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Research Article

On State Ideals and State Relative Annihilators in De Morgan State Residuated Lattices

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The concept of state has been considered in commutative and noncommutative logical systems, and their properties are at the center for the development of an algebraic investigation of probabilistic models for those algebras. This article mainly focuses on the study of the lattice of state ideals in De Morgan state residuated lattices (DMSRLs). First, we prove that the lattice of all state ideals $\mathcal{SI}(L)$ of a DMSRL (L, τ) is a coherent frame. Then, we characterize the DMSRL for which the lattice $\mathcal{SI}(L)$ is a Boolean algebra. In addition, we bring in the concept of state relative annihilator of a given nonempty subset with respect to a state ideal in DMSRL and investigate various properties. We prove that state relative annihilators are a particular kind of state ideals. Finally, we investigate the notion of prime state ideal in DMSRL and establish the prime state ideal theorem.

1. Introduction

It is known to all that the algebraic research on logical systems has considerable applications. In particular, it plays a meaningful role in artificial intelligence, which make computer simulate human being in dealing with fuzzy and uncertain information. In a large number of multivalued logic and fuzzy logic of algebraic systems, the residuated lattice is an important class, which was brought in by Ward and Dilworth in Ref. [1] as a generalization of ideal lattices of rings. Then, in 2018, Liviu-Constantin Holdon introduced an important variety of this structure called De Morgan residuated lattice, which comprises salient subclasses of residuated lattices such as Boolean algebras, BL-algebras, MTL-algebras, Stonean residuated lattices, and regular residuated lattices (MV-algebras, IMTL-algebras) (see Ref. [2]). Furthermore, he considered ideals and annihilators in this new structure.

The concept of internal state also called state operator was introduced by Flaminio and Montagna ([3, 4]) by adding a unary operation τ, which preserves the usual properties of states to the language of MV-algebras. Since then, several authors deeply investigated this topic in other algebraic structures (see [5–8]). State residuated lattices were initiated by He et al. [9]. They introduced the concept of state operators on residuated lattices and investigated some related properties. Moreover, they inserted the notion of state filter in state residuated lattices. Following this study, Kondo and Kawaguchi recently studied generalized state operators on state residuated lattices (see [10, 11]). In 2017, Dehghani and Forouzesh [12] made a deep investigation on state filters in state residuated lattices and inducted the concept of prime state filters in state residuated lattices and proved the prime state filter theorem. Woumfo et al. [13] introduced the notion of state ideal in state residuated lattices and established that the lattice of state ideals is a complete lattice. In this article, stimulated by the previous research on the structure of De Morgan residuated lattices and by the importance of the theory of ideals and annihilators in MV-algebras, BL-algebra, and Stonean residuated lattices (see [14–17]), we analyze the notions of state ideal and state relative annihilator in a new class of state residuated lattices called De Morgan state residuated lattices.

This article is divided into four sections: in the first one, we describe some preliminaries comprising the basic definitions, some rules of calculus and theorems that are needed in the sequel. Section 2 is the part in which we consider the algebraic structure of the set $\mathcal{SI}(L)$ of all state ideals in a DMSRL (L, τ). It is shown that $(\mathcal{SI}(L), \subseteq)$ is a coherent frame. Also, we characterize the DMSRL for which the lattice $\mathcal{SI}(L)$ is a Boolean algebra. In Section 3, we bring in the notion of state annihilator of a nonempty set X with respect to a state ideal I in a DMSRL and analyze some of its properties. We demonstrate that state relative annihilators are a particular kind of state ideals. In the last section, we put emphasis on the notion of prime state ideals in a DMSRL. We prove the prime state ideal theorem and investigate some allied properties.

2. Preliminaries

We summarize here some fundamental definitions and results about residuated lattices. Readers can obtain more details in Refs. [1, 18–21].

A nonempty set L with four binary operations $\wedge, \vee, \odot, \longrightarrow$ and two constants $0, 1$ is called a bounded integral commutative residuated lattice or shortly residuated lattice if the following axioms are verified:

(C1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(C2) $(L, \odot, 1)$ is a commutative monoid (with the unit element 1);

(C3) For all $x, y \in L$, $x \odot y \leq z \Leftrightarrow x \leq y \longrightarrow z$.

The following notations of residuated lattices will be used:

L will stand for $(L, \wedge, \vee, \odot, \longrightarrow, 0, 1)$, a residuated lattice and for any $x \in L$ and $n \in \mathbb{N}^*$, $x\prime := x \longrightarrow 0$, $x\prime\prime := (x\prime)\prime$, $x^0 := 1$ and $x^n := x^{n-1} \odot x$.

In Refs. [2,9,14,21], we have the following definitions and examples:

(1) A residuated lattice satisfying the divisibility condition (div): $x \wedge y = x \odot (x \longrightarrow y)$ is called a R ℓ-monoid.

(2) A residuated lattice satisfying the prelinearity condition (pre): $(x \longrightarrow y) \vee (y \longrightarrow x) = 1$ is called a MTL-algebra.

(3) A residuated lattice satisfying prelinearity and divisibility conditions is called a BL-algebra.

(4) A residuated lattice satisfying the double negation condition (dn): $x\prime\prime = x$ is called a regular residuated lattice.

(5) A BL-algebra satisfying the double negation condition is called an MV-algebra.

(6) A MTL-algebra satisfying the double negation condition is called an IMTL-algebra.

(7) A residuated lattice satisfying the Boolean properties $x \vee x\prime = 1$ and $x \wedge x\prime = 0$ is called a Boolean algebra;

(8) A residuated lattice satisfying the Stone property $x' \vee x'' = 1$ is called a Stonean residuated lattice;

(9) A residuated lattice satisfying the De Morgan property $(x \wedge y)' = x' \vee y'$ is called a De Morgan residuated lattice.

We shall notice from (see [2]) that Boolean algebras, BL-algebras, MTL-algebras, Stonean residuated lattices, and regular residuated lattices (MV-algebras, IMTL-algebras) are particular important subclasses of De Morgan residuated lattices.

Example 1. Let $L = \{0, a, b, c, d, e, f, g, 1\}$ endowed with the Hasse diagram and Caley tables (Figure 1):

Then, $L = (L, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ is a nonregular residuated lattice. One can easily check that L is a De Morgan residuated lattice, which is Stonean.

Example 2. Let $L = \{0, p, a, b, c, d, e, f, q, 1\}$ with his Hasse diagram and Cayley tables of $\odot$ and $\longrightarrow$ be the following (Figure 2):

Then, $L = (L, \wedge, \vee, \odot, \longrightarrow, 0, 1)$ is a residuated lattice.

One can readily verify that L is a De Morgan residuated, which is regular. But L is not a BL-algebra, L is not a MV-algebra, L is not a Boolean algebra, L is not a Stonean residuated lattice, L is not a MTL-algebra, and L is not a IMTL-algebra.

The following basic arithmetic of residuated lattices will be used for any $x, y, z \in L$ (see [19,22]):

(RL1): $1 \longrightarrow x = x$, $x \longrightarrow x = 1$, $x \longrightarrow 1 = 1$, $0 \longrightarrow x = 1$;

(RL2): $x \leq y \Leftrightarrow x \longrightarrow y = 1$;

(RL3): $x \longrightarrow y = y \longrightarrow x = 1 \Leftrightarrow x = y$;

(RL4): if $x \leq y$, then $y \longrightarrow z \leq x \longrightarrow z$, $z \longrightarrow x \leq z \longrightarrow y$, $x \odot z \leq y \odot z$ and $y\prime \leq x\prime$;

(RL5): $x \odot (x \longrightarrow y) \leq y$; $x \odot (x \longrightarrow y) \leq x \wedge y$;

(RL6): $x \odot y \leq x \wedge y \leq x$, $y \leq x \vee y$; $x \leq y \longrightarrow x$; $x \odot y \leq x \longrightarrow y, y \longrightarrow x$;

(RL7): $(x \odot y)\prime\prime = x\prime\prime \odot y\prime\prime$, $(x \vee y)\prime = x\prime \wedge y\prime$ and $(x \wedge y)\prime \geq x\prime \vee y\prime$;

(RL8): $0\prime = 1$, $1\prime = 0$;

(RL9): $x \leq x\prime\prime \leq x\prime \longrightarrow x$;

(RL10): $x \longrightarrow y \leq y\prime \longrightarrow x\prime$;

(RL11): $x''' = x'$, $(x \odot y)\prime = x \longrightarrow y\prime = y \longrightarrow x\prime = x\prime\prime \longrightarrow y\prime$;

(RL12): $x \odot x\prime = 0$, $x \odot y = 0 \Leftrightarrow x \leq y\prime$; $x \odot 0 = 0$;

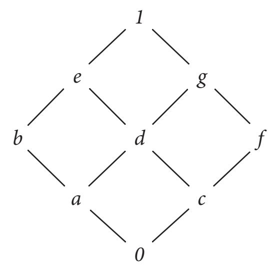
(RL13): $x\prime \longrightarrow y \leq (x\prime \odot y\prime)\prime$;

(RL14): $x \longrightarrow (x \wedge y) = x \longrightarrow y$;

(RL15): $x \odot y = x \odot (x \longrightarrow x \odot y)$.

Let L be a residuated lattice, and we set $x \oslash y = x\prime \longrightarrow y$, for every $x, y \in L$.

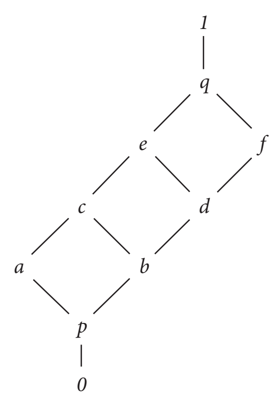
Definition 1 (see [19]). An ideal of L is a nonempty subset I of L, which satisfies the following conditions for every $x, y \in L$:



$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

FIGURE 1: A De Morgan residuated lattice which is Stonean.



$\odot$	0	p	a	b	c	d	e	f	q	1
0	0	0	0	0	0	0	0	0	0	0
p	0	0	0	0	0	0	0	0	0	p
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	0	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
q	0	0	a	b	c	d	e	f	q	q
1	0	p	a	b	c	d	e	f	q	1

$\rightarrow$	0	p	a	b	c	d	e	f	q	1
0	1	1	1	1	1	1	1	1	1	1
p	q	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
q	p	p	a	b	c	d	e	f	1	1
1	0	p	a	b	c	d	e	f	q	1

FIGURE 2: A De Morgan residuated lattice which is not Stonean.

(I1) If $y \in I$ and $x \leq y$, then $x \in I$;

(I2) If $x, y \in I$, then $x \oslash y \in I$.

By the fact that $\oslash$ is neither commutative nor associative in residuated lattices and in order to get an operation with properties closed to addition properties, Buşneag et al. defined a commutative and associative operation $\oplus$ in residuated lattices as follows: $x \oplus y = (x' \odot y')'$, for every $x, y \in L$.

Note that in Ref. [2], it was shown that a nonempty subset I of a residuated lattice L is an ideal of L if and only if the following conditions hold for any $x, y \in L$:

(I3) If $y \in I$ and $x \leq y$, then $x \in I$;

(I4) If $x, y \in I$, then $x \oplus y \in I$.

The set of all ideals of a residuated lattice L will be denoted by $\mathcal{I}(L)$.

For a nonempty subset X of a residuated lattice L, the ideal generated by X is $\langle X \rangle : = \{a \in L: a \leq x_1 \oplus x_2 \oplus, \ldots, \oplus x_n, \text{for some } n \in \mathbb{N}^*, x_i \in X, \text{for} 1 \leq i \leq n\}$ [14].

Here are some properties of the operation $\oplus$ (see [13,14]). Let L be a residuated lattice. For any $x, y, z, t \in L$, we have

(P1): $x \oplus y = x' \longrightarrow y'' = y' \longrightarrow x''$;

(P2): $x \oplus x' = 1$, $x \oplus 0 = x''$, $x \oplus 1 = 1$;

(P3): $x \oplus y = y \oplus x$, $x, y \leq x \oplus y$;

(P4): $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(P5): If $x \leq y$, then $x \oplus z \leq y \oplus z$;

(P6): If $x \leq y$ and $z \leq t$, then $x \oplus z \leq y \oplus t$.

Let L be a residuated lattice. For any $x \in L$ and $n \in \mathbb{N}$, we define $0x = 0$, $1x = x$, and $nx = (n-1)x \oplus x$, for $n \geq 2$.

Then, the following relations hold for any $x, y \in L$ and $m, n \geq 2$:

(P7): $m \leq n \Rightarrow mx \leq nx$. In particular, $x \leq nx$;

(P8): $x \leq y \Rightarrow mx \leq my$;

(P9): $n(x \oplus y) = nx \oplus ny$;

(P10): $x \oplus ny \leq n(x \oplus y)$;

(P11): $x \oslash y \leq x \oplus y$.

Lemma 1. *In any De Morgan residuated lattice L, the following relations hold for any $x, y, z \in L$ AND $m, n \geq 2$[2]:*

(P12): $(x \wedge y)'' = x'' \wedge y''$;

(P13): $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$;

(P14): $x \wedge (ny) \leq n(x \wedge y)$;

(P15): $(mx) \wedge (ny) \leq mn(x \odot y) \leq mn(x \wedge y)$;

(P16): $x \wedge (y \oplus z) \leq (x \wedge y) \oplus (x \wedge z)$.

Now, we give some necessary results for the sequel about lattices and frames. It is worth noting that the main references for frames theory are the following books [23, 24].

A lattice $(L, \wedge, \vee)$ is called Brouwerian if it satisfies the equality $x \wedge (\bigvee_{k \in K} y_k) = \bigvee_{k \in K} (x \wedge y_k)$ (whenever the arbitrary joins exist), for any $x, y_k \in L$, $k \in K$.

Definition 2. We call frame a complete lattice L that satisfies the infinite distributive law $x \wedge \vee A = \vee \{x \wedge a: a \in A\}$, for all $x \in L$ and $A \subseteq L$[25].

Remark 1

(1) Every Brouwerian lattice $(L, \wedge, \vee)$ is distributive;

(2) A frame is a complete Brouwerian lattice.

An element x of a complete lattice L is called compact if for all $A \subseteq L$, $a \leq \vee A$ implies that $a \leq \vee H$ for some finite $H \subseteq A$ (see [9,26]). We will denote by $\mathcal{C}(L)$ the set of all compact elements of a complete lattice L.

Proposition 1. *Let L be a frame and $x \in L$ [27]. Then, $x \in \mathcal{C}(L)$ if for all $A \subseteq L$, $x = \vee A$ implies that $x = \vee H$ for some finite $H \subseteq A$.*

Definition 3. A frame L is called coherent if the following conditions hold [27]:

(i) $\mathcal{C}(L)$ is a sublattice of L; that is, for all $x, y \in L$, if $x, y \in \mathcal{C}(L)$, then $x \wedge y, x \vee y \in \mathcal{C}(L)$;

(ii) For all $x \in L$, $x = \bigvee_{k \in K} x_k$, with $x_k \in \mathcal{C}(L)$.

Recall that if L is a lattice with 0 its bottom element, and $x \in L$, then $y \in L$ is said to be a pseudocomplement of x if $x \wedge y = 0$ and for every $z \in L$, $x \wedge z = 0$ implies $z \leq y$. L is called pseudocomplemented if every element has a pseudocomplement. Every frame is pseudocomplemented (see [28]).

For every $x, y \in L$, we call a relative pseudocomplement of x with respect to y, the greatest element (if it exists) $z \in L$ such that $x \wedge z \leq y$.

In what follows, we recall some results about state residuated lattices, which will be used in the sequel.

Definition 4 (see [9]). A map $\tau: L \longrightarrow L$ is said to be a state operator on L if the following conditions hold for any $x, y \in L$ [25]:

(SO1): $\tau(0) = 0$;

(SO2): $x \longrightarrow y = 1$ implies $\tau(x) \longrightarrow \tau(y) = 1$;

(SO3): $\tau(x \longrightarrow y) = \tau(x) \longrightarrow \tau(x \wedge y)$;

(SO4): $\tau(x \odot y) = \tau(x) \odot \tau(x \longrightarrow (x \odot y))$;

(SO5): $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$;

(SO6): $\tau(\tau(x) \longrightarrow \tau(y)) = \tau(x) \longrightarrow \tau(y)$;

(SO7): $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$;

(SO8): $\tau(\tau(x) \wedge \tau(y)) = \tau(x) \wedge \tau(y)$.

The pair (L, τ) is called a state residuated lattice.

We shall notice that for any residuated lattice L, the identity map id_L is a state operator on L, which is an endomorphism, but in general, a state operator τ is not an endomorphism.

Definition 5. Let (L, τ) be a state residuated lattice [9, 13]. An ideal I of L is said to be a state ideal of (L, τ) if $\tau(I) \subseteq I$

(i.e., for all $x \in L$, $x \in I \Rightarrow \tau(x) \in I$). Similarly, a filter F of L is called a state filter if $\tau(F) \subseteq F$.

$\mathcal{SI}(L)$ will stand for the set of all state ideals of (L, τ). It is obvious that $\{0\}, L \in \mathcal{SI}(L) \subseteq \mathcal{I}(L)$.

For computational issues, we will use the following properties (see [9, 13]).

Let (L, τ) be a state residuated lattice. Then, for any $x, y \in L$, for all $n \geq 1$, we have

(SO9): $\tau(1) = 1$;

(SO10): $x \leq y$ implies $\tau(x) \leq \tau(y)$;

(SO11): $\tau(x\prime) = (\tau(x))\prime$;

(SO12): $\tau(x \odot y) \geq \tau(x) \odot \tau(y)$ and if $x \odot y = 0$, then $\tau(x \odot y) = \tau(x) \odot \tau(y)$;

(SO13): $\tau(x \odot y\prime) \geq \tau(x) \odot (\tau(y))\prime$ and if $x \leq y$, then $\tau(x \odot y\prime) = \tau(x) \odot (\tau(y))\prime$;

(SO14): $\tau(x \longrightarrow y) \leq \tau(x) \longrightarrow \tau(y)$. Particularly, if x, y are comparable, then $\tau(x \longrightarrow y) = \tau(x) \longrightarrow \tau(y)$;

(SO15): If τ is faithful, then $x < y$ implies $\tau(x) < \tau(y)$;

(SO16): $\tau^2(x) = \tau(\tau(x)) = \tau(x)$;

(SO17): $\tau(L) = Fix(\tau)$, where $Fix(\tau) = \{x \in L \colon \tau(x) = x\}$;

(SO18): $\tau(L)$ is a subalgebra of L;

(SO19): $\ker(\tau) = \{x \in L \colon \tau(x) = 1\}$ is a state filter of (L, τ);

(SO20): $\operatorname{coker}(\tau) = \{x \in L \colon \tau(x) = 0\}$ is a state ideal of (L, τ);

(SO21): $(\tau(x))\prime\prime = \tau(x\prime\prime)$;

(SO22): $\tau(x \oplus y) \leq \tau(x) \oplus \tau(y)$;

(SO23): If $x, y \in \tau(L)$, then $x \oplus y \in \tau(L)$.

(SO24): $\tau(nx) \leq n\tau(x)$.

For any nonempty subset X of L, we denote by $\langle X \rangle_\tau$ the state ideal of (L, τ) generated by X,=; that is, $\langle X \rangle_\tau$ is the smallest state ideal of (L, τ) containing X, and for an element $a \in L$, $\langle a \rangle_\tau \colon = \langle \{a\} \rangle_\tau$ is called the principal state ideal of (L, τ). If $I \in \mathcal{SI}(L)$ and $a \notin I$, we denote by $\langle I, a \rangle_\tau \colon = \langle I \cup \{a\} \rangle_\tau$.

The next theorem gives the concrete description of the state ideal generated by a nonempty subset of a state residuated lattice.

Theorem 1 (see [13]). *Let X be a nonempty subset of L, $I, I_1, I_2 \in \mathcal{SI}(L)$ and $a \in L \backslash I$. Then,*

(1) $\langle X \rangle_\tau = \{x \in L \colon x \leq n_1(x_1 \oplus \tau(x_1)) \oplus \ldots \oplus n_k(x_k \oplus \tau(x_k))$, for some $k \in \mathbb{N}^*$, $x_i \in X$, $n_i \in \mathbb{N}^*$, for $1 \leq i \leq k\}$;

(2) $\langle a \rangle_\tau = \{x \in L \colon x \leq n(a \oplus \tau(a))$, for some $n \in \mathbb{N}^*\}$;

(3) $\langle I, a \rangle_\tau = \{x \in L \colon x \leq i \oplus n(a \oplus \tau(a))$, for some $i \in I$ and $\in \mathbb{N}^*\}$;

(4) $I_1 \vee I_2 \colon = \langle I_1 \cup I_2 \rangle_\tau = \{x \in L \colon x \leq i_1 \oplus i_2$, with $i_1 \in I_1$ and $i_2 \in I_2\}$.

Lemma 2 (see [13]). *Let (L, τ) be a state residuated lattice. For all $a, b \in L$, we have*

(5) $a \leq b \Rightarrow \langle a \rangle_\tau \subseteq \langle b \rangle_\tau$;

(6) $\langle \tau(a) \rangle_\tau \subseteq \langle a \rangle_\tau$;

(7) $\langle a \oplus \tau(a) \rangle_\tau = \langle a \rangle_\tau$;

(8) $\langle (a \oplus \tau(a)) \wedge (b \oplus \tau(b)) \rangle_\tau \subseteq \langle a \rangle_\tau \cap \langle b \rangle_\tau$;

(9) $\langle a \rangle_\tau \vee \langle b \rangle_\tau = \langle a \vee b \rangle_\tau = \langle a \oplus b \rangle_\tau$.

Proposition 2. *Let (L, τ) be a state residuated lattice [13]. Then,*

$(\mathcal{SI}(L), \subseteq)$ is a bounded complete lattice with the bottom element $\{0\}$ and the top element L.

Now, we introduce the concept of De Morgan state residuated lattice.

Definition 6. A state residuated lattice (L, τ) is called De Morgan if L is a De Morgan residuated lattice. More precisely, a De Morgan state residuated lattice is a De Morgan residuated lattice endowed with a state operator.

The following remarks give the relationship between the above definition and the notion of state-morphism operator studied on universal algebras in Ref. [29], also with the notion of generalized state in Ref. [18].

Remark 2. Let L be a De Morgan residuated lattice.

(1) If a map $\tau \colon L \longrightarrow L$ is an idempotent De Morgan-endomorphism (i.e., τ is an endomorphism of L such that $\tau^2 = \tau$), then τ is a state operator on L and it is said to be a state-morphism operator. Therefore, the couple (L, τ) is called a De Morgan state-morphism residuated lattice. So, by taking a state-morphism operator τ (which is a particular type of state operator), our theorems can be extended to the general setting of universal algebras as in Ref. [29].

(2) Let τ be a state operator on L. From Definition 1.9, we have

(SO1) $\tau(0) = 0$;

(SO3) $\tau(x \longrightarrow y) = \tau(x) \longrightarrow \tau(x \wedge y)$, for any $x, y \in L$.

Thus, from Ref. [18], Proposition 8.1 (iii) and Definition 8.1 (1), τ is a generalized Bosbach state of type 1. Moreover, from (SO10), we have $x \leq y$ implies $\tau(x) \leq \tau(y)$, for any $x, y \in L$. Therefore, any state operator τ on a De Morgan residuated lattice L can be seen as an ordering-preserving generalized Bosbach state of type 1.

A state operator τ is called cofaithful if $\operatorname{coker}(\tau) = \{0\}$ and uncofaithful otherwise.

Example 3. Set $L = \{0; a; b; 1\}$ with $0 < a < b < 1$. Then, L is De Morgan residuated lattice that is a BL-algebra but not an MV-algebra with the operations (Figure 3):

Let define the unary operator τ on L by

⊙	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

→	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

FIGURE 3: Operations of the De Morgan residuated lattice of exemple 3.

$$\tau(x) = \begin{cases} 0, & \text{if x = 0,} \\ a, & \text{if x = a;} \\ 1, & \text{if x} \in \{b, 1\}. \end{cases} \quad (1)$$

One can easily check that τ is a state operator on L. Hence, the couple (L, τ) is a DMSRL. In addition, $\operatorname{coker}(\tau) = \{0\}$ and τ verified the following properties: $\tau(x \longrightarrow y) = \tau(x) \longrightarrow \tau(y)$ and $\tau(x \odot y) = \tau(x) \odot \tau(y)$, for any $x, y \in L$. Therefore, τ is a cofaithful De Morgan state-morphism operator. Furthermore, the state ideals of (L, τ) are $\{0\}$ and L.

Example 4. Let L_1 and L_2 be two nontrivial De Morgan residuated lattices and h: $L_1 \longrightarrow L_2$ a homomorphism. We define the map $\begin{array}{l}\tau_h: L_1 \times L_2 L_1 \times L_2 \\ (x, y) \mapsto \tau_h(x, y) = (x, h(x)).\end{array}$ Then, one can check that τ_h is an idempotent endomorphism on $L_1 \times L_2$. So, τ_h is a state-morphism operator and the couple $(L_1 \times L_2, \tau_h)$ is a De Morgan state-morphism residuated lattice. In addition, $\operatorname{coker}(\tau_h) = \{0\} \times L_2$. Hence, τ_h is a uncofaithful state operator.

Remark 3.

(1) For every De Morgan residuated lattice L, (L, id_L) is a De Morgan state residuated lattice. That is, a De Morgan residuated lattice L can be seen as a De Morgan state residuated lattice. One can see that all ideals of L is a state ideal of (L, id_L).

(2) Let L be a De Morgan residuated lattice, S a sub-De Morgan residuated lattice of L and τ: $L \longrightarrow L$ be a state operator on L such that $\tau(S) \subseteq S$. Then, the restriction of τ on S is a state operator on S. We can see that $\operatorname{coker}(\tau/S) = S \cap \operatorname{coker}(\tau)$ is a state ideal of S.

This means that if a state operator preserves a substructure, then its restriction to this substructure is a state operator.

(3) Let $\{L_i: i \in I\}$ be a nonempty family of De Morgan residuated lattices and $\{\tau_i: i \in I\}$ a family of state operators such that τ_i is a state operator on L_i for each $i \in I$. We define the map $\begin{array}{l}\tau: \prod_{i \in I} L_i \longrightarrow \prod_{i \in I} L_i \\ (x_i)_{i \in I} \mapsto (\tau_i(x_i))_{i \in I}.\end{array}$

Then, τ is a state operator on $\prod_{i \in I} L_i$. We can check that $\operatorname{coker}(\tau) = \prod_{i \in I} \operatorname{coker}(\tau_i)$.

So, if we have state operators on structures, then we can define a state operator on the product and compute its cokernel easily.

Lemma 3. *If (L, τ) is a De Morgan state residuated lattice, then for all $a, b \in L$, we have $\langle (a \oplus \tau(a)) \wedge (b \oplus \tau(b)) \rangle_\tau = \langle a \rangle_\tau \cap \langle b \rangle_\tau$.*

Proof. Let $a, b \in L$. Then, from Lemma 2 (8), we have $\langle (a \oplus \tau(a)) \wedge (b \oplus \tau(b)) \rangle_\tau \subseteq \langle a \rangle_\tau \cap \langle b \rangle_\tau$. Now, let $x \in \langle a \rangle_\tau \cap \langle b \rangle_\tau$. Then, $x \in \langle a \rangle_\tau$ and $x \in \langle b \rangle_\tau$. Hence, from Theorem 1 (2), there exists $m, n \geq 1$ such that, $x \leq m(a \oplus \tau(a))$ and $x \leq n(b \oplus \tau(b))$. Therefore,

$$\begin{aligned} x &\leq m(a \oplus \tau(a)) \wedge n(b \oplus \tau(b)) \\ &\overset{(P15)}{\leq} mn(a \oplus \tau(a)) \wedge (b \oplus \tau(b))) \\ &\overset{(P3)}{\leq} mn((a \oplus \tau(a)) \wedge (b \oplus \tau(b))) \oplus \tau((a \oplus \tau(a)) \wedge (b \oplus \tau(b))) \end{aligned} \quad (2)$$

That is, $x \in \langle (a \oplus \tau(a)) \wedge (b \oplus \tau(b)) \rangle_\tau$. Hence, $\langle a \rangle_\tau \cap \langle b \rangle_\tau \subseteq \langle (a \oplus \tau(a)) \wedge (b \oplus \tau(b)) \rangle_\tau$. Thus, $\langle (a \oplus \tau(a)) \wedge (b \oplus \tau(b)) \rangle_\tau = \langle a \rangle_\tau \cap \langle b \rangle_\tau$. □

3. The Lattice of All State Ideals of a DMSRL

In this section, we focus on the study of the algebraic structure of the set $\mathcal{SI}(L)$ of all state ideals of a De Morgan state residuated lattice (L, τ).

From now on, unless otherwise specified, (L, τ) will always denote a De Morgan state residuated lattice $(L, \vee, \wedge, \odot, \longrightarrow, 0, 1)$; that is, L is a De Morgan residuated lattice and τ is a state operator on L.

Proposition 3. *$(\mathcal{SI}(L), \subseteq)$ is a Brouwerian lattice.*

Proof. Let K be an index set, $I \in \mathcal{SI}(L)$, and $\{I_k\}_{k \in K}$ be a family of state ideals of (L, τ). We will show that $I \wedge (\bigvee_{k \in K} I_k) = \bigvee_{k \in K} (I \wedge I_k)$. That is,

$I \cap \langle \bigcup_{k \in K} I_k \rangle_\tau = \langle \bigcup_{k \in K} (I \cap I_k) \rangle_\tau$. Clearly, $\langle \bigcup_{k \in K} (I \cap I_k) \rangle_\tau \subseteq I \cap \langle \bigcup_{k \in K} I_k \rangle_\tau$.

Let $x \in I \cap \langle \bigcup_{k \in K} I_k \rangle_\tau$. Then, $x \in I$ and $x \in \langle \bigcup_{k \in K} I_k \rangle_\tau$. It follows that there exist $k_1, k_2, \ldots, k_m \in K$, $x_{k_j} \in I_{k_j}$, $1 \leq j \leq$, such that $x \leq x_{k_1} \oplus x_{k_2} \oplus \ldots \oplus x_{k_m}$. Then, $x = x \wedge (x_{k_1} \oplus x_{k_2} \oplus \ldots \oplus x_{k_m}) \overset{(P16)}{\leq} (x \wedge x_{k_1}) \oplus (x \wedge x_{k_2}) \oplus \ldots \oplus (x \wedge x_{k_m})$. Since $I, I_{k_j} \in \mathcal{SI}(L)$, we have $x \wedge x_{k_j} \in I \cap I_{k_j}$, for every $1 \leq j \leq m$. We deduce that $x \in \langle \bigcup_{k \in K} (I \cap I_k) \rangle_\tau$. Hence, $I \cap \langle \bigcup_{k \in K} I_k \rangle_\tau \subseteq \langle \bigcup_{k \in K} (I \cap I_k) \rangle_\tau$, that is, $I \wedge (\bigvee_{k \in K} I_k) = \bigvee_{k \in K} (I \wedge I_k)$. Therefore, $(\mathcal{SI}(L), \subseteq)$ is a Brouwerian lattice. □

Theorem 2. *The lattice $(\mathcal{SI}(L), \subseteq)$ is a frame.*

Proof. From Proposition 2, $(\mathcal{SI}(L), \subseteq)$ is a complete lattice. From Proposition 3, $(\mathcal{SI}(L), \subseteq)$ is a Brouwerian lattice. Combining them, we have by Remark 1 (2) that $(\mathcal{SI}(L), \subseteq)$ is a frame.

In the following result, we establish a concrete description of the right adjoint of the map:

$$\begin{aligned} I_1 \cap \_ &: \mathcal{SI}(L) \longrightarrow \mathcal{SI}(L) \\ I &\mapsto (I_1 \cap \_)(I) = I_1 \cap I \end{aligned} \tag{3}$$

Now, for any $I_1, I_2 \in \mathcal{SI}(L)$, we put $I_1 \longrightarrow I_2 = \{x \in L: I_1 \cap \langle x \rangle_\tau \subseteq I_2\}$. □

Proposition 4. *In the frame $(\mathcal{SI}(L), \subseteq)$, for any $I_1, I_2, I \in \mathcal{SI}(L)$, we have*

(1) $I_1 \longrightarrow I_2 \in \mathcal{SI}(L)$

(2) $I_1 \cap I \subseteq I_2 \Leftrightarrow I \subseteq I_1 \longrightarrow I_2$, that is, $I_1 \longrightarrow I_2 = \sup\{I \in \mathcal{SI}(L): I_1 \cap I \subseteq I_2\}$ and $I_1 \longrightarrow \_: \mathcal{SI}(L) \longrightarrow \mathcal{SI}(L)$, $I \mapsto (I_1 \longrightarrow \_)(I) = I_1 \longrightarrow I$ is the right adjoint of $I_1 \cap \_: \mathcal{SI}(L) \longrightarrow \mathcal{SI}(L)$;

(3) $I_1 \longrightarrow I_2 = \{x \in L: i \wedge n(x \oplus \tau(x)) \in I_2$, for all $i \in I_1$ and $n \in \mathbb{N}^*\}$.

Proof

(1) We will show that $I_1 \longrightarrow I_2$ is a state ideal of (L, τ). We have $I_1 \longrightarrow I_2 \neq \emptyset$. Indeed, $\langle 0 \rangle_\tau = \{0\}$ and $I_1 \cap \langle 0 \rangle_\tau = \{0\} \subseteq I_2$. Hence, $0 \in I_1 \longrightarrow I_2$.

Now, let $x, y \in I_1 \longrightarrow I_2$, then $I_1 \cap \langle x \rangle_\tau \subseteq I_2$ and $I_1 \cap \langle y \rangle_\tau \subseteq I_2$. We obtain $(I_1 \cap \langle x \rangle_\tau) \vee (I_1 \cap \langle y \rangle_\tau) \subseteq I_2$. From Theorem 2, it follows that $(I_1 \cap \langle x \rangle_\tau) \vee (\langle y \rangle_\tau) \subseteq I_2$, which implies (by Lemma 3 (9)) that $I_1 \cap \langle x \oplus y \rangle_\tau \subseteq I_2$. Thus, $x \oplus y \in I_1 \longrightarrow I_2$.

Assume $x \in I_1 \longrightarrow I_2$ and $y \leq x$. Then, $I_1 \cap \langle x \rangle_\tau \subseteq I_2$ and $\langle y \rangle_\tau \overset{\text{Lemma1.12 (9)}}{\subseteq} \langle x \rangle_\tau$. It follows that $I_1 \cap \langle y \rangle_\tau \subseteq I_1 \cap \langle x \rangle_\tau \subseteq I_2$. Hence, $y \in I_1 \longrightarrow I_2$. Finally, let $x \in I_1 \longrightarrow I_2$; then, $I_1 \cap \langle x \rangle_\tau \subseteq I_2$. Since $I_1 \cap \langle \tau(x) \rangle_\tau \subseteq I_1 \cap \langle x \rangle_\tau \subseteq I_2$, we obtain that $\tau(x) \in I_1 \longrightarrow I_2$. Therefore, $I_1 \longrightarrow I_2$ is a state ideal of (L, τ). That is, $I_1 \longrightarrow I_2 \in \mathcal{SI}(L)$.

(2) Now, we show that $I_1 \cap I \subseteq I_2 \Leftrightarrow I \subseteq I_1 \longrightarrow I_2$, for any $I, I_1, I_2 \in \mathcal{SI}(L)$. Assume $I_1 \cap I \subseteq I_2$ and $x \in I$, we obtain $I_1 \cap \langle x \rangle_\tau \subseteq I_1 \cap I \subseteq I_2$ (since $\tau(x) \in I$)). It follows that $x \in I_1 \longrightarrow I_2$. That is, $I \subseteq I_1 \longrightarrow I_2$. Conversely, let $I \subseteq I_1 \longrightarrow I_2$ and $x \in I_1 \cap I$. Then, we have $x \in I \subseteq I_1 \longrightarrow I_2$. That is, $I_1 \cap \langle x \rangle_\tau \subseteq I_2$. Since $x \in I_1 \cap \langle x \rangle_\tau \subseteq I_2$, we deduce that $x \in I_2$, which implies $I_1 \cap I \subseteq I_2$. Therefore, $I_1 \longrightarrow I_2 = \sup\{I \in \mathcal{SI}(L): I_1 \cap I \subseteq I_2\}$ and $I_1 \longrightarrow \_: \mathcal{SI}(L) \longrightarrow \mathcal{SI}(L) I \mapsto (I_1 \longrightarrow \_)(I) = I_1 \longrightarrow I$ is the right adjoint of $I_1 \cap \_: \mathcal{SI}(L) \longrightarrow \mathcal{SI}(L)$.

(3) Set $A = \{x \in L: i \wedge n(x \oplus \tau(x)) \in I_2$, for all $i \in I_1$ and $n \in \mathbb{N}^*\}$. First, let $x \in I_1 \longrightarrow I_2$. Then, $I_1 \cap \langle x \rangle_\tau \subseteq I_2$. For $n \geq 1$, and $i \in I_1$, we have $n(x \oplus \tau(x)) \in \langle x \rangle_\tau$ (since $x, \tau(x) \in \langle x \rangle_\tau$)). It follows that $i \wedge n(x \oplus \tau(x)) \in I_1 \cap \langle x \rangle_\tau$, which implies that $i \wedge n(x \oplus \tau(x)) \in I_2$. Hence, $x \in A$.

Conversely, let $x \in A$ and $t \in I_1 \cap \langle x \rangle_\tau$. Then, there exists $n \in \mathbb{N}^*$ such that $t \leq n(x \oplus \tau(x))$. Hence, $t = t \wedge n(x \oplus \tau(x)) \in I_2$. That is, $I_1 \cap \langle x \rangle_\tau \subseteq I_2$. Hence, $x \in I_1 \longrightarrow I_2$. Therefore, $I_1 \longrightarrow I_2 = \{x \in L: i \wedge n(x \oplus \tau(x)) \in I_2$, for all $i \in I_1$ and $n \in \mathbb{N}^*\}$.

For every $I \in \mathcal{SI}(L)$, we put $I\prime = I \longrightarrow \{0\} = \{x \in L: \langle x \rangle_\tau \cap I = \{0\}\}$. Then, from Proposition 1 (3), we have the following corollary. □

Corollary 1. *For all $I \in \mathcal{SI}(L)$, we have $I\prime = \{x \in L: i \wedge n(x \oplus \tau(x)) = 0$, for all $i \in I$ and $n \in \mathbb{N}^*\}$.*

Theorem 3. *Let $I \in \mathcal{SI}(L)$. Then, $\mathcal{C}(\mathcal{SI}(L)) = \{\langle x \rangle_\tau: x \in L\}$.*

Proof. ⇒). Assume $I \in \mathcal{C}(\mathcal{SI}(L))$. Set $H = \{\langle x \rangle_\tau: x \in L\}$. Since $I = \bigvee_{x \in I} \langle x \rangle_\tau$, there are $\{x_i\}_{1 \leq i \leq n}$ such that $I = \langle x_1 \rangle_\tau \vee \langle x_2 \rangle_\tau \vee \ldots \vee \langle x_n \rangle_\tau$ (Proposition 1). By Lemma 2 (9), we have $I = \langle x_1 \oplus x_2 \oplus \ldots \oplus x_n \rangle_\tau$. Thus, $I \in H$, that is, $\mathcal{C}(\mathcal{SI}(L)) \subseteq H$.

⇐). Let $I \in H$. Then, there exists $x \in L$ such that $I = \langle x \rangle_\tau$. Assume $\{I_k\}_{k \in K} \subseteq \mathcal{SI}(L)$ and $I = \langle x \rangle_\tau \subseteq \bigvee_{k \in K} \{I_k\}$. Then, $x \in \bigvee_{k \in K} \{I_k\} = \langle \bigcup_{k \in K} I_k \rangle_\tau$. It follows that there exist $k_j \in K$, $x_{k_j} \in I_{k_j}$, for all $1 \leq k \leq m$ such that $x \leq x_{k_1} \oplus x_{k_2} \oplus \ldots \oplus x_{k_m}$. That is, $x \in \langle I_{k_1} \cup I_{k_2} \cup \ldots \cup I_{k_m} \rangle_\tau = I_{k_1} \vee I_{k_2} \vee \ldots \vee I_{k_m}$. Hence, $I = \langle x \rangle_\tau \subseteq I_{k_1} \vee I_{k_2} \vee \ldots \vee I_{k_m}$. Hence, $I \in \mathcal{C}(\mathcal{SI}(L))$, that is, $H \subseteq \mathcal{C}(\mathcal{SI}(L))$. Therefore, $\mathcal{C}(\mathcal{SI}(L)) = \{\langle x \rangle_\tau: x \in L\}$. □

Remark 4. Theorem 3 means that a state ideal I is a compact element of the frame $\mathcal{SI}(L)$ if and only if it is principal.

Theorem 4. *The lattice $(\mathcal{SI}(L), \subseteq)$ is a coherent frame.*

Proof.

(i) From Theorem 2, we have that $(\mathcal{SI}(L), \subseteq)$ is a frame.

(ii) From Theorem 3, we obtain that $\mathcal{C}(\mathcal{SI}(L)) = \{\langle x \rangle_\tau: x \in L\}$. By Lemma 3, we have $\langle x \rangle_\tau \wedge \langle y \rangle_\tau = \langle (x \oplus \tau(x)) \wedge (y \oplus \tau(y)) \rangle_\tau$, and by Lemma 2 (9), we have $\langle x \rangle_\tau \vee \langle y \rangle_\tau = \langle x \oplus y \rangle_\tau$ for all $x, y \in L$. Therefore, $(\mathcal{C}(\mathcal{SI}(L)), \subseteq)$ is a sublattice of $(\mathcal{SI}(L), \subseteq)$.

(iii) For any $I \in \mathcal{SI}(L)$, we have $I = \bigvee_{x \in I} \langle x \rangle_\tau$.

(i), (ii), and (iii) combined with Definition 3 imply $(\mathcal{SI}(L), \subseteq)$ is a coherent frame.

We have immediately the following results. □

Corollary 2. *The lattice $(\mathcal{SI}(L), \subseteq)$ is pseudocomplemented. Clearly, for all $I \in \mathcal{SI}(L)$, we have that $I\prime = I \longrightarrow \{0\} = \{x \in L: I \cap \langle x \rangle_\tau = \{0\}\}$ is the pseudocomplement of I.*

According to Definition 3 and Corollary 2, we have the following result in a De Morgan residuated lattice L.

Corollary 3. *(1) $(\mathcal{I}(L), \subseteq)$ is a coherent frame;*

(2) $(\mathcal{I}(L), \subseteq)$ *is a pseudocomplemented lattice.*

We recall that a Heyting algebra ([30]) is a lattice $(L, \vee, \wedge)$ *with 0 such that for every* $x, y \in L$*, there exists an element* $x \longrightarrow y \in L$ *(call the pseudocomplement of* x *with respect to* y*) such that for every* $z \in L, x \wedge z \leq y \Leftrightarrow z \leq x \longrightarrow y$ *(i.e.,* $x \longrightarrow y = sup\{z \in L\colon x \wedge z \leq y\}$*).*

Remark 5. From Proposition 4, $(\mathcal{SI}(L), \vee, \wedge, \prime, \{0\})$ is a Heyting algebra, where for $I \in \mathcal{SI}(L)$, $I\prime = I \longrightarrow \{0\} = \{x \in L\colon \langle x\rangle_\tau \cap I = \{0\}\}$.

Proposition 5. *For any* $a, b \in L$*, we have*

(1) $(\langle a\rangle_\tau)\prime = \{x \in L\colon (a \oplus \tau(a)) \wedge (x \oplus \tau(x)) = 0\}$;

(2) $(\langle a\rangle_\tau)\prime \cap (\langle b\rangle_\tau)\prime = (\langle a \oplus b\rangle_\tau)\prime$.

Proof.

(1) We have $(\langle a\rangle_\tau)\prime = \langle a\rangle_\tau \longrightarrow \{0\} = \{x \in L\colon \langle a\rangle_\tau \cap \langle x\rangle_\tau = \{0\}\} \overset{\text{Lemma1.19}}{=} \{x \in L\colon \langle (a \oplus \tau(a)) \wedge (x \oplus \tau(x))\rangle_\tau = \{0\}\} = \{x \in L\colon (a \oplus \tau(a)) \wedge (x \oplus \tau(x)) = 0\}$.

(2) Let $x \in (\langle a\rangle_\tau)\prime \cap (\langle b\rangle_\tau)\prime$. Then, by (1), we have $(a \oplus \tau(a)) \wedge (x \oplus \tau(x)) = 0$ and $(b \oplus \tau(b)) \wedge (x \oplus \tau(x)) = 0$. In addition, by Lemma 1 (P16), we have $((a \oplus \tau(a)) \oplus (b \oplus \tau(b))) \wedge (x \oplus \tau(x)) \leq ((a \oplus \tau(a)) \wedge (x \oplus \tau(x))) \oplus ((b \oplus \tau(b)) \wedge (x \oplus \tau(x))) = 0$; hence, $((a \oplus \tau(a)) \oplus (b \oplus \tau(b))) \wedge (x \oplus \tau(x)) = 0$. Then, by Lemma 1 (P16), $(2((a \oplus \tau(a)) \oplus (b \oplus \tau(b)))) \wedge (x \oplus \tau(x)) \leq 2((a \oplus \tau(a)) \oplus (b \oplus \tau(b))) \wedge (x \oplus \tau(x))) = 0$. Moreover, by (RL6) and (SO22), $\left(\begin{smallmatrix} 2((a \oplus \tau(a)) \oplus (b \oplus \tau(b))) = ((a \oplus \tau(a)) \oplus (b \oplus \tau(b))) \oplus ((a \oplus \tau(a)) \oplus (b \oplus \tau(b))) \\ \geq a \oplus b \oplus \tau(a) \oplus \tau(b) \geq a \oplus b \oplus \tau(a \oplus b) \end{smallmatrix}\right.$.

Hence, $(a \oplus b \oplus \tau(a \oplus b)) \wedge (x \oplus \tau(x)) = 0$. Then, by (1), we have $x \in (\langle a \oplus b\rangle_\tau)\prime$. Therefore, $(\langle a\rangle_\tau)\prime \cap (\langle b\rangle_\tau)\prime \subseteq (\langle a \oplus b\rangle_\tau)\prime$.

Conversely, let $x \in (\langle a \oplus b\rangle_\tau)\prime$. Then, by (1), $((a \oplus b) \oplus \tau(a \oplus b))) \wedge (x \oplus \tau(x)) = 0$, we have $a \leq a \oplus b$, so $\tau(a) \leq \tau(a \oplus b)$. It follows that $a \oplus \tau(a) \leq (a \oplus b) \oplus \tau(a \oplus b)$. We obtain $(a \oplus \tau(a)) \wedge (x \oplus \tau(x)) \leq ((a \oplus b) \oplus \tau(a \oplus b)) \wedge (x \oplus \tau(x)) = 0$ and thus, $(a \oplus \tau(a)) \wedge (x \oplus \tau(x)) = 0$. Analogously, $(b \oplus \tau(b)) \wedge (x \oplus \tau(x)) = 0$. Therefore, $x \in (\langle a\rangle_\tau)\prime \cap (\langle b\rangle_\tau)\prime$. □

Lemma 4. *Let* (L, τ) *be a state residuated lattice,* $x \in L$ *and* $n \in \mathbb{N}, n \geq 2$*. Then, the following hold:*

(P17) $(nx)\prime\prime = nx$;

(P18) $x\prime\prime \oplus \tau(x\prime\prime) = x \oplus \tau(x)$.

Proof. (P17): By induction, we have $2x = x \oplus x = (x\prime \odot x\prime)\prime$. It follows that $(2x)\prime\prime = (x\prime \odot x\prime)\prime\prime\prime = (x\prime \odot x\prime)\prime = 2x$. Let $k \in \mathbb{N}$ such that $k \geq 2$. Assume that $(kx)\prime\prime = kx$. Then, we have $(k+1)x = kx \oplus x \overset{\text{hypotesis}}{=} (kx)\prime\prime \oplus x = ((kx)\prime\prime\prime \odot x\prime)\prime = ((kx)\prime \odot x\prime)\prime$. It follows that $((k+1)x)\prime\prime = ((kx)\prime \odot x\prime)\prime\prime\prime = ((kx)\prime \odot x\prime)\prime = (kx) \oplus x = (k+1)x$. Therefore, $(nx)\prime\prime = nx$ for all $n \in \mathbb{N}, n \geq 2$.

(P18): We have $x\prime\prime \oplus \tau(x\prime\prime) \overset{(SO21)}{=} x\prime\prime \oplus (\tau(x))\prime\prime = (x\prime\prime\prime \odot (\tau(x))\prime\prime\prime)\prime \overset{(RL11)}{=} (x\prime \odot (\tau(x))\prime)\prime = x \oplus \tau(x)$. □

Theorem 5. *Let* (L, τ) *be a De Morgan state residuated lattice. The following are equivalent:*

(i) $(\mathcal{SI}(L), \vee, \wedge, \prime, \{0\}, L)$ *is a Boolean algebra;*

(ii) Every state ideal of (L, τ) *is principal, and for every* $x \in L$*, there is* $n \in \mathbb{N}^*$*, such that* $(x \oplus \tau(x)) \wedge ((n(x \oplus \tau(x)))\prime \oplus \tau((n(x \oplus \tau(x)))\prime)) = 0$.

Proof. $(i) \Rightarrow (ii)$. Assume that $(\mathcal{SI}(L), \vee, \wedge, \prime, \{0\}, L)$ is a Boolean algebra. Then, for every $I \in \mathcal{SI}(L)$, $I \vee I\prime = L$; thus, $1 \in I \vee I\prime$. But according to Theorem 1 (4), $I \vee I\prime\colon = \langle I \cup I\prime\rangle_\tau = \{x \in L\colon x \leq y \oplus z, \text{ with } y \in I \text{ and } z \in I\prime\}$. Hence, there are $y \in I$ and $z \in I\prime$ such that $y \oplus z = 1$. We will prove that $I = \langle y\rangle_\tau$. Since $y \in I$, it follows that $\langle y\rangle_\tau \subseteq I$. According to Corollary 1, $I\prime = \{a \in L\colon x \wedge n(x \oplus \tau(x)) = 0, \text{for all } x \in I \text{ and } n \in \mathbb{N}^*\}$. Thus, $x \wedge n(z \oplus \tau(z)) = 0$, for all $x \in I$ and $n \in \mathbb{N}^*$. Then $x \odot z \leq x \wedge z = 0$ for every $x \in I$, that is, $x \odot z = 0$ for every $x \in I$. Hence, $x\prime\prime \longrightarrow z\prime \overset{(RL11)}{=} (x \odot z)\prime = 1$, that is, $x\prime\prime \leq z\prime$. Since $y \oplus z = 1$, we obtain that $n(y \oplus \tau(y)) \oplus z = 1$ for every $n \in \mathbb{N}^*$ (because $y \oplus z \leq n(y \oplus \tau(y)) \oplus z$). Hence, by (P1), $(n(y \oplus \tau(y)))\prime \longrightarrow z\prime\prime = 1$, that is, $(n(y \oplus \tau(y)))\prime \leq z\prime\prime$. It follows that $z\prime \overset{(RL11)}{=} z\prime\prime\prime \overset{(RL4)}{\leq} (n(y \oplus \tau(y)))\prime\prime \overset{(P17)}{=} n(y \oplus \tau(y))$. Hence, $z\prime \leq n(y \oplus \tau(y))$. Thus, we obtain that $x \overset{(RL9)}{\leq} x\prime\prime \leq z\prime \leq n(y \oplus \tau(y))$, that is, $x \leq n(y \oplus \tau(y))$, for every $x \in I$, that is, $I \subseteq \langle y\rangle_\tau$. Therefore, $I = \langle y\rangle_\tau$.

Now, let $x \in L$, since $(\mathcal{SI}(L), \vee, \wedge, \prime, \{0\}, L)$ is a Boolean algebra, we have $L = \langle x\rangle_\tau \vee (\langle x\rangle_\tau)\prime = \{t \in L\colon t \leq y \oplus n(x \oplus \tau(x)), \text{ for some } n \in \mathbb{N}^*, y \in (\langle x\rangle_\tau)\prime\}$. Hence, there exists $y \in (\langle x\rangle_\tau)\prime$ and $n \in \mathbb{N}^*$, such that $y \oplus n(x \oplus \tau(x)) = 1$. Since $y \in (\langle x\rangle_\tau)\prime$, then by Proposition 5 (1), $(x \oplus \tau(x)) \wedge (y \oplus \tau(y)) = 0$. From $y \oplus n(x \oplus \tau(x)) = 1$, we deduce that $y\prime \longrightarrow (n(x \oplus \tau(x)))\prime\prime = 1$, which implies $y\prime \leq (n(x \oplus \tau(x)))\prime\prime \overset{(P17)}{=} n(x \oplus \tau(x))$. Thus, by (RL4), $(n(x \oplus \tau(x)))\prime \leq y\prime\prime$, which implies $\tau((n(x \oplus \tau(x)))\prime) \overset{(SO10)}{\leq} \tau(y\prime\prime)$. Hence, $(n(x \oplus \tau(x)))\prime \oplus \tau((n(x \oplus \tau(x)))\prime) \overset{(P6)}{\leq} y\prime\prime \oplus \tau(y\prime\prime) \overset{(P18)}{=} y \oplus \tau(y)$. It follows that $(x \oplus \tau(x)) \wedge ((n(x \oplus \tau(x)))\prime \oplus \tau((n(x \oplus \tau(x)))\prime)) \leq (x \oplus \tau(x)) \wedge (y \oplus \tau(y)) = 0$. Therefore, $(x \oplus \tau(x)) \wedge ((n(x \oplus \tau(x)))\prime \oplus \tau((n(x \oplus \tau(x)))\prime)) = 0$.

$(ii) \Rightarrow (i)$. By Remark 5, $(\mathcal{SI}(L), \vee, \wedge, \prime, \{0\})$ is a Heyting algebra. In order to prove that $(\mathcal{SI}(L), \vee, \wedge, \prime, \{0\}, L)$ is a Boolean algebra, it is enough to prove that for every $I \in \mathcal{SI}(L)$, we have $I\prime = \{0\} \Leftrightarrow I = L$ (according to ([30])). Let $I \in \mathcal{SI}(L)$ with $I\prime = \{0\}$. By the hypothesis, every state ideal is principal. Hence, there is $x \in L$ such that $I = \langle x\rangle_\tau$. Thus, $(\langle x\rangle_\tau)\prime = \{0\}$. Moreover, there is $n \in \mathbb{N}^*$ such that $(x \oplus \tau(x)) \wedge ((n(x \oplus \tau(x)))\prime \oplus \tau((n(x \oplus \tau(x)))\prime)) = 0$. By Proposition 5 (1), it follows that $(n(x \oplus \tau(x)))\prime \in (\langle x\rangle_\tau)\prime = \{0\}$. Thus, $(n(x \oplus \tau(x)))\prime = 0$, that is, $n(x \oplus \tau(x)) \overset{(P17)}{=} (n(x \oplus \tau(x)))\prime\prime = 1$. Since $n(x \oplus \tau(x)) \in \langle x\rangle_\tau$, we deduce that $1 \in \langle x\rangle_\tau = I$. Therefore, $I = \langle x\rangle_\tau = L$. □

4. State Relative Annihilators in DMSRL

Consider a state filter F of a state residuated lattice (L,τ), Pengfei et al. defined the notion of co-annihilator of a nonempty subset X of L with respect to F by $X_I^{\perp_\tau} = \{a \in L\colon \tau(a)\vee x \in F, \text{for all } x \in X\}$, which is a state filter. Inspired by this work, in this section, we introduce the notion of state relative annihilator of a nonempty set X with respect to a state ideal I in a De Morgan state residuated lattice (L,τ) and study some properties.

Definition 7. Let (L,τ) be a De Morgan state residuated lattice, I a state ideal of (L,τ), and X a nonempty subset of L.

The τ-relative annihilator of X with respect to I is defined by $X_I^{\perp_\tau} = \{a \in L\colon \tau(a)\wedge x \in I, \text{for all } x \in X\}$.

We have $0 \in X_I^{\perp_\tau}$. Hence, $X_I^{\perp_\tau} \neq \varnothing$. Particularly, the following facts hold:

(1) If $\tau = id_L$, $I = \{0\}$, then $X^{\perp} := X_{\{0\}}^{\perp_{id_L}} = \{a \in L\colon a\wedge x = 0, \text{for all } x \in X\}$, and it is called the annihilator of X in L (see [2]);

(2) If $\tau = id_L$, then $X_I^{\perp} := X_I^{\perp_{id_L}} = \{a \in L\colon a\wedge x \in I, \text{for all } x \in X\}$, and it is called the relative annihilator of X with respect to I (see [2]);

(3) If $X = \{x\}$, then $x_I^{\perp_\tau} := \{x\}_I^{\perp_\tau} = \{a \in L\colon \tau(a)\wedge x \in I\}$;

(4) If $I = \{0\}$, then $X^{\perp_\tau} := X_{\{0\}}^{\perp_\tau} = \{a \in L\colon \tau(a)\wedge x = 0, \text{for all } x \in X\}$.

Example 5. Consider the De Morgan residuated lattice of Example 2. Assume that τ is a state operator on L. Then, $\tau(0) = 0$. Also, we have $b\odot b = 0$, which implies by (SO12) that $0 = \tau(0) = \tau(b\odot b) = \tau(b)\odot\tau(b)$. Hence by the table of $\odot$, $\tau(b) \in \{0,p,b\}$. Suppose that $\tau(b) = 0$. Since $p \le b$, by (SO10) $\tau(p) \le \tau(b) = 0$, that is, $\tau(p) = 0$. Moreover, $\tau(f) = \tau(c \longrightarrow d) \overset{(SO3)}{=} \tau(c) \longrightarrow \tau(c\wedge d) = \tau(c) \longrightarrow \tau(b) = \tau(c) \longrightarrow 0 = (\tau(c))' \overset{(SO11)}{=} \tau(c') = \tau(d)$. Thus, $\tau(f) = \tau(d)$. In addition, $\tau(e) = \tau(f \longrightarrow e) = \tau(f) \longrightarrow \tau(f\wedge e) = \tau(f) \longrightarrow \tau(d) = \tau(f) \longrightarrow \tau(f) \overset{(RL1)}{=} 1$, that is, $\tau(e) = 1$. Since $e \le q \le 1$, by (SO10) $1 = \tau(e) \le \tau(q) \le \tau(1)$, that is, $\tau(q) = \tau(1) = 1$. Also on the one hand, we have $1 = \tau(e) = \tau(e \longrightarrow c) \overset{(SO14)}{=} \tau(e) \longrightarrow \tau(c) = 1 \longrightarrow \tau(c) \overset{(RL1)}{=} \tau(c)$. Hence, $\tau(c) = 1$. On the other hand, we have $1 = \tau(e) = \tau(c \longrightarrow a) \overset{(SO14)}{=} \tau(c) \longrightarrow \tau(a) = 1 \longrightarrow \tau(a) \overset{(RL1)}{=} \tau(a)$. Thus, $\tau(a) = 1$. We have $d\odot a = 0$, which implies that $0 = \tau(0) = \tau(d\odot a) \overset{(SO12)}{=} \tau(d)\odot\tau(a) = \tau(d)\odot 1 = \tau(d)$. It follows that $\tau(f) = \tau(d) = 0$. Therefore, we obtain that the map τ defined on L by

$$\tau(x) = \begin{cases} 0, & \text{if } x \in \{0,p,b,d,f\}; \\ 1, & \text{if } x \in \{a,c,e,q,1\}. \end{cases} \tag{4}$$

Is a state operator on L. Thus, (L,τ) is a De Morgan state residuated lattice. One can easily check that $\{0\}$, $\{0,p\}$, $\{0,p,a\}$, $\{0,p,c\}$, $\{0,p,b,d,f\}$, L are ideals of L and $\{0\}$, $I = \{0,p\}$, $J = \{0,p,b,d,f\}$ and L are state ideals of (L,τ). In addition, $\text{coker}(\tau) = \{0,p,b,d,f\} \neq \{0\}$. Thus, τ is a uncofaithful state operator.

We have $0_I^{\perp_\tau} = p_I^{\perp_\tau} = L$ and $a_I^{\perp_\tau} = c_I^{\perp_\tau} = d_I^{\perp_\tau} = e_I^{\perp_\tau} = f_I^{\perp_\tau} = q_I^{\perp_\tau} = 1_I^{\perp_\tau} = \{0,p,b,d,f\} = J$.

Let $X = \{0,P,b\}$, we have $X_I^{\perp_\tau} = \{0,p,b,d,f\} = J$.

Example 6. Let $L = \{0,n,a,b,c,d,1\}$ with his Hasse diagram and Cayley tables of $\odot$ and $\longrightarrow$ be the following (Figure 4):

Then , $L = (L,\wedge,\vee,\odot,\longrightarrow,0,1)$ is a nonregular De Morgan residuated lattice, which is Stonean. Now suppose that τ is a state operator on $L[2]$. Then, $\tau(0) = 0$ and by (SO3), $\tau(b) = \tau(c \longrightarrow b) = \tau(c) \longrightarrow (c\wedge b) = \tau(c) \longrightarrow \tau(a)$. If $\tau(c) = 1$, we have $\tau(b) = 1 \longrightarrow \tau(a) = \tau(a)$, that is, $\tau(b) = \tau(a)$. Since $c \le d \le 1$, by (SO10), we have $1 = \tau(c) \le \tau(d) \le \tau(1)$, that is, $\tau(d) = \tau(1) = 1$. Also, $1 = \tau(d) = \tau(a \longrightarrow n) \overset{(SO14)}{=} \tau(a) \longrightarrow \tau(n)$, that is, $\tau(a) \longrightarrow \tau(n) = 1$. Hence, $\tau(a) \overset{(RL2)}{\le} \tau(n)$. Since, $n \le a$, we have $\tau(n) \le \tau(a)$. Thus, $\tau(n) = \tau(a) = \tau(b) = \lambda$, for some $\lambda \in L$. By (SO7) and (SO8), we have, for $* \in \{\vee,\wedge\}$, $\tau(\tau(a) * \tau(b)) = \tau(a) * \tau(b) \Leftrightarrow \tau(\lambda) = \tau(\lambda * \lambda) = \lambda * \lambda = \lambda$. That is, $\tau(\lambda) = \lambda$. Thus, $\lambda \in \{0,n,a,b,1\}$. In addition, if $\lambda = a$, we have $\tau(\tau(a)\odot\tau(b)) = \tau(a\odot a) = \tau(n) = a \neq n = a\odot a = \tau(a)\odot\tau(b)$. Hence, by (SO5), $\lambda \neq a$. By some calculations, we obtain that for $\lambda = 1$, the map τ defined on L by

$$\tau(x) = \begin{cases} 0, & \text{if } x \in \{0\}; \\ 1, & \text{if } x \in \{n,a,b,c,d,1\}, \end{cases} \tag{5}$$

is a state operator on L. Thus, (L,τ) is a De Morgan state residuated lattice. In addition, $\text{coker}(\tau) = \{0\}$. Hence, τ is cofaithful.

We have $0^{\perp_\tau} = L$ and $n^{\perp_\tau} = a^{\perp_\tau} = b^{\perp_\tau} = c^{\perp_\tau} = d^{\perp_\tau} = 1^{\perp_\tau} = \{0\}$.

For any $X \subseteq L$, $X \neq \{0\}$, we have $X^{\perp_\tau} = \{0\}$.

Example 7. Consider the De Morgan residuated lattice of Example 1. Then, from Remark 3, (L,id_L) is a De Morgan state residuated lattice. If $I = \{0\}$ and $X = \langle f\rangle_{id_L} = \{0,c,f\}$, it is easy to check that $X^{\perp} := X_{\{0\}}^{\perp_{id_L}} = \{0,a,b\} = \langle b\rangle_{id_L} = \langle b\rangle$.

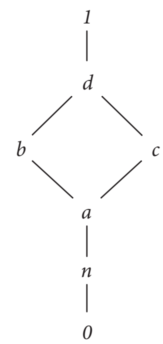
Theorem 5. *Let I be a state ideal of (L,τ). Given a nonempty subset X of L, $X_I^{\perp_\tau}$ is a state ideal of (L,τ).*

Proof. Let $a,b \in X_I^{\perp_\tau}$. Then, $\tau(a)\wedge x \in I$ and $\tau(b)\wedge x \in I$, for all $x \in X$. Hence, $(\tau(a)\wedge x)\oplus(\tau(b)\wedge x) \in I$ because I is an ideal. Since $\tau(a\oplus b)\wedge x \overset{(SO22)}{\le} (\tau(a)\oplus\tau(b))\wedge x \overset{(P16)}{\le} (\tau(a)\wedge x)\oplus(\tau(b)\wedge x) \in I$, we have $\tau(a\oplus b)\wedge x \in I$. That is, $a\oplus b \in X_I^{\perp_\tau}$.

Let $a \in X_I^{\perp_\tau}$ and $b \in L$ such that $b \le a$. Then, $\tau(b) \le \tau(a)$, from (SO10). It follows that $\tau(b)\wedge x \le \tau(a)\wedge x$, for all $x \in X$. Since $\tau(a)\wedge x \in I$ and I is and ideal, $\tau(b)\wedge x \in I$, for all $x \in X$. Therefore, $b \in X_I^{\perp_\tau}$.

Let $a \in X_I^{\perp_\tau}$. Since $\tau(\tau(a)) = \tau^2(a) \overset{(SO16)}{=} \tau(a)$, we have $\tau(\tau(a))\wedge x = \tau(a)\wedge x \in I$, for all $x \in X$. Hence, $\tau(a) \in X_I^{\perp_\tau}$. Therefore, $X_I^{\perp_\tau}$ is a state ideal of (L,τ). □

Corollary 4. *Let I be a state ideal of (L,τ). Given a nonempty subset X of L, we have*



$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	n	n	n	n	n	n
a	0	n	n	n	n	n	a
b	0	n	n	b	n	b	b
c	0	n	n	n	c	c	c
d	0	n	n	b	c	d	d
1	0	n	a	b	c	d	1

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	0	1	1	1	1	1	1
a	0	d	1	1	1	1	1
b	0	c	c	1	c	1	1
c	0	b	b	b	1	1	1
d	0	a	a	b	c	1	1
1	0	n	a	b	c	d	1

FIGURE 4: A non regular De Morgan residuated lattice.

(1) $X^{\perp_\tau}$ is a state ideal of (L,τ);

(2) $X_I^{\perp}$ and $X^{\perp}$ are state ideals of L;

(3) For all $x \in L$, $x_I^{\perp_\tau}$ is a state ideal of (L,τ) and $I \subseteq x_I^{\perp_\tau}$.

Proposition 6. *Let I, J be state ideals of (L,τ). Given nonempty subsets X, Y of L, we have*

(1) $I \subseteq J \Rightarrow X_I^{\perp_\tau} \subseteq X_J^{\perp_\tau}$;

(2) $X \subseteq Y \Rightarrow Y_I^{\perp_\tau} \subseteq X_I^{\perp_\tau}$;

(3) $(\bigcup_{k \in K} X_k)_I^{\perp_\tau} = \bigcap_{k \in K} (X_k)_I^{\perp_\tau}$;

(4) $X^{\perp_\tau}_{(\bigcap_{k \in K} I_k)} = \bigcap_{k \in K} X_{I_k}^{\perp_\tau}$;

(5) $\langle X \rangle_I^{\perp_\tau} = X_I^{\perp_\tau}$. Particularly, $\varnothing_I^{\perp_\tau} = 0_I^{\perp_\tau} = L$;

(6) $coker(\tau) \subseteq X^{\perp_\tau}$, $L^{\perp_\tau} = coker(\tau)$;

(7) $X_I^{\perp_\tau} = \bigcap_{x \in X} x_I^{\perp_\tau}$;

(8) $X^{\perp_\tau} = \bigcap_{x \in X} x^{\perp_\tau}$;

(9) $X_I^{\perp_\tau} = L$ if $X \subseteq I$. Specifically, $0_I^{\perp_\tau} = I_I^{\perp_\tau} = L$;

(10) $\bigcap_{k \in K} (X_k)_I^{\perp_\tau} \subseteq (\bigcap_{k \in K} X_k)_I^{\perp_\tau}$;

(11) $I \subseteq X_I^{\perp_\tau}$;

(12) $X^{\perp_\tau} \subseteq X_I^{\perp_\tau}$.

Proof

(1) Let $I \subseteq J$ and $a \in X_I^{\perp_\tau}$. Then, $\tau(a) \wedge x \in I$, for all $x \in X$. Thus, $\tau(a) \wedge x \in J$, for all $x \in X$, that is, $a \in X_J^{\perp_\tau}$. Therefore, $X_I^{\perp_\tau} \subseteq X_J^{\perp_\tau}$.

(2) Let $X \subseteq Y$ and $a \in Y_I^{\perp_\tau}$. Then, $\tau(a) \wedge x \in I$, for all $x \in Y$. Hence, $\tau(a) \wedge x \in I$ for all $x \in X$, which implies $a \in X_I^{\perp_\tau}$. Therefore, $Y_I^{\perp_\tau} \subseteq X_I^{\perp_\tau}$.

(3) Since $X_k \subseteq \bigcup_{k \in K} X_k$, it follows from (2) that $(\bigcup_{k \in K} X_k)_I^{\perp_\tau} \subseteq (X_k)_I^{\perp_\tau}$ for all $k \in K$. We deduce that $(\bigcup_{k \in K} X_k)_I^{\perp_\tau} \subseteq \bigcap_{k \in K} (X_k)_I^{\perp_\tau}$.
Conversely, let $a \in \bigcap_{k \in K} (X_k)_I^{\perp_\tau}$; we have $a \in (X_k)_I^{\perp_\tau}$ for all $k \in K$. Hence, $\tau(a) \wedge x_k \in I$ for all $x_k \in X_k$ and $k \in K$, which implies $a \in (\bigcup_{k \in K} X_k)_I^{\perp_\tau}$. That is, $\bigcap_{k \in K} (X_k)_I^{\perp_\tau} \subseteq (\bigcup_{k \in K} X_k)_I^{\perp_\tau}$. Therefore, $(\bigcup_{k \in K} X_k)_I^{\perp_\tau} = \bigcap_{k \in K} (X_k)_I^{\perp_\tau}$.

(4) We have $a \in X^{\perp_\tau}_{(\bigcap_{k \in K} I_k)} \Leftrightarrow (\tau(a) \wedge x \in \bigcap_{k \in K} I_k, \text{for all } x \in X) \Leftrightarrow (\tau(a) \wedge x \in I_k, \text{for all } x \in X \text{ and } k \in K) \Leftrightarrow (a \in X_{I_k}^{\perp_\tau}, \text{for all } k \in K) \Leftrightarrow a \in \bigcap_{k \in K} X_{I_k}^{\perp_\tau}$.
Thus, $X^{\perp_\tau}_{(\bigcap_{k \in K} I_k)} = \bigcap_{k \in K} X_{I_k}^{\perp_\tau}$.

(5) Since $X \subseteq \langle X \rangle$, by (2), we obtain $\langle X \rangle_I^{\perp_\tau} \subseteq X_I^{\perp_\tau}$. Conversely, let $a \in X_I^{\perp_\tau}$ and $z \in \langle X \rangle$. Then, $\tau(a) \wedge x \in I$, for all $x \in X$. Since $z \in \langle X \rangle$, there are $n \in \mathbb{N}^*$ and $x_1, x_2, \ldots, x_n \in X$ such that $z \leq x_1 \oplus x_2 \oplus, \ldots, \oplus x_n$. It follows that $\tau(a) \wedge z \leq \tau(a) \wedge (x_1 \oplus x_2 \oplus, \ldots, \oplus x_n) \leq (\tau(a) \wedge x_1) \oplus (\tau(a) \wedge x_2) \oplus, \ldots, \oplus (\tau(a) \wedge x_n)$. But $\tau(a) \wedge x_i \in I$, for all $1 \leq i \leq n$. Thus, $\tau(a) \wedge z \in I$, which implies that $a \in \langle X \rangle_I^{\perp_\tau}$. Therefore, $\langle X \rangle_I^{\perp_\tau} = X_I^{\perp_\tau}$.

(6) Let $a \in \mathrm{coker}(\tau)$. Then, $\tau(a) = 0$. Since $\tau(a) \wedge x \leq \tau(a) = 0$, for all $x \in X$, we have $\tau(a) \wedge x = 0$, for all $x \in X$. That is, $a \in X^{\perp_\tau}$. Thus, $\mathrm{coker}(\tau) \subseteq X^{\perp_\tau}$. Taking $X = L$, we have $\mathrm{coker}(\tau) \subseteq L^{\perp_\tau}$. Now, let $a \in L^{\perp_\tau}$, and then, $\tau(a) \wedge x = 0$, for all $x \in L$. In particular, taking $x = \tau(a)$, we have $\tau(a) = 0$, which implies that $a \in \mathrm{coker}(\tau)$, that is, $L^{\perp_\tau} \subseteq \mathrm{coker}(\tau)$. Therefore, $\mathrm{coker}(\tau) \subseteq X^{\perp_\tau}$ and $L^{\perp_\tau} = \mathrm{coker}(\tau)$.

(7) According to (3), we have $X_I^{\perp_\tau} = (\bigcup_{x\in X}\{x\})_I^{\perp_\tau} = \bigcap_{x\in X} x_I^{\perp_\tau}$.

(8) Taking $I=\{0\}$ in (1), we obtain $X^{\perp_\tau} = \bigcap_{x\in X} x^{\perp_\tau}$.

(9) Assume $X_I^{\perp_\tau} = L$, and let $x\in X$. Then, $x=\tau(1)\wedge x\in I$, that is, $x\in I$.

Conversely, suppose $X\subseteq I$, and $a\in L$. Then, for any $x\in X$, we have $x\in I$. Since $\tau(a)\wedge x\le x\in I$, we obtain $\tau(a)\wedge x\in I$. Hence, $a\in X_I^{\perp_\tau}$, and therefore, $X_I^{\perp_\tau}=L$.

(10) Since $\bigcap_{k\in K} X_k\subseteq X_k$ for each $k\in K$, it follows from (2) that $(X_k)_I^{\perp_\tau}\subseteq(\bigcap_{k\in K} X_k)_I^{\perp_\tau}$ for all $k\in K$. Thus, $\bigcap_{k\in K}(X_k)_I^{\perp_\tau}\subseteq(\bigcap_{k\in K}X_k)_I^{\perp_\tau}$.

(11) Let $a\in I$. Since I is a state ideal, we have $\tau(a)\in I$. In addition, $\tau(a)\wedge x\le\tau(a)\in I$, for all $x\in X$. Then, $\tau(a)\wedge x\in I$, for all $x\in X$. That is $a\in X_I^{\perp_\tau}$. Therefore, $I\subseteq X_I^{\perp_\tau}$.

(12) Let I be an ideal of L. We have $a\in X^{\perp_\tau}$ implies $\tau(a)\wedge x=0\in I$, for any $x\in X$. Hence, $\tau(a)\wedge x\in I$, for any $x\in X$. Thus, $a\in X_I^{\perp_\tau}$. Therefore, $X^{\perp_\tau}\subseteq X_I^{\perp_\tau}$. □

Example 8. Consider the De Morgan state residuated lattice (L,τ) of Example 6. By taking $X=\{c\}$ and $Y=\{d\}$, for $I=\{0\}$, we have $X_I^{\perp_\tau}=Y_I^{\perp_\tau}=\{0\}$. It follows that $X_I^{\perp_\tau}\cap Y_I^{\perp_\tau}=\{0\}$. But $(X\cap Y)_I^{\perp_\tau}=\varnothing_I^{\perp_\tau}=L\nsubseteq\{0\}=X_I^{\perp_\tau}\cap Y_I^{\perp_\tau}$. So, the inclusion may be strict for (10).

Set $Z=\{0,n,a,b\}$, then $Z=\{0\}\cup\{n\}\cup\{a\}\cup\{b\}$. It follows by (3) that $Z_I^{\perp_\tau}=0_I^{\perp_\tau}\cap n_I^{\perp_\tau}\cap a_I^{\perp_\tau}\cap b_I^{\perp_\tau}=L\cap\{0\}\cap\{0\}\cap\{0\}=\{0\}$.

Remark 6. For the reader's convenience, we shall notice that for $\tau=id_L$ in the above Proposition 6, we get all the properties of relative annihilators established in De Morgan residuated lattices in ([2], Proposition 4.51). Also, for $\tau=id_L$ and $I=\{0\}$, we get all the important results about annihilators in De Morgan residuated lattices investigated in Ref. [2] (pages 462–472).

The following properties of relative annihilators always hold for $\tau=id_L$ but may not in general be the case for some state operators of De Morgan residuated lattices.

Proposition 7. *Let L be a De Morgan residuated lattice and I, J, K be ideals of L and X, Y two nonempty subsets of L. Then,*

(1) $J_I^\perp\cap J\subseteq I$;

(2) $J\cap K\subseteq I\Leftrightarrow K\subseteq J_I^\perp$;

(3) $X\cap X_I^\perp\subseteq I$;

(4) $I\subseteq X$ *if* $X\cap X_I^\perp=I$. *Particularly,* $X_I^\perp\cap(X_I^\perp)_I^\perp=I$;

(5) $Y_I^\perp\cup X_I^\perp\subseteq Y^\perp_{(X_I^\perp)}, X^\perp_{(Y_I^\perp)}$ *and* $Y^\perp_{(X_I^\perp)}\cup X^\perp_{(Y_I^\perp)}\subseteq(X\wedge Y)_I^\perp$, *where* $X\wedge Y=\{x\wedge y\colon x\in X, y\in Y\}$;

(6) $L_I^\perp=I$ *and* $1_I^\perp=I$;

(7) $X\subseteq(X_I^\perp)_I^\perp$. *Particularly,* $(I_I^\perp)_I^\perp=I$ *and* $(L_I^\perp)_I^\perp=L$;

(8) $X_I^\perp=((X_I^\perp)_I^\perp)_I^\perp$;

(9) $X^\perp_{(X_I^\perp)}\subseteq X_{(X_{(Y_I^\perp)})}$;

(10) If $X\subseteq I$, *then* $X^\perp_{(X_I^\perp)}=X^\perp_{(X^\perp_{(X_I^\perp)})}=X_{(X_{(Y_I^\perp)})}=L$, *whenever* $X\subseteq Y$.

Proof. For the proof of (1) and (2), see [2] (Proposition 4.51 (9) and (10)).

(3) Let $a\in X\cap X_I^\perp$. Then, for any $x\in X$, we have $a\wedge x\in I$. Particularly, for $x=a$, we obtain $a=a\wedge a\in I$, implying $a\in I$. Thus, $X\cap X_I^\perp\subseteq I$.

(4) Firstly, $X\cap X_I^\perp\subseteq I$, from (3). Conversely, assume $I\subseteq X$. We know that $I\subseteq X_I^\perp$, from Proposition 6 (11). Therefore, $I\subseteq X\cap X_I^\perp$. Hence, $X\cap X_I^\perp=I$.

In another hand, assume $X\cap X_I^\perp=I$. Then, $I=X\cap X_I^\perp\subseteq X$

(5) We have $I\subseteq Y_I^\perp$. Applying Proposition 6 (1), we obtain $X_I^\perp\subseteq X^\perp_{(Y_I^\perp)}$. Since from (1), $Y_I^\perp\subseteq X^\perp_{(Y_I^\perp)}$, it follows that $Y_I^\perp\cup X_I^\perp\subseteq X^\perp_{(Y_I^\perp)}$. Similarly, we obtain $Y_I^\perp\cup X_I^\perp\subseteq Y^\perp_{(X_I^\perp)}$.

In addition, let $a\in Y^\perp_{(X_I^\perp)}$, and $z\in X\wedge Y$. Then, there are $x\in X$, $y\in Y$ such that $z=x\wedge y$. Since, $a\wedge y\in X_I^\perp$, we have $a\wedge z=a\wedge(x\wedge y)=(a\wedge x)\wedge y=(a\wedge y)\wedge x\in I$, which means that $a\wedge z\in I$. Hence, $a\in(X\wedge Y)_I^\perp$, that is, $Y^\perp_{(X_I^\perp)}\subseteq(X\wedge Y)_I^\perp$. Analogously, we show that $X^\perp_{(Y_I^\perp)}\subseteq(X\wedge Y)_I^\perp$. Therefore, $Y^\perp_{(X_I^\perp)}\cup X^\perp_{(Y_I^\perp)}\subseteq(X\wedge Y)_I^\perp$.

(6) From Proposition 6 (11), we have $I\subseteq L_I^\perp$. Conversely, for any $x\in L_I^\perp$, we have $x=1\wedge x\in I$. Hence, $x\in I$. Similarly, we obtain $1_I^\perp=I$.

(7) Let $x\in X$. For any $a\in X_I^\perp$, we have $a\wedge x\in I$, which implies $x\in(X_I^\perp)_I^\perp$.

(8) By (7), we have $X_I^\perp\subseteq((X_I^\perp)_I^\perp)_I^\perp$. On the other hand, applying Proposition 6 (2) to the above (7), we obtain $((X_I^\perp)_I^\perp)_I^\perp\subseteq X_I^\perp$.

(9) We always have $I\subseteq Y_I^\perp$, from Proposition 6 (11). Applying Proposition 6 (1) twice, we obtain $X^\perp_{(X_I^\perp)}\subseteq X_{(X_{(Y_I^\perp)})}$.

(10) Now assume $X\subseteq I$. First of all, we show that $X^\perp_{(X_I^\perp)}=L$.

We know that $X\subseteq I\subseteq X_I^\perp$, that is, $X\subseteq X_I^\perp$, which implies $X^\perp_{(X_I^\perp)}=L$, from Proposition 6 (9).

In addition, since $I\subseteq X_I^\perp$, applying Proposition 6 (1) twice, we have $X^\perp_{(X_I^\perp)}\subseteq X^\perp_{(X^\perp_{(X_I^\perp)})}$. Therefore, $L=X^\perp_{(X_I^\perp)}\subseteq X^\perp_{(X^\perp_{(X_I^\perp)})}$, which implies $X^\perp_{(X^\perp_{(X_I^\perp)})}=L$. Thus, $X^\perp_{(X_I^\perp)}\subseteq X^\perp_{(X^\perp_{(X_I^\perp)})}=L$.

Moreover, suppose $X\subseteq Y$. Then, from Proposition 6 (2), $Y_I^\perp\subseteq X_I^\perp$. This implies $X_{(X_{(Y_I^\perp)})}\overset{(Prop.3.71)}{\subseteq}X^\perp_{(X^\perp_{(X_I^\perp)})}=X^\perp_{(X_I^\perp)}$.

Since $X^\perp_{(X_I^\perp)}\overset{(9)}{\subseteq}X_{(X_{(Y_I^\perp)})}$, it follows that $X_{(X_{(Y_I^\perp)})}=X^\perp_{(X_I^\perp)}$. □

Example 8. Consider the De Morgan state residuated lattice (L,τ) of Example 5. $I=\{0,p\}$ and $J=\{0,p,b,d,f\}$ are two state ideals of (L,τ). We will show that item (3) and (4) do not hold with τ.

(3) Set $X=\{b\}$, and then, $X_I^{\perp_\tau}=J$. But $X\cap X_I^{\perp_\tau}=\{b\}\nsubseteq I$.

(4) Let $X=\{0,P,b\}$, we obtain $X_I^{\perp_\tau}=J$. $I\subseteq X$, but $X\cap X_I^{\perp_\tau}=X\cap J=X\neq I$.

Theorem 6. *Let I be an ideal of L and X a nonempty set of L. Then, we have* $\langle X\rangle\longrightarrow I=\{x\in L\colon \langle X\rangle\cap\langle x\rangle\subseteq I\}=X_I^{\perp}$ *in the frame* $(I(L),\subseteq)$.

Proof. Let $x\in\langle X\rangle\longrightarrow I$. Then, $\langle X\rangle\cap\langle x\rangle\subseteq I$. It follows from Proposition 6 (5) and Proposition 7 (1) that $x\in\langle X\rangle\subseteq\langle X\rangle_I^{\perp_\tau}=X_I^{\perp}$. Therefore, we obtain $\langle X\rangle\longrightarrow I\subseteq X_I^{\perp}$.

Conversely, let $x\in X_I^{\perp}$. Since $\langle X\rangle_I^{\perp}=X_I^{\perp}$, we have $x\in\langle X\rangle_I^{\perp}$, which implies that $\langle x\rangle\subseteq\langle X\rangle_I^{\perp}$. By Proposition 7 (2), we have $\langle X\rangle\cap\langle x\rangle\subseteq I$. It follows that $x\in\langle X\rangle\longrightarrow I$. Hence, $X_I^{\perp}\subseteq\langle X\rangle\longrightarrow I$. Therefore, we obtain $\langle X\rangle\longrightarrow I=X_I^{\perp}$ in the frame $(I(L),\subseteq)$.

In Ref. [31], the authors investigated the concept of f-relative annihilators in residuated lattices, where f is an endomorphism. Knowing that the notion of endomorphism already existed in universal algebras and due to the fact that any idempotent endomorphism is a state operator, it seems reasonable that all the results of this section can be extended to a general universal algebras and in particular to residuated lattice, where the state operator τ is replaced by an idempotent endomorphism. □

5. Prime State Ideals in DMSRL

This section proceeds with the study of the notion of prime state ideals in De Morgan state residuated lattices. We establish the prime state ideal theorem and make investigation for some related properties.

Proposition 8. *Let P be a proper state ideal of* (L,τ). *Then, the following statements are equivalent:*

(1) If $P_1,P_2\in\mathcal{SI}(L)$ *and* $P=P_1\cap P_2$, *then* $P=P_1$ *or* $P=P_2$;

(2) If $P_1,P_2\in\mathcal{SI}(L)$ *and* $P_1\cap P_2\subseteq P$, *then* $P_1\subseteq P$ *or* $P_2=P$;

(3) If $a,b\in L$ *so that* $(a\oplus\tau(a))\wedge(b\oplus\tau(b))\in P$, *then* $a\in P$ *or* $b\in P$.

Proof. (1)⇒(2) Let $P_1,P_2\in\mathcal{SI}(L)$ and $P_1\cap P_2\subseteq P$. Then, $(P_1\cap P_2)\vee P=P$. From Proposition 3, the lattice $(\mathcal{SI}(L),\subseteq)$ is Brouwerian, so it is distributive. It follows that $(P_1\vee P)\cap(P_2\vee P)=P$. Now by (1), $P_1\vee P=P$ or $P_2\vee P=P$. Thus, $P_1\subseteq P$ or $P_2\subseteq P$.

(1)⇒(3). Let $a,b\in L$. Assume $(a\oplus\tau(a))\wedge(b\oplus\tau(b))\in P$. Set $(P_1=\langle P,a\rangle_\tau$ and $(P_2=\langle P,b\rangle_\tau$. Obviously, $P\subseteq P_1\cap P_2$. Let $x\in P_1\cap P_2$, then by Theorem 1 (3), there are $l,k\in P$ and $m,n\geq 1$ such that $x\leq k\oplus m(a\oplus\tau(a))$ and $x\leq l\oplus n(b\oplus\tau(b))$. Hence, we have

$$
\begin{aligned}
x &\leq (k\oplus m(a\oplus\tau(a)))\wedge(l\oplus n(b\oplus\tau(b)))\\
&\overset{(P16)}{\leq} ((k\oplus m(a\oplus\tau(a)))\wedge l)\oplus((k\oplus m(a\oplus\tau(a)))\wedge n(b\oplus\tau(b)))\\
&\overset{(P16)}{\leq} (k\wedge l)\oplus(m(a\oplus\tau(a))\wedge l)\oplus(k\wedge n(b\oplus\tau(b)))\oplus(m(a\oplus\tau(a))\wedge n(b\oplus\tau(b)))\\
&\overset{(P15)}{\leq} (k\wedge l)\oplus(l\wedge m(a\oplus\tau(a)))\oplus(k\wedge n(b\oplus\tau(b)))\oplus mn((a\oplus\tau(a))\wedge(b\oplus\tau(b))).
\end{aligned}
\tag{6}
$$

But $(k\wedge l)$, $(l\wedge m(a\oplus\tau(a)))$, $(k\wedge n(b\oplus\tau(b)))$, $mn(a\oplus\tau(a))\wedge(b\oplus\tau(b))\in P$. Thus, $x\in P$. Hence, $P=P_1\cap P_2$. Therefore, by (1), $P=P_1$ or $P=P_2$, that is, $a\in P$ or $b\in P$.

(3)⇒(1) Let $P_1,P_2\in\mathcal{SI}(L)$ such that $P=P_1\cap P_2$. Suppose that $P\neq P_1$ and $P\neq P_2$ and let $a\in P_1/P$ and $b\in P_2/P$. Then, $(a\oplus\tau(a))\wedge(b\oplus\tau(b))\in P_1\cap P_2=P$, that is, a contradiction. Thus, $P=P_1$ or $P=P_2$. □

Definition 8. A proper state ideal P of (L,τ) is said to be prime if it satisfies one of the equivalent conditions of Proposition 8.

We denote the set of all prime state ideals of (L,τ) by $Spect_\tau(L)$.

The following example shows a prime state ideal of (L,τ), which is not a prime ideal of L.

Example 10. Consider the De Morgan state residuated lattice (L,τ) of Example 5. $\{0\}$, $\{0,p\}$, $\{0,p,a\}$, $\{0,p,c\}$, $\{0,p,b,d,f\}$, L are ideals of L, and the state ideals of (L,τ) are $\{0\}$, $I=\{0,p\}$, $J=\{0,p,b,d,f\}$ and L. The ideal I is not a prime ideal of L, because $\{0,p\}=\{0,p,a\}\cap\{0,p,c\}$ but $\{0,p\}\neq\{0,p,a\}$ and $\{0,p\}\neq\{0,p,c\}$. Still as a state ideal of (L,τ), $\{0,p\}$ is a prime state ideal (according to Proposition 8, (1)).

Now, we give one of our main theorems.

Theorem 7 *(Prime state ideal theorem) Let F be a filter in the lattice* $(L,\subseteq)$, *and I be a state ideal of* (L,τ) *such that* $I\cap F=\varnothing$. *Then, there is a prime state ideal P of* (L,τ) *such that* $I\subseteq P$ *and* $P\cap F=\varnothing$.

Proof. Set $\mu(I)=\{J\colon J\in\mathcal{SI}(L), I\subseteq J \text{ and } J\cap F=\varnothing\}$. We have $I\in\mathcal{SI}(L)$, $I\subseteq I$ and $I\cap F=\varnothing$. Then, $I\in\mu(I)$, that is, $\mu(I)\neq\varnothing$. It is easy to check that the set $\mu(I)$ is inductively ordered by inclusion. Hence by Zorn's lemma, it has a maximal element P. Let's show that that $P\in Spect_\tau(L)$. Since $P\in\mu(I)$, it follows that P is a proper state ideal and $P\cap F=\varnothing$. Let $x,y\in L$, such that $(a\oplus\tau(a))\wedge(b\oplus\tau(b))\in P$.

Assume $a \notin P$ and $b \notin P$, and consider the sets $\langle P, a\rangle_\tau$ and $\langle P, b\rangle_\tau$. Then, P is strictly contained in $\langle P, a\rangle_\tau$ and $\langle P, b\rangle_\tau$ and the maximality of P implies that $\langle P, a\rangle_\tau \notin \mu(I)$ and $\langle P, b\rangle_\tau \notin \mu(I)$. Thus, $\langle P, a\rangle_\tau \cap F \neq \varnothing$ and $\langle P, b\rangle_\tau \cap F \neq \varnothing$. Let $x \in \langle P, a\rangle_\tau \cap F$ and $y \in \langle P, b\rangle_\tau \cap F \neq \varnothing$. According to Theorem 1 (3), there are $k, l \in P$ and $m, n \geq 1$ such that $x \leq k \oplus m(a \oplus \tau(a))$ and $y \leq l \oplus n(b \oplus \tau(b))$. Then,

$$\begin{aligned} x \wedge y &\leq (k \oplus m(a \oplus \tau(a))) \wedge (l \oplus n(b \oplus \tau(b))) \\ &\overset{(P16)}{\leq} ((k \oplus m(a \oplus \tau(a))) \wedge l) \oplus (k \oplus m(a \oplus \tau(a))) \wedge n(b \oplus \tau(b)) \\ &\overset{(P16)}{\leq} (k \wedge l) \oplus (m(a \oplus \tau(a)) \wedge l) \oplus (k \wedge n(b \oplus \tau(b))) \oplus (m(a \oplus \tau(a))) \wedge n(b \oplus \tau(b)) \\ &\overset{(P15)}{\leq} (k \wedge l) \oplus (l \wedge m(a \oplus \tau(a))) \oplus (k \wedge n(b \oplus \tau(b))) \oplus mn((a \oplus \tau(a)) \wedge (b \oplus \tau(b))). \end{aligned} \tag{7}$$

But $(k \wedge l)$, $(l \wedge m(a \oplus \tau(a)))$, $(k \wedge n(b \oplus \tau(b)))$, $mn((a \oplus \tau(a)) \wedge (b \oplus \tau(b))) \in P$; hence, $x \wedge y \in P$. Moreover, since F is a filter of the lattice $(L, \subseteq)$, it follows that $x \wedge y \in F$, and therefore, $P \cap F \neq \varnothing$, which is a contradiction. Therefore, $P \in Spect_\tau(L)$. □

Proposition 9. *Let I be a proper state ideal of (L, τ). Then, there is a maximal state ideal M of (L, τ) such that $I \subseteq M$.*

Proof. Let us consider the set $\theta(I) = \{J\colon J \text{ is a proper state ideal such that } I \subseteq J\}$. I is a proper state ideal of (L, τ) and $I \subseteq I$. So, $I \in \theta(I)$, that is, $\theta(I) \neq \varnothing$. One can easily see that $\theta(I)$ is inductively ordered by inclusion. Then, by Zorn's lemma, $\theta(I)$ has a maximal element M. Let show that $\theta(I)$ is a maximal state ideal of (L, τ). In fact, if N is a proper state ideal of (L, τ) such that $M \subseteq N$, then $N \in \theta(I)$ and the maximality of M implies that $N = M$. □

Proposition 10. *Let $a \in L$, $a > 0$. Then, there is a prime state ideal P of (L, τ) such that $a \notin P$.*

Proof. Since $\{0\}$ is a state ideal and $\{0\} \cap [a) = \varnothing$ (where $[a)$ is the filter generated by $\{a\}$ in the lattice $(L, \subseteq)$). Hence by Theorem 7, there exists a prime state ideal P such that $P \cap \langle a\rangle = \varnothing$. Thus, $a \notin P$. □

6. Conclusion

In this article, we have introduced the notion of De Morgan state residuated lattice (DMSRL) and made investigations on certain related properties and examples. We have proved that the lattice of all state ideals $(\mathcal{SI}(L), \subseteq)$ of a DMSRL is a coherent frame, and for any state ideal I, the pseudo-complement of I is $I\prime = I \longrightarrow \{0\} = \{x \in L\colon I \cap \langle x\rangle_\tau = \{0\}\} = \{x \in L\colon i \wedge n(x \oplus \tau(x)) = 0, \text{for all } i \in I \text{ and } n \in \mathbb{N}^*\}$. Furthermore, we have illustrated that the set of compacts element of the sublattice $\mathcal{SI}(L)$ is $\mathscr{C}(\mathcal{SI}(L)) = \{\langle x\rangle_\tau\colon x \in L\}$. Also, we characterized the DMSRL for which the lattice $\mathcal{SI}(L)$ is a Boolean algebra. In addition, we brought in the concept of state relative annihilator of a given nonempty subset with respect to a state ideal in DMSRL and investigated some of its properties. We proved that state relative annihilators are a particular kind of state ideals. After all, we have studied the notion of prime state ideals in DMSRL and established the prime state ideal theorem.

Knowing that in universal algebras, any idempotent endomorphism is a state operator [29], it seems reasonable that all the results in this article can be extended to the more general class of residuated lattice and even in general case of universal algebras, where the state operator τ is replaced by an idempotent endomorphism. So, our further work will consist to look this direction. In the same view as the work in Ref. [32], another way on this topic will focus on the study of the lattices of L-fuzzy state ideals in De Morgan state residuated lattices.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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