi(x)) + |u(x)|\right), \text{for } x \in \Omega a.e, \\ &\le \delta' A\left(\max\left\{A^{-1}(\psi(x)), |u(x)|\right\}\right) \text{for } x \in \Omega a.e,\end{aligned} \tag{74}$$

where δ and δ' are constants. Thus, $h(x) = \delta' A(\max\{A^{-1}(\psi(x)), |u(x)|\})$. In consequence, it yields that $\widetilde{A}(|v_{n_{k_j}} - v|) = \widetilde{A}(|a(|u_{n_{k_j}}|)u_{n_{k_j}} - a(|u|)u|) \longrightarrow 0$ in $L^1(\Omega)$, as $j \longrightarrow \infty$. Due to arbitrariness of $(v_{n_k})_k$, we deduce that $\widetilde{A}(|v_n - v|) = \widetilde{A}(|a(|u_n|)u_n - a(|u|)u|) \longrightarrow 0$ in $L^1(\Omega)$, as $n \longrightarrow \infty$. Hence, $a(|u_n|)u_n \longrightarrow a(|u|)u$ in $L_{\widetilde{A}}(\Omega)$.

(ii) The proof is analogous to (i). ☐

Lemma 16. *The functional $\Phi_A(u) = \int_\Omega A(|u|)dx$ is convex, of class C^1 and sequentially weakly lower semicontinuous in X. Furthermore, if $(u_n)_n$ is a sequence of elements in X and u belongs to X such that u_n converge weakly to $u \in X$, then $\Phi_A'(u_n) \longrightarrow \Phi_A'(u)$ in X'.*

Proof. The convexity of Φ_A follows from the convexity of the positive function $t \longmapsto A(|t|)$ defined on $\mathbb{R}$. The function $t \longmapsto A(|t|)$ is also continuous and vanishes in zero. By Lemma 3 in [16], for all $k > 1$, for $\varepsilon > 0$, such that, $0 < \varepsilon < (1/k)$, we have

$$|A(|a+b|) - A(|a|)| \le \varepsilon \varphi_\varepsilon(a) + \psi_\varepsilon(b), \text{for all } a, b \in \mathbb{R}, \tag{75}$$

where $\varphi_\varepsilon(t) = A(k|t|) - kA(|t|)$, $\psi_\varepsilon(t) = 2A(|C_\varepsilon t|)$, for all $t \in \mathbb{R}$ with $(1/C_\varepsilon) = \varepsilon(k-1)$.

Let $(u_n)_n$ be a sequence of X and u an element of X such that $u_n \longrightarrow u$ in X. u belongs to $L_A(\Omega)$ by Lemma 8 and we have $\int_\Omega A(|u|)\mathrm{d}x < \infty$.$u_n \longrightarrow u$ in X implies $u_n \longrightarrow u$ in $H_0^1(\Omega)$. So, $u_n \longrightarrow u$ a.e in Ω. By (35), properties of function ζ_1 and Lemma 8, we have

$$\begin{aligned} 0 &\le \int_\Omega \varphi_\varepsilon(u_n - u)dx = \int_\Omega A\big(k|u_n - u|\big) - kA\big(|u_n - u|\big)dx \\ &\le \zeta_1\big(k\|u_n - u\|_A\big) \le k^q \zeta_1\big(\|u_n - u\|_A\big) \le k^q \zeta_1\big(\|u_n - u\|_X\big). \end{aligned} \tag{76}$$

As $u_n \longrightarrow u$ in X, there exists $M > 0$ such that for all integer n,

$$0 \le \int_\Omega \varphi_\varepsilon(u_n - u)dx \le k^q M. \tag{77}$$

Since u belongs to $L_A(\Omega)$, $A(|\theta u|) \in L^1(\Omega)$ for all $\theta \in \mathbb{R}$ and thus,

$$0 \le \int_\Omega \psi_\varepsilon(u)\mathrm{d}x < \infty, \text{for all } \varepsilon > 0. \tag{78}$$

Hence, by Theorem 2 in [16], it follows that $\Phi_A(u_n) \longrightarrow \Phi_A(u)$, as $n \longrightarrow \infty$. This assures the continuity of the functional Φ_A. It then follows that Φ_A is sequentially weakly lower semicontinuous in X by Corollary 3.9 in [15]. Furthermore, Φ_A is Gateaux-differentiable in X and for all $u, \varphi \in X$,

$$\langle \Phi_A'(u), \varphi\rangle = \int_\Omega a(|u(x)|)u(x)\varphi(x)dx. \tag{79}$$

Let $(u_n) \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X as $n \longrightarrow \infty$. Let us show that $\Phi_A'(u_n) \longrightarrow \Phi_A'(u)$ in X'. By Hölder's inequality, we have

$$\begin{aligned} \|\Phi_A'(u_n) - \Phi_A'(u)\|_{X'} &= \sup_{\substack{\varphi\in X\\ \|\varphi\|_X=1}} \left|\int_\Omega \left[a\big(|u_n|\big)u_n - a(|u|)u\right]\varphi \mathrm{d}x\right| \\ &\le 2 \sup_{\substack{\varphi\in X\\ \|\varphi\|_X=1}} \left\|a\big(|u_n|\big)u_n - a(|u|)u\right\|_{\widetilde{A}}\|\varphi\|_A \le 2\left\|a\big(|u_n|\big)u_n - a(|u|)u\right\|_{\widetilde{A}}. \end{aligned} \tag{80}$$

Since $u_n \rightharpoonup u$ in X, by Lemma 8 and Proposition 3, it follows that $\|a(|u_n|)u_n - a(|u|)u\|_{\widetilde{A}} \longrightarrow 0$ and consequently, $\Phi_A'(u_n) \longrightarrow \Phi_A'(u)$ in X'. In particular, this shows that Φ_A is class C^1 in X and the proof of the Lemma is finished. □

Lemma 17. *The functional $\Phi_B(u) = \int_\Omega B(|u|)dx$ is convex, of class C^1 and sequentially weakly lower semicontinuous in X . Moreover, if $(u_n)_n$ is a sequence of elements in X and $u \in X$ such that $u_n \rightharpoonup u \in X$ as $n \longrightarrow \infty$, then $\Phi_B'(u_n) \rightharpoonup \Phi_B'(u)$ in X' .*

Proof. The convexity of Φ_B follows from the convexity of the positive function $t \longmapsto B(|t|)$ defined on $\mathbb{R}$. The proof of continuity of Φ_B in X is analogous to the proof of continuity of Φ_A in X in the previous Lemma. In fact, by Lemma 3 in [16], for all $k > 1$, for $\varepsilon > 0$, such that, $0 < \varepsilon < (1/k)$, we have

$$|B(|x + y|) - B(|x|)| \le \varepsilon\varphi_\varepsilon(x) + \psi_\varepsilon(y), \quad \text{for all } x, y \in \mathbb{R}, \tag{81}$$

where $\varphi_\varepsilon(t) = B(k|t|) - kB(|t|)$, $\psi_\varepsilon(t) = 2B(|C_\varepsilon t|)$, forall$t \in \mathbb{R}$ with $(1/C_\varepsilon) = \varepsilon(k-1)$.

Let $(u_n)_n \subset X$ and $u \in X$ such that $u_n \longrightarrow u$ in X. By the definition of the space X, we have $X \longrightarrow L_B(\Omega)$, $\int_\Omega B(|u|)\mathrm{d}x < \infty$, and $u_n \longrightarrow u$ in $L_B(\Omega)$. $u_n \longrightarrow u$ in X implies $u_n \longrightarrow u$ in $H_0^1(\Omega)$. So, $u_n \longrightarrow u a.e$ in Ω. By (35) and properties of function ζ_2, we have

$$0 \le \int_\Omega \varphi_\varepsilon(u_n - u)dx \le k^r \zeta_2\big(\|u_n - u\|_X\big). \tag{82}$$

As $u_n \longrightarrow u$ in X, there exists $M > 0$ such that for all integer n, $0 \le \int_\Omega \varphi_\varepsilon(u_n - u)dx \le k^r M$. Since u belongs to $L_B(\Omega)$, $B(|\theta u|) \in L^1(\Omega)$ for all $\theta \in \mathbb{R}$ and thus,

$$0 \le \int_\Omega \psi_\varepsilon(u)dx < \infty, \quad \text{for all } \varepsilon > 0. \tag{83}$$

Hence, by Theorem 2 in [16], it follows that $\Phi_B(u_n) \longrightarrow \Phi_B(u)$ as $n \longrightarrow \infty$. This assures the continuity of the functional Φ_B in X. It then follows that Φ_B is sequentially weakly lower semicontinuous in X by Corollary 3.9 of [15]. Furthermore, Φ_B is Gateaux-differentiable in X and for all $u, \varphi \in X$,

$$\langle \Phi_B'(u), \varphi\rangle = \int_\Omega b(|u(x)|)u(x)\varphi(x)\mathrm{d}x. \tag{84}$$

Let $(u_n) \subset X$ and $u \in X$ such that $u_n \longrightarrow u$ in X as $n \longrightarrow \infty$. Let us show that $\Phi_B'(u_n) \longrightarrow \Phi_B'(u)$ in X'. By Hölder's inequality, we have

$$\|\Phi_B'(u_n) - \Phi_B'(u)\|_{X'} = \sup_{\substack{\varphi\in X\\ \|\varphi\|_X=1}} \left|\int_\Omega \left[b\big(|u_n|\big)u_n - b(|u|)u\right]\varphi \, dx\right| \le 2\left\|b\big(|u_n|\big)u_n - b(|u|)u\right\|_{\widetilde{B}}. \tag{85}$$

According to Proposition 3 (ii), it follows that $\|b(|u_n|)u_n - b(|u|)u\|_{\widetilde{B}} \longrightarrow 0$ and consequently, $\Phi_B'(u_n) \longrightarrow \Phi_B'(u)$ in X'. Thus, Φ_B is class C^1 in X.

Let us show the last part of this Lemma. Let $(u_n) \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X. We shall proof that $\Phi_B'(u_n) \overset{*}{\rightharpoonup} \Phi_B'(u)$ in X' as $n \longrightarrow \infty$. Let $\varphi \in X$. For any integer n, let us set $v_n = b(|u_n|)u_n$ and consider an arbitrary subsequence $(v_{n_k})_k$ of the sequence (v_n). We have $v_{n_k} = b(|n_k|)u_{n_k}$, where $(u_{n_k})_k$ is a subsequence of (u_n). Then, $u_{n_k} \rightharpoonup u$ weakly in X when $k \longrightarrow \infty$. By Lemma 3, it follows that $u_{n_k} \longrightarrow u$ in $L^2(\Omega)$. Thus, there exists a subsequence $(u_{n_{k_j}})_j$ of $(u_{n_k})_k$ such that $u_{n_{k_j}} \longrightarrow u$ a.e in Ω and $|u_{n_{k_j}}| \leq g$ a.e in Ω, where $g \in L^1(\Omega)$. In consequence, for all $\varphi \in X$, $v_{n_{k_j}}\varphi \longrightarrow b(|u|)u\varphi$ a.e in Ω and there exists $h \in L^1(\Omega)$ such that $|b(|u_{n_{k_j}}|)u_{n_{k_j}}\varphi - b(|u|)u\varphi| \leq h$ a.e in Ω. In fact, by (30) and (36) we have

$$\begin{aligned}\left|\left[b\left(\left|u_{n_{k_j}}\right|\right)u_{n_{k_j}} - b(|u|)u\right]\varphi\right| &\leq \left(b\left(\left|u_{n_{k_j}}\right|\right)\left|u_{n_{k_j}}\right| + b(|u|)|u|\right)|\varphi|,\\ &\leq rB(1)|\varphi|\left\{\max\left(\left|u_{n_{k_j}}\right|^{q-1}, \left|u_{n_{k_j}}\right|^{r-1}\right) + \max\left(|u|^{q-1}, |u|^{r-1}\right)\right\} a.e in \Omega,\\ &\leq rB(1)|\varphi|\left\{\max\left(g^{q-1}, g^{r-1}\right) + \max\left(|u|^{q-1}, |u|^{r-1}\right)\right\} a.e in \Omega.\end{aligned} \tag{86}$$

Thus, $h = rB(1)|\varphi|\left\{\max(g^{q-1}, g^{r-1}) + \max(|u|^{q-1}, |u|^{r-1})\right\} \in L^1(\Omega)$. Hence,

$$\int_\Omega \left|\left(v_{n_{k_j}} - b(|u|)u\right)\varphi\right| dx = \int_\Omega \left|\left[b\left(\left(u_{n_{k_j}}\right)\right)u_{n_{k_j}} - b(|u|)u\right]\varphi\right| dx \longrightarrow 0. \tag{87}$$

In conclusion, $\Phi_B'(u_n) \overset{*}{\rightharpoonup} \Phi_B'(u)$ in X', as $n \longrightarrow \infty$, since $(v_{n_k})_k$ is an arbitrary subsequence of the sequence (v_n).

For any $(x, u) \in \Omega \times \mathbb{R}$, let us set

$$f(x, u) = \lambda a(|u|)u - b(|u|)u, \tag{88}$$

$$\begin{aligned}F(x, u) &= \int_0^u f(x, v(x))dx\\ &= \lambda A(|u|) - B(|u|).\end{aligned} \tag{89}$$

□

Lemma 18. *For any fixed $u \in X$, the functional $\mathcal{F}_u$: $X \longrightarrow \mathbb{R}$, $v \longmapsto \int_\Omega f(x, u(x))v(x)dx$ is in X' (X' is the Banach dual space of X). In particular, if $v_n \rightharpoonup v in X$, then $\mathcal{F}_u(v_n) \longrightarrow \mathcal{F}_u(v)$.*

Proof. Let us take $u \in X$. It is easy to show that $\mathcal{F}_u$ is linear. Moreover, for all $v \in X$, by using Hölder inequality and imbedding properties of X in $L_A(\Omega)$ and X in $L_B(\Omega)$, we have

$$\begin{aligned}\left|\mathcal{F}_u(v)\right| &\leq \lambda \int_\Omega a(|u|)|u||v|dx + \int_\Omega b(|u|)|u||v|dx \leq 2\lambda\|a(|u|)|u|\|_{\widetilde{A}}\|v\|_A\\ &\quad + 2\|b(|u|)|u|\|_{\widetilde{B}}\|v\|_B \leq 2\left(C_A\lambda\|a(|u|)|u|\|_{\widetilde{A}} + 2\|b(|u|)|u|\|_{\widetilde{B}}\right)\|v\|_X.\end{aligned} \tag{90}$$

Thus, we conclude that $\mathcal{F}_u$ is continuous on X. Then, it follows that, $v_n \rightharpoonup v in X$ yields $\mathcal{F}_u(v_n) \longrightarrow \mathcal{F}_u(v)$. □

Lemma 19. *The functional Φ_λ is of class C^1 and sequentially weakly lower semicontinuous in X, that is if $u_n \rightharpoonup u$ in X, then*

$$\Phi_\lambda(u) \leq \liminf_{n \longrightarrow \infty} \Phi_\lambda(u_n). \tag{91}$$

Proof. Lemmas 14–17 imply $\Phi_\lambda \in C^1(X)$. Let $(u_n)_n \subset X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X. The definition of Φ_λ and (89) allow us to write

$$\Phi_\lambda(u)-\Phi_\lambda(u_n)=\frac{1}{2}\left(\|u\|^2_{H^1_0(\Omega)}-\|u_n\|^2_{H^1_0(\Omega)}\right)+\frac{1}{2}\left(\|u\|_2^2-\|u_n\|_2^2\right)+\int_\Omega[F(x,u_n(x))-F(x,u(x))]. \tag{92}$$

Since $u_n \rightharpoonup u$ in X, Lemmas 14 and 15 yield respectively that

$$\|u\|^2_{H^1_0(\Omega)}\le \liminf_{n\longrightarrow\infty}\|u_n\|^2_{H^1_0(\Omega)}, \tag{93}$$

$$\|u\|_2^2\le \liminf_{n\longrightarrow\infty}\|u_n\|_2^2. \tag{94}$$

Hence, by (92), we get

$$\limsup_{n\longrightarrow\infty}[\Phi_\lambda(u)-\Phi_\lambda(u_n)]\le \limsup_{n\longrightarrow\infty}\int_\Omega[F(x,u_n(x))-F(x,u(x))]. \tag{95}$$

By (88) and (89), for all $s\in[0,1]$,

$$\begin{aligned}F_u(x,u+s(u_n-u))&=\frac{\partial F}{\partial u}(x,u+s(u_n-u))\\&=f(x,u+s(u_n-u))\\&=f(x,u)+(u_n-u)\int_0^s f_u(x,u+t(u_n-u))dt.\end{aligned} \tag{96}$$

where $f_u(x,u)=(\partial f/\partial u)(x,u)=\lambda[|u|a'(|u|)+a\ \ (|u|)]-[|u|b'(|u|)+b(|u|)]$, forall$u\in X$.

Multiplying (96) by (u_n-u) and integrating the result over [0,1] with respect to s, we obtain

$$F(x,u_n)-F(x,u)=f(x,u)(u_n-u)+(u_n-u)^2\int_0^1\left(\int_0^s f_u(x,u+t(u_n-u))dt\right)ds. \tag{97}$$

According to the assumptions $(\mathscr{A}_2)$ and $(\mathscr{B}_2)$, we have

$$\text{for all } u\in X,\ f_u(x,u)\le(q-1)[\lambda a(|u|)-b(|u|)]. \tag{98}$$

Afterwards, (29) and (30) imply

$$\text{for all } u\in X\smallsetminus\{0\},\ f_u(x,u)\le q(q-1)[\lambda A(|u|)-B(|u|)]|u|^{-2}. \tag{99}$$

By (24), for all $u\in X\smallsetminus\{0\}$, we have

$$[\lambda A(|u(x)|)-B(|u(x)|)]|u|^{-2}\le\begin{cases}\lambda A(1)|u(x)|^{p-2}-B(1)|u(x)|^{r-2}, & \text{for } x\in\Omega_u,\\ \lambda A(1)|u(x)|^{q-2}-B(1)|u(x)|^{r-2}, & \text{for } x\in\Omega_u'.\end{cases} \tag{100}$$

Let $s=p-2$ or $s=q-2$. By applying Young's inequality $ab\le(a^\alpha/\alpha)+(b^\beta/\beta)$ for $a,b>0$ and $\alpha>1$ and $\beta>1$, we have

$$\lambda A(1)|u(x)|^s\le\frac{s}{r-2}B(1)|u(x)|^{r-2}+\frac{r-2-s}{r-2}\left(\frac{\lambda A(1)}{(B(1))^{(s/(r-2))}}\right)^{((r-2)/(r-2-s))}, \tag{101}$$

where we take $a = (B(1))^{(s/(r-2))}|u(x)|^s$, $b = \lambda A(1)(B(1))^{-(s/(r-2))}$, $\alpha = ((r-2)/s) > 1$, and $\beta = ((r-2)/(r-2-s)) > 1$. Inequality (101) yields

$$\lambda A(1)|u(x)|^s - B(1)|u(x)|^{r-2} \leq B(1)|u(x)|^{r-2}\left(\frac{s}{r-2} - 1\right) + \frac{r-2-s}{r-2}\left(\frac{\lambda A(1)}{(B(1))^{(s/(r-2))}}\right)^{((r-2)/(r-2-s))}, \tag{102}$$

$$\leq \lambda^{((r-2)/(r-2-s))}\frac{(r-2-s)}{r-2}\left(\frac{A(1)}{(B(1))^{s/(r-2)}}\right)^{((r-2)/(r-2-s))}, \tag{103}$$

being $s < r-2$. Hence, (100) yields

$$[\lambda A(|u(x)|) - B(|u(x)|)]|u(x)|^{-2} \leq \begin{cases} \lambda^{((r-2)/(r-p))}\dfrac{r-p}{r-2}\left(\dfrac{A(1)}{(B(1))^{((p-2)/(r-2))}}\right)^{((r-2)/(r-p))}, & \text{for } x \in \Omega_u, \\ \lambda^{((r-2)/(r-q))}\dfrac{r-q}{r-2}\left(\dfrac{A(1)}{(B(1))^{((q-2)/(r-2))}}\right)^{((r-2)/(r-q))}, & \text{for } x \in \Omega_u'. \end{cases} \tag{104}$$

As $> (r-p/r-2)(r-q/r-2)$, it follows, from (104), that

$$[\lambda A(|u(x)|) - B(|u(x)|)]|u(x)|^{-2} \leq \frac{r-p}{r-2}c_\lambda', \quad \text{for all } x \in \Omega, \tag{105}$$

where

$$c_\lambda' = \max\left\{\lambda^{((r-2)/(r-p))}\left(\frac{A(1)}{(B(1))^{((p-2)/(r-2))}}\right)^{((r-2)/(r-p))}, \lambda^{((r-2)/(r-q))}\left(\frac{A(1)}{(B(1))^{((p-2)/(r-2))}}\right)^{((r-2)/(r-q))}\right\}. \tag{106}$$

From (104) and (105), we obtain

$$\text{for all } u \in X, f_u(x,u) \leq \frac{q(q-1)(r-p)}{r-2}c_\lambda'. \tag{107}$$

Consequently, (97) yields

$$\begin{aligned} \int_\Omega [F(x,u_n) - F(x,u)]dx &\leq \int_\Omega f(x,u)(u_n - u)\mathrm{d}x + q(q-1)(r-p)c_\lambda'/2(r-2)\int_\Omega (u_n-u)^2\mathrm{d}x \\ &\leq \mathcal{F}_u(u_n-u) + \frac{q(q-1)(r-p)c_\lambda'}{2(r-2)}\|u_n-u\|_2^2. \end{aligned} \tag{108}$$

Since $u_n \rightharpoonup u$ in X, Lemmas 3 and 18 yield respectively $\mathcal{F}_u(u_n-u) \longrightarrow 0$ and $\|u_n-u\|_2 \longrightarrow 0$. Thus, $\limsup_{n\longrightarrow\infty}\int_\Omega [F(x,u_n(x)) - F(x,u(x))]\mathrm{d}x \leq 0$ and then, by (95), we get the claim (91). □

5. Existence of Weak Solutions of (ε_λ) for Large Values of λ

Let

$$\bar{\lambda} = \inf_{\substack{u \in X \\ \Phi_A(u)=1}} f \left\{ \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + |u|^2 \right) + \int_\Omega B(|u(x)|)\mathrm{d}x \right\} \text{with } \Phi_A(u) = \int_\Omega A(|u(x)|)\mathrm{d}x. \tag{109}$$

We remark that $\bar{\lambda} > 0$. Indeed, by the fact that $\Phi_A(u) = 1 \Rightarrow \|u\|_A = 1$, (48) implies that

$$\int_\Omega \left(|\nabla u|^2 + |u|^2 \right)\mathrm{d}x \geq \|u\|^2_{H^1_0(\Omega)} \geq \frac{1}{C_A^2}, \quad \text{for all } u \in X \text{ with } \Phi_A(u) = 1. \tag{110}$$

By the property (37), Lemma 5 and the monotony of function ζ_2, we have

$$\int_\Omega B(|u(x)|)\mathrm{d}x \geq \zeta_2(\|u\|_B) \geq \zeta_2\left(\frac{1}{\mathscr{H}}\right), \quad \text{for all } u \in X \text{ with } \Phi_A(u) = 1. \tag{111}$$

Thus,

$$\bar{\lambda} \geq \frac{1}{2C_A^2} + \zeta_2\left(\frac{1}{\mathscr{H}}\right) > 0. \tag{112}$$

Lemma 20. *For all $\lambda > \bar{\lambda}$, the functional Φ_λ admits in X a global nontrivial non-negative minimizer m with negative energy, that is $\Phi_\lambda(m) < 0$.*

Proof. For each $\lambda > 0$, Lemmas 11 and 19 and Corollary 3.23 in [15] assure to Φ_λ the existence of a global minimizer $m \in X$, that is

$$\Phi_\lambda(m) = \inf_{v \in X} \Phi_\lambda(v). \tag{113}$$

So, m is a solution of (ε_λ). Let us prove that m is nontrivial as soon as $\lambda > \bar{\lambda}$. For that we shall just prove that $\inf_{v \in X} \Phi_\lambda(v) < 0$. Let $\lambda > \bar{\lambda}$. By the definition of $\bar{\lambda}$, there exists a function $\varphi \in X$, with $\int_\Omega A(|u(x)|)\mathrm{d}x = 1$, such that

$$\lambda \int_\Omega A(|u(x)|)\mathrm{d}x = \lambda > \frac{1}{2} \int_\Omega \left(|\nabla \varphi|^2 + |\varphi(x)|^2 \right) + \int_\Omega B(|\varphi(x)|)\mathrm{d}x \geq \bar{\lambda}. \tag{114}$$

Hence,

$$\frac{1}{2} \int_\Omega \left(|\nabla \varphi|^2 + |\varphi(x)|^2 \right) - \lambda \int_\Omega A(|u(x)|)\mathrm{d}x + \int_\Omega B(|\varphi(x)|)\mathrm{d}x < 0 \tag{115}$$

and then

$$\Phi_\lambda(\varphi) < 0. \tag{116}$$

Therefore,

$$\Phi_\lambda(m) = \inf_{v \in X} \Phi_\lambda(v) \leq \Phi_\lambda(\varphi) < 0. \tag{117}$$

Hence, it follows that, for $\lambda > \bar{\lambda}$, equation (ε_λ) has a nontrivial weak solution $m \in X$ with $\Phi_\lambda(m) < 0$. To complete the proof, we can assume $m \geq 0$ a.e in Ω because $|m| \in X$ and $\Phi_\lambda(m) = \Phi_\lambda(|m|)$.

Let us define

$$\tilde{\lambda} = \inf\{\lambda > 0 : (\varepsilon_\lambda) \text{has an on trivial weak solution}\}. \tag{118}$$

By Lemma 20, it is clear that this definition is meaningful. Furthermore, it follows easily that $\bar{\lambda} > \tilde{\lambda}$. ☐

Theorem 3. *For any $\lambda > \tilde{\lambda}$, the problem (ε_λ) admits a nontrivial non-negative weak solution $u_\lambda \in X$.*

Proof. Fix $\lambda > \tilde{\lambda}$. By definition of $\tilde{\lambda}$, there exists $h \in (\tilde{\lambda}, \lambda)$ such that Φ_h has a nontrivial critical point $u_h \in X$. Without loss of generality, we assume that $u_h \geq 0 a.e$ in Ω, since $|u_h|$ is also a solution of (ε_h). We can easily see that u_h is a subsolution for (ε_λ). Let us consider the following minimization problem:

$$\inf_{v \in \mathscr{C}} \Phi_\lambda(v) \text{where } \mathscr{C} = \{v \in X : \text{for all} x \in \Omega, v(x) \geq u_h(x)\}. \tag{119}$$

We remark easily that $\mathscr{C}$ is a closed and convex set. Thus, $\mathscr{C}$ is weakly closed. Moreover, being Φ_λ is coercive in X by Lemma 11, it follows that it is coercive in $\mathscr{C}$. Finally, Φ_λ is sequentially weakly lower semicontinuous in X and so in $\mathscr{C}$.

Hence, Corollary 3.23 of [15] implies that there exists $u_\lambda \in \mathscr{C}$ such that

$$\Phi_\lambda(u_\lambda) = \inf_{v\in\mathscr{C}} \Phi_\lambda(v). \tag{120}$$

Now, we shall prove that u_λ is a solution of (ε_λ). Let $\vartheta \in C_0^\infty(\Omega)$ and $\epsilon > 0$. Let us set $\vartheta_\epsilon = \max\{0, u_h - u_\lambda - \epsilon\vartheta\}$ and $\omega_\epsilon = u_\lambda + \epsilon\vartheta + \vartheta_\epsilon$. It is clear that $\omega_\epsilon \in \mathscr{C}$ and

$$0 \le \langle \Phi_\lambda'(u_\lambda), \omega_\varepsilon - u_\lambda\rangle = \varepsilon\langle \Phi_\lambda'(u_\lambda), \vartheta\rangle + \langle \Phi_\lambda'(u_\lambda), \vartheta_\varepsilon\rangle. \tag{121}$$

Hence

$$\langle \Phi_\lambda'(u_\lambda), \vartheta\rangle \ge -\frac{1}{\varepsilon}\langle \Phi_\lambda'(u_\lambda), \vartheta_\varepsilon\rangle. \tag{122}$$

Let us set

$$\Omega_\varepsilon = \{x \in \Omega \colon u_\lambda(x) + \varepsilon\vartheta(x) \le u_h(x) < u_\lambda(x)\}. \tag{123}$$

It is obvious that $\Omega_\epsilon \subset \operatorname{supp}\vartheta$. Since, u_h is a subsolution of (ε_λ) and $\vartheta_\epsilon \ge 0$, it turns out that $\langle \Phi_\lambda'(u_\lambda), \vartheta_\epsilon\rangle \le 0$. Consequently, as

$$\begin{gathered}\langle \Phi_\lambda'(u_\lambda), \vartheta_\varepsilon\rangle = \langle \Phi_\lambda'(u_h), \vartheta_\varepsilon\rangle + \langle \Phi_\lambda'(u_\lambda) - \Phi_\lambda'(u_h), \vartheta_\varepsilon\rangle,\\ f(u(x)) = \lambda a(|u(x)|)u(x) - b(|u(x)|)u(x),\end{gathered} \tag{124}$$

We have

$$\begin{aligned}\langle \Phi_\lambda'(u_\lambda), \vartheta_\varepsilon\rangle &\le \langle \Phi_\lambda'(u_\lambda) - \Phi_\lambda'(u_h), \vartheta_\varepsilon\rangle\\ &\le \int_{\Omega_\varepsilon} (\nabla u_\lambda - \nabla u_h)\cdot\nabla(u_h - u_\lambda - \varepsilon\vartheta)dx + \int_{\Omega_\varepsilon}(u_\lambda - u_h)(u_h - u_\lambda - \varepsilon\vartheta)\mathrm{d}x\\ &\quad - \int_{\Omega_\varepsilon}[f(u_\lambda) - f(u_h)]\cdot(u_h - u_\lambda - \varepsilon\vartheta)dx\\ &\le \int_{\Omega_\varepsilon} (\nabla u_\lambda - \nabla u_h)\cdot\nabla(-\varepsilon\vartheta)dx + \int_{\Omega_\varepsilon}(u_\lambda - u_h)(u_h - u_\lambda - \varepsilon\vartheta)dx\\ &\quad - \int_{\Omega_\varepsilon}[f(u_\lambda) - f(u_h)]\cdot(u_h - u_\lambda - \varepsilon\vartheta)\mathrm{d}x.\end{aligned} \tag{125}$$

(i) For $\Omega_\varepsilon = \varnothing$, it is clear that $\langle \Phi_\lambda'(u_\lambda), \vartheta_\epsilon\rangle \le 0$. Hence by (122), we have $\langle \Phi_\lambda'(u_\lambda), \vartheta\rangle \ge 0$.

(ii) Also, when $\Omega_\varepsilon \ne \varnothing$, we have $0 \le u_h - u_\lambda - \varepsilon\vartheta = u_h - u_\lambda + \varepsilon|\vartheta| < \varepsilon|\vartheta|$ in Ω_ε because in this case $\vartheta \le 0$ in Ω_ϵ. Then, (125) yields

$$\langle \Phi_\lambda'(u_\lambda), \vartheta_\varepsilon\rangle \le \varepsilon\left(\int_{\Omega_\varepsilon}(\nabla u_h - \nabla u_\lambda)\cdot\nabla(\vartheta)\mathrm{d}x + \int_{\Omega_\varepsilon}(u_\lambda - u_h)|\vartheta|\mathrm{d}x + \int_{\Omega_\varepsilon}\left|f(u_\lambda) - f(u_h)\right|\cdot|\vartheta|dx\right) \le \varepsilon\int_{\Omega_\varepsilon}\psi(x)\mathrm{d}x, \tag{126}$$

where $\psi(x) = (\nabla u_h - \nabla u_\lambda).\nabla\vartheta + [(u_\lambda - u_h) + |f(u_\lambda) - f(u_h)|]|\vartheta|$. We claim that $\psi \in L^1(\operatorname{supp}\vartheta)$. Indeed, ∇u_h and ∇u_λ are in $[L^2(\Omega)]^N$ and u_λ and u_h are in $L^1_{loc}(\Omega)$ since $X \subset H_0^1(\Omega)$. Moreover, by $(\mathscr{A}_1)$ and $(\mathscr{B}_1)$, the functions $t \longmapsto a(t)t$ and $t \longmapsto b(t)t$ are continuous in $\mathbb{R}_+$. Thus, the functions $a(|u_\lambda|)|u_\lambda|$, $a(|u_h|)|u_h|$, $b(|u_\lambda|)|u_\lambda|$, and $b(|u_h|)|u_h|$ are continuous in Ω. Consequently, $a(|u_\lambda|)|u_\lambda|$, $a(|u_h|)|u_h|$, $b(|u_\lambda|)|u_\lambda|$, and $b(|u_h|)|u_h|$ belong to $L^1_{\mathrm{loc}}(\Omega)$. Finally, $|f(u_\lambda) - f(u_h)| \in L^1_{\mathrm{loc}}(\Omega)$ since $|f(u_\lambda) - f(u_h)| \le \lambda(a(|u_\lambda|)|u_\lambda| + a(|u_h|)|u_h|) + (b(|u_\lambda|)|u_\lambda| + b(|u_h|)|u_h|)$. Therefore, the claim is obtained. Thus,

$$\lim_{\varepsilon\longrightarrow 0^+}\int_{\Omega_\varepsilon}\psi(x)\mathrm{d}x = 0, \tag{127}$$

since $|\Omega_\epsilon| \longrightarrow 0$ as $\epsilon \longrightarrow 0^+$. Then, (126) implies that $\langle \Phi_\lambda'(u_\lambda), \vartheta_\varepsilon\rangle \le {}^\circ(\varepsilon)$ as $\epsilon \longrightarrow 0^+$. So, by (122), it follows that $\langle \Phi_\lambda'(u_\lambda), \vartheta\rangle \ge {}^\circ(1)$ as $\epsilon \longrightarrow 0^+$.

We deduce that, $\langle \Phi_\lambda'(u_\lambda), \vartheta\rangle \ge 0$ for all $\vartheta \in C_0^\infty(\Omega)$, that is $\langle \Phi_\lambda'(u_\lambda), \vartheta\rangle = 0$ for all $\vartheta \in C_0^\infty(\Omega)$. Thus, u_λ is a weak solution of (ε_λ) in X because X is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_X$. Finally, u_λ is nontrivial and non-negative, since $u_\lambda > u_h$. □

Lemma 21. $\tilde{\lambda} = \hat{\lambda}$.

Proof. This proof is the same to that one did in the step 5 of the proof of (Theorem 1.1 in [7]). In fact, by Theorem 3, we have $\tilde{\lambda} \ge \hat{\lambda}$. Indeed, for all λ such that (ε_λ) a weak nontrivial solution, we have $\lambda > \widehat{\lambda}$ and thus, $\widetilde{\lambda} \geq \widehat{\lambda}$. Let us show that $\widetilde{\lambda} \leq \widehat{\lambda}$. Suppose that $\widetilde{\lambda} > \widehat{\lambda}$. For $\lambda < \widetilde{\lambda}$, the problem (ε_λ) cannot admit a nontrivial solution $u \in X$ since this would contradict the minimality of $\widetilde{\lambda}$. Thus, for all $\lambda \in [\widehat{\lambda}, \widetilde{\lambda})$ the unique solution of (ε_λ) is $u \equiv 0$. This assertion is still again impossible because it would contradict the maximality of $\widehat{\lambda}$. Hence, $\widetilde{\lambda} = \widehat{\lambda}$. □

Theorem 4. *The problem $(\varepsilon_{\widehat{\lambda}})$ admits a nontrivial non-negative weak solution in X.*

Proof. Let $(\lambda_n)_{n\in\mathbb{N}^*}$ be a strictly decreasing sequence converging to $\widehat{\lambda}$ and $u_n \in X$ be a nontrivial non-negative weak solution of $(\varepsilon_{\lambda_n})$forall$n \in \mathbb{N}^*$. For all n, we have

$$\int_\Omega \nabla u_n \nabla v \mathrm{d}x = -\int_\Omega u_n v \mathrm{d}x + \lambda_n \int_\Omega a(|u_n|) u_n v \mathrm{d}x - \int_\Omega b(|u_n|) u_n v \mathrm{d}x, \quad \text{for all } v \in X. \tag{128}$$

By (29), (30), (49), and Lemma 4 we have

$$\|u\|_{H_0^1(\Omega)} + q\zeta_2(\|u_n\|_B) \leq q\lambda_n \zeta_1(\|u_n\|_A). \tag{129}$$

Furthermore, by inequality (60) and the monotony of ζ_1 there exists a constant K_A such that

$$\|u_n\|_{H_0^1(\Omega)} + q\zeta_2(\|u_n\|_B) \leq K_A q C_{\lambda_n}, \quad \text{for all } n \in \mathbb{N}^*, \tag{130}$$

where

$$C_{\lambda_n} = \max\left\{\lambda_n^{1+pr/2(r-p)}\left(\frac{A(1)}{(B(1))^{p/r}}\right)^{pr/2(r-p)}, \lambda_n^{1+qr/2(r-p)}\left(\frac{A(1)}{(B(1))^{p/r}}\right)^{qr/2(r-p)}, \lambda_n^{1+pr/2(r-q)}\left(\frac{A(1)}{(B(1))^{q/r}}\right)^{pr/2(r-q)}, \lambda_n^{1+qr/2(r-q)}\left(\frac{A(1)}{(B(1))^{q/r}}\right)^{qr/2(r-q)}\right\}, \quad \text{for all } n \in \mathbb{N}^*. \tag{131}$$

Thus, by monotony of sequence $(\lambda_n)_{n\in\mathbb{N}^*}$, we get

$$\|u_n\|_{H_0^1(\Omega)} + q\zeta_2(\|u_n\|_B) \leq K_A q C_{\lambda_1}, \quad \text{for all } n \in \mathbb{N}^*. \tag{132}$$

Therefore, the sequences $(\|u_n\|_{H_0^1(\Omega)})_n$ and $(\|u_n\|_B)_n$ are bounded, and it yields that the sequence $(\|u_n\|_X)_n$ is bounded. According to Theorem 3.18 in [15], Propositions 1 and 2 and Lemma 8 allow to extract, from the sequence $(u_n)_n$, a subsequence still relabeled $(u_n)_n$ and satisfying

$$u_n \rightharpoonup u \text{ in} X, u_n \longrightarrow u \text{in} L_A(\Omega), u_n \rightharpoonup u \text{ in} L_B(\Omega), u_n \longrightarrow u \; a.\text{e in}\Omega, \nabla u_n \rightharpoonup \nabla u \text{ in} \left[L^2(\Omega)\right]^N, \tag{133}$$

for some $u \in X$. We claim that u, which is clearly non-negative by (133), is the solution we are looking for. In fact, for all $v \in X$,

$$\int_\Omega \nabla u_n \nabla v \mathrm{d}x \longrightarrow \int_\Omega \nabla u \nabla v \mathrm{d}x, \tag{134}$$

as $n \longrightarrow \infty$, since $\nabla u_n \rightharpoonup \nabla u \text{ in} [L^2(\Omega)]^N$ by (133). Since $u_n \rightharpoonup u$ in X, Lemma 15 yields in particular that for all $v \in X$,

$$\int_\Omega u_n v \mathrm{d}x \longrightarrow \int_\Omega u v \mathrm{d}x, \tag{135}$$

as $n \longrightarrow \infty$. Moreover, Lemmas 16 and 17 imply that for all $v \in X$

$$\int_\Omega a(|u_n|) u_n v \mathrm{d}x \longrightarrow \int_\Omega a(|u|) u v \mathrm{d}x, \qquad \int_\Omega b(|u_n|) u_n v \mathrm{d}x \longrightarrow \int_\Omega b(|u|) u v \mathrm{d}x, \tag{136}$$

as $n \longrightarrow \infty$. By passing to the limit in (128) as $n \longrightarrow \infty$, we get by (134)–(136)

$$\int_\Omega \nabla u \nabla v \mathrm{d}x + \int_\Omega u v \mathrm{d}x = \widehat{\lambda} \int_\Omega a(|u|) u v \mathrm{d}x - \int_\Omega b(|u|) u v \mathrm{d}x\Omega, \tag{137}$$

for all $v \in X$. Hence, u is a weak non-negative solution of $(\varepsilon_{\widehat{\lambda}})$. It remains to show that the solution u is nontrivial. Since $u_n \rightharpoonup u$ in X by (133), Lemma 8 yields in particular that $\|u\|_A = \lim \|u_n\|_A$. Moreover, (52) applied to each $u_n \equiv 0$, implies that

$$\|u_n\|_A \geq \min\left\{\left(qC_A^2\right)^{1/(2-p)} \lambda_n^{1/(2-p)}, \left(qC_A^2\right)^{1/(2-q)} \lambda_n^{1/(2-q)}\right\}. \tag{138}$$

Thus, by passing to the limit as $n \longrightarrow \infty$, we get

$$\|u\|_A \geq \min\left\{\left(qC_A^2\right)^{1/(2-p)}(\widehat{\lambda})^{1/(2-p)}, \left(qC_A^2\right)^{1/(2-q)}(\widehat{\lambda})^{1/(2-q)}\right\}, \tag{139}$$

since $\lambda_n \searrow \widehat{\lambda}$ and $\widehat{\lambda} > 0$. Consequently, u is nontrivial and non-negative by (133).

Proof of Theorem 1. Section 3, Lemma 20, and Theorems 3 and 4 show the existence of $\widehat{\lambda} > 0$ such that for all $\lambda \geq \widehat{\lambda}$, (ε_λ) admits at least a nontrivial non-negative weak solution in X . □

6. Second Solution for Large Values of the Parameter

By variational methods, we prove, in this section, that the Dirichlet problem (ε_λ) admits at least two nontrivial weak solutions if λ is sufficiently large.

Lemma 22. *For any $m \in X \setminus \{0\}$ and $\lambda > 0$ there exist $\varrho \in (0, \|m\|_{H_0^1(\Omega)})$ and $\alpha = \alpha(\varrho) > 0$ such that $\Phi_\lambda(u) \geq \alpha$ for all $u \in X$, with $\|u\|_{H_0^1(\Omega)} = \varrho$.*

Proof. Let $m \in X \setminus \{0\}$ and $\lambda > 0$. Let u be in X. By (35) and (48) and monotony of the function ζ_1,

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda\int_\Omega A(|u|)dx \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda\zeta_1(\|u\|_A) \geq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda\zeta_1\left(C_A\|u\|_{H_0^1(\Omega)}\right)\\ &\geq \left[\frac{1}{2} - \lambda\max\left\{C_A^p\|u\|^{p-2}_{H_0^1(\Omega)}, C_A^q\|u\|^{q-2}_{H_0^1(\Omega)}\right\}\right]\|u\|^2_{H_0^1(\Omega)}.\end{aligned} \tag{140}$$

So, to find the result of the Lemma, it is enough to take ϱ such that

$$\% \ 0 < \varrho < \min\left\{\left(2\lambda C_A^p\right)^{1/(2-p)}, \left(2\lambda C_A^q\right)^{1/(2-q)}, \|m\|_{H_0^1(\Omega)}\right\}, \tag{141}$$

$$\alpha = \alpha(\varrho) = \left[\frac{1}{2} - \lambda\max\left\{C_A^p\varrho^{p-2}, C_A^q\varrho^{q-2}\right\}\right]\varrho^2 > 0. \tag{142}$$

In Lemma 20, we have shown that for all $\lambda > \overline{\lambda}$, the problem (ε_λ) admits a nontrivial non-negative weak solution $m \in X$ which is above all a global minimizer of Φ_λ in X, with $\Phi_\lambda(m) < 0$. In this section, we are looking for a second nontrivial weak solution of (ε_λ) when $\lambda > \overline{\lambda}$. □

Lemma 23. *There exists $\widehat{\kappa} > 0$ such that*

$$\text{for all}\,(x, y) \in \mathbb{R}^2,\ (b(|x|)x - b(|y|)y)(x - y) \geq \widehat{\kappa}B(|x - y|) \geq 0. \tag{143}$$

Proof

(i) If $x = y$, the inequality is verified for all $\widehat{\kappa} > 0$.

(ii) If $x \neq 0$ and $y = 0$ (or $x = 0$ and $y \neq 0$), the inequality is obtained by (30) and $\widehat{\kappa} = q > 0$.

(iii) Suppose now that $x \neq y$ with $x \neq 0$ and $x \neq 0$. By hypothesis $(\mathscr{B}_1)$, the function $t \longmapsto b(|t|)t$ is strictly increased in $[0; +\infty)$ and consequently it is strictly increased in $(-\infty; +\infty)$ because it is an odd function. Thus, for $x \neq y$ with $x \neq 0$ and $x \neq 0$, $(b(|x|)x - b(|y|)y)(x - y) > 0$ and

$$\varphi(x, y) = \frac{(b(|x|)x - b(|y|)y)(x - y)}{B(|x - y|} > 0. \tag{144}$$

Let

$$\kappa = \inf_{\substack{(x,y)\in(\mathbb{R}^*)^2\\ x\neq y}} \varphi(x, y) > 0. \tag{145}$$

In conclusion, taking $\widehat{\kappa} = \max\{q, \kappa\}$, we get the result of this Lemma. □

Theorem 5 (see Theorem A.3 in [10]). *Let $(X, \|\cdot\|)$ and $(X, \|.\|_E)$ be two Banach spaces such that $X \longrightarrow E$. Let Φ: $X \longrightarrow \mathbb{R}$ be a C^1 functional with $\Phi(0) = 0$. Suppose that there exist $\varrho, \alpha > 0$ and $m \in X$ such that $\|m\|_E > \varrho$, $\Phi(m) < \alpha$, and $\Phi(u) \geq \alpha$ for all $u \in X$ with $\|u\|_E = \varrho$. Then, there exists a sequence $(u_n)_n \in X$ such that for all n*

$$c \leq \Phi(u_n) \leq c + \frac{1}{n^2}\,\text{and}\left\|\Phi'(u_n)\right\|_{X'} \leq \frac{2}{n}, \tag{146}$$

where

$$\begin{aligned}c &= \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi(\gamma(t)),\\ \Gamma &= \{\gamma \in C([0,1]; X)\colon \gamma(0) = 0, \gamma(1) = m\},\end{aligned} \tag{147}$$

Proof. One can find the proof of this theorem in the section (Appendix A in [10]).

Proof of Theorem 2. Let λ be a strictly positive fixed number such that $\lambda > \overline{\lambda}$. Then, let $m \in X$ be the global minimizer of Φ_λ given by Lemma 20. Thanks to Lemma 22 and the fact that $\Phi_\lambda(m) < 0$, the assumptions of Theorem 5 which is a variant of Ekeland's variational principle are fulfilled for the energy functional Φ_λ of (ε_λ). Hence, there exists a sequence $(u_n)_n \subset X$ such that

$$\begin{aligned} \Phi_\lambda(u_n) &\longrightarrow c, \\ \|\Phi'(u_n)\|_{X'} &\longrightarrow 0, \end{aligned} \tag{148}$$

where

$$\begin{aligned} c &= \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)), \\ \Gamma &= \{\gamma \in C([0,1]; X)\colon \gamma(0) = 0, \gamma(1) = m\}. \end{aligned} \tag{149}$$

In the sequel, we shall prove that the sequence $(u_n)_n$ strongly converges to some u in X and that u is a second nontrivial non-negative weak solution of (ε_λ). □

Step 1. By Lemma 11 (coercivity of Φ_λ on X), the sequence $(u_n)_n$ is bounded in X and consequently, $(u_n)_n$ is bounded in $H_0^1(\Omega)$. For the sequel, let us use the argument of the proof of Theorem 4. Hence, by Propositions 1 and 2 and Lemma 8 we can extract from the sequence $(u_n)_n$, a subsequence still relabeled $(u_n)_n$ and satisfying

$$u_n \rightharpoonup u \text{ in } X, u_n \longrightarrow u \text{ in } L_A(\Omega), u_n \rightharpoonup u \text{ in } L_B(\Omega), u_n \longrightarrow u \; a.\text{e in } \Omega, \nabla u_n \rightharpoonup \nabla u \text{ in } \left[L^2(\Omega)\right]^N, \tag{150}$$

for some $u \in X$. It remains to show that u is a nontrivial solution of (ε_λ), with $u \neq m$. We have

$$\langle \Phi_\lambda'(u_n), v\rangle = \int_\Omega \nabla u_n \nabla v \mathrm{d}x - \int_\Omega \left[-u_n + \lambda a(|u_n|)u_n - b(|u_n|)u_n\right] v \mathrm{d}x, \text{ for all } n, \text{ for all } v \in X. \tag{151}$$

By the same argument used in the proof Theorem 4, we have, for all $v \in X$,

$$\begin{aligned} \int_\Omega \nabla u_n \nabla v \, dx \longrightarrow \int_\Omega \nabla u \nabla v \mathrm{d}x, \int_\Omega u_n v \, dx \longrightarrow \int_\Omega uv \mathrm{d}x, \int_\Omega a(|u_n|)u_n v \mathrm{d}x \\ \longrightarrow \int_\Omega a(|u|)uv \mathrm{d}x, \int_\Omega b(|u_n|)u_n v \mathrm{d}x \longrightarrow \int_\Omega b(|u|)uv \mathrm{d}x, \end{aligned} \tag{152}$$

as $n \longrightarrow \infty$. Hence, passing to the limit as $n \longrightarrow \infty$ in (151), using and the fact that $\langle \Phi_\lambda'(u_n), v\rangle \longrightarrow 0$ as $n \longrightarrow \infty$ for all $v \in X$, we get

$$\begin{aligned} \int_\Omega \nabla u_n \nabla v \mathrm{d}x + \int_\Omega uv \mathrm{d}x = \lambda \int_\Omega a(|u|)uv \mathrm{d}x \\ - \int_\Omega b(|u|)uv \mathrm{d}x, \text{forall} v \in X, \end{aligned} \tag{153}$$

so u is a weak solution of (ε_λ).

Step 2. We claim that

$$\mathcal{I}_a(n) = \int_\Omega \left(a(|u_n|)u_n - a(|u|)u\right)(u_n - u)\mathrm{d}x \longrightarrow 0, \tag{154}$$

as $n \longrightarrow \infty$. By hypothesis $(\mathscr{A}_1)$, the function $t \longmapsto a(|t|)t$ is strictly increased in $[0; +\infty)$ and consequently, it is strictly increased in $(-\infty; +\infty)$ because it is an odd function. Thus, we have

$$\mathcal{I}_a(n) \geq 0. \tag{155}$$

By (150), $u_n \longrightarrow u$ in $L_A(\Omega)$. Thus, $a(|u_n|)u_n \longrightarrow a(|u|)u)$ in $L_{\tilde{A}}(\Omega)$ by Proposition 3. Therefore, by Hölder inequality, we get

$$0 \leq \int_\Omega (a)(|u_n|)u_n - a(|u|)u)(u_n - u)\mathrm{d}x \leq 2\left\|a(|u_n|)u_n - a(|u|)u\right\|_{\tilde{A}} \|u_n - u\|_A \longrightarrow 0, \tag{156}$$

as $n \longrightarrow \infty$. This gives the proof of the claim.

Step 3. In this step, we show that $\|u_n - u\|_X \longrightarrow 0$ as $n \longrightarrow \infty$. Let us set

$$\mathcal{I}_b(n) = \int_\Omega \big(b(|u_n|)u_n\big) - b(|u|)u)(u_n - u)\mathrm{d}x. \tag{157}$$

For the same arguments, as for $\mathcal{I}_a(n)$, we have

$$\mathcal{I}_b(n) \geq 0. \tag{158}$$

Note that $\langle \Phi_\lambda'(u_n) - \Phi_\lambda'(u), u_n - u\rangle \longrightarrow 0$ as $n \longrightarrow \infty$, since $u_n \longrightarrow u$ in X and $\Phi_\lambda'(u_n) \longrightarrow 0$ in X' as $n \longrightarrow \infty$. Hence, by (154)

$$\|u_n - u\|^2_{H_0^1(\Omega)} + \|u_n - u\|_2^2 + \mathcal{I}_b(n) = \langle \Phi_\lambda'(u_n) - \Phi_\lambda'(u), u_n - u\rangle + \lambda \mathcal{I}_a(n) \longrightarrow 0, \tag{159}$$

as $n \longrightarrow \infty$. Furthermore, we have

$$\|u_n - u\|^2_{H_0^1(\Omega)} \longrightarrow 0, \tag{160}$$

as $n \longrightarrow \infty$. By (143) and (159), it follows that

$$\int_\Omega B(|u_n - u|)\mathrm{d}x \leq \frac{1}{\widetilde{\kappa}}\mathcal{I}_b(n) \longrightarrow 0, \tag{161}$$

as $n \longrightarrow \infty$. Thus, by section 8.13 in [11],

$$\|u_n - u\|_B \longrightarrow 0 \,\text{as}\, n \longrightarrow \infty, \tag{162}$$

since the N-function B satisfied the Δ_2-condition. Finally,

$$\|u_n - u\|_X \longrightarrow 0 \,\text{as}\, n \longrightarrow \infty, \tag{163}$$

by (160) and (162).

Step 4. Since $u_n \longrightarrow u$ in X and $\Phi_\lambda \in C^1(X)$, we get $\Phi_\lambda(u) = c = \lim_{n\longrightarrow\infty}\Phi_\lambda(u_n)$. So, u is a second independent nontrivial weak solution of (ε_λ), with $\Phi_\lambda(u) = c > 0 > \Phi_\lambda(m)$. We can assume $u \geq 0$ a.e. in Ω, since $|u|$ is also a solution of (ε_λ) due to $\Phi_\lambda(|u|) = \Phi_\lambda(u)$. This concludes the proof.

7. Conclusion

In this paper we studied the nonexistence, the existence, and multiplicity results for nontrivial weak solutions of the semilinear elliptic Dirichlet problem with (ε_λ) involving a positive parameter λ. Our main results are obtained in the Theorems 1 and 2 by using variational methods.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Ambrosetti, H. Brezis, and G. Cerami, “Combined effects of concave and convex nonlinearities in some elliptic problems,” *Journal of Functional Analysis*, vol. 122, no. 2, pp. 519–543, 1994.

[2] S. Alama and G. Tarantello, “Elliptic problems with nonlinearities indefinite in sign,” *Journal of Functional Analysis*, vol. 141, no. 1, pp. 159–215, 1996.

[3] D. G. De Figueiredo, J.-P. Gossez, and P. Ubilla, “Local superlinearity and sublinearity for indefinite semilinear elliptic problems,” *Journal of Functional Analysis*, vol. 199, no. 2, pp. 452–467, 2003.

[4] A. Kristály and G. Morosanu, “New competition phenomena in Dirichlet problems,” *Journal de Mathematiques Pures et Appliquees*, vol. 94, no. 6, pp. 555–570, 2010.

[5] S. Li and Z. Q. Wang, “Mountain pass theorem in order intervals and multiple solutions for semilinear elliptic Dirichlet problems,” *Journal d’Analyse Mathematique*, vol. 81, no. 1, pp. 373–396, 2000.

[6] A. Ambrosetti, J. Garcia Azorero, and I. Peral, “Multiplicity results for some nonlinear elliptic equations,” *Journal of Functional Analysis*, vol. 137, no. 1, pp. 219–242, 1996.

[7] R. Filippucci, P. Pucci, and R, V. dulescu, ̆a Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions,” *Communications in Partial Differential Equations*, vol. 33, no. 4-6, pp. 706–717, 2008.

[8] P. Pucci and V. Radulescu, “Combined effects in quasilinear elliptic problems with lack of compactness,” *Rendiconti Lincei. Matematica e Applicazioni*, vol. 22, pp. 189–205, 2011.

[9] L. Baldelli and R. Filippucci, “Existence of solutions for critical (p, q)-Laplacian equations in $\mathbb{R}^N$,” *Communications in Contemporary Mathematics*, vol. 2022, Article ID 2150109, 2022.

[10] G. Autuori and P. Pucci, “Existence of entire solutions for a class of quasilinear elliptic equations,” *Nonlinear Differential Equations and Applications NoDEA*, vol. 20, no. 3, pp. 977–1009, 2013.

[11] R. A. Adams and J. F. Fournier, *Sobolev Spaces*, Academic Press, Cambridge, MA, USA, 2003.

[12] M. A. Krasnosielskii and Y. B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, Holland, 1961.

[13] G. Talenti, “Best constant in Sobolev inequality,” *Annali di Matematica Pura ed Applicata*, vol. 110, no. 1, pp. 353–372, 1976.

[14] N. Fukagai, M. Ito, and K. Narukawa, “Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on $\mathbb{R}^N$,” *Funkcialaj Ekvacioj*, vol. 49, no. 2, pp. 235–267, 2006.

[15] H. Brezis, “Functional Analysis, Sobolev Spaces and Partial Differential Equations,” *Universitext*, Springer, New York, NY, USA, 2011.

[16] H. Brezis and E. Lieb, “A relation between pointwise convergence of functions and convergence of functionals,” *Proceedings of the American Mathematical Society*, vol. 88, no. 3, pp. 486–490, July 1983.