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Research Article

A Necessary and Sufficient Condition for $\mathbf{k} = \mathbb{Q}(\sqrt{4\mathbf{n}^2 + 1})$ to Have Class Number $\omega(\mathbf{n}) + \mathbf{c}$

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In this paper, we give a necessary and sufficient condition for real quadratic field $k = \mathbb{Q}(\sqrt{4n^2 + 1})$ to have class number $\omega(n) + c$ where $c \in \{1, 2\}$ and $\omega(n)$ is the number of odd prime divisors of n.

1. Introduction

Let d be a positive square-free integer and let $h(d)$ and C_k denote the class number and the class group of a real quadratic field $k = \mathbb{Q}(\sqrt{d})$, respectively.

The class number problem of quadratic fields is one of the most intriguing unsolved problems in algebraic number theory, and it has been the object of attention for many years of research.

In [1], Gauss had conjectured that there exist exactly nine imaginary quadratic fields of class number 1. Later, this was solved after diverse works of Stark, Heegner, and Baker. On the other hand, it was conjectured by Gauss that there are infinitely many real quadratic fields with class number one. This conjecture is still open.

Many fruitful research studies have been conducted in this direction (see [2–7]). In particular, we mention some results showing finiteness of class number one real quadratic fields in special families of real quadratic fields. Biro (see [2, 3])proved in 2003 two important results: Yokoi's conjecture which asserts that $h(m^2 + 4) = 1$ only for six values of $m = 1, 3, 5, 7, 13, 17$ and Chowla's conjecture which says that $h(4m^2 + 1) = 1$ only for six values of $m = 1, 2, 3, 5, 7, 13$.

Another line of work, in these contexts, is finding bounds for class number of a number field. Hasse [8] and Yokoi [9, 10] studied lower bounds for class numbers of certain real quadratic fields. Mollin [11, 12] generalized their results for certain real quadratic and biquadratic fields. Recently, Chakraborty et al. [13] derived a lower bound for $\mathbb{Q}(\sqrt{n^2 + 1})$. Also, Mishra [14] gave a lower bound for $\mathbb{Q}(\sqrt{n^2 + r})$, where $r = 1, 4$. For a given fixed number h, it is interesting to find necessary and sufficient conditions for a real quadratic field to have class number h, see [15–18]. In particular, Yokoi [18] showed that the class number of $k = \mathbb{Q}(\sqrt{4n^2 + 1})$ is 1 if and only if $n^2 - t(t + 1)$ (with $1 \le t \le n - 1$) is a prime. In this work, we use the lower bound for $h(4n^2 + 1)$ to give a necessary and sufficient condition for $k = \mathbb{Q}(\sqrt{4n^2 + 1})$ to have class number $\omega(n) + c$ where $c \in \{1, 2\}$.

2. Preliminaries

Let k be a real quadratic field and $\zeta_k(s)$ be its Dedekind zeta function. Siegel [19] developed a method of computing $\zeta_k(1 - 2n)$, where n is a positive integer. By specializing Siegel's formula for a real quadratic field, we obtain the following result.

Theorem 1 (Zagier, [20]). *Let k be a real quadratic field with discriminant D. Then,*

$$\zeta_k(-1) = \frac{1}{60} \sum_{\substack{|t| < \sqrt{D} \\ t^2 \equiv D \,(\mathrm{mod}\, 4)}} \sigma_1\left(\frac{D - t^2}{4}\right), \tag{1}$$

where $\sigma_1(r)$ denotes the sum of divisors of r.

However, there is another method, due to Lang, of computing special values of $\zeta_k(s)$ if k is a real quadratic field.

Let $k=\mathbb{Q}(\sqrt{d})$ be a real quadratic field of discriminant D and H an ideal class of k. Let I be any integral ideal belonging to H^{-1} with an integral basis $\{r_1, r_2\}$. We put

$$\delta(\mathrm{I}) = r_1 r_2' - r_1' r_2, \tag{2}$$

where r_1' and r_2' are the conjugates of r_1 and r_2, respectively.

Let ε be the fundamental unit of k. Then, $\{\varepsilon r_1, \varepsilon r_2\}$ is also integral basis of I, and thus we can find a matrix $M=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ with integer entries satisfying

$$\varepsilon\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M.\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \tag{3}$$

The entries of M are given by

$$\begin{aligned} a &= tr\left(\frac{r_1 r_2' \varepsilon}{\delta(\mathrm{I})}\right), b = tr\left(\frac{r_1 r_1' \varepsilon'}{\delta(\mathrm{I})}\right), \\ c &= tr\left(\frac{r_2 r_2' \varepsilon}{\delta(\mathrm{I})}\right), d = tr\left(\frac{r_1 r_2' \varepsilon'}{\delta(\mathrm{I})}\right). \end{aligned} \tag{4}$$

Moreover, $\det M = N(\varepsilon)$ and $\mathrm{bc}\neq 0$. See ([13, 14, 21]). Now, we can state Lang's formula.

Theorem 2 (Lang, [22]). *By keeping the abovementioned notation, we have*

$$\begin{aligned} \zeta_k(-1,\mathrm{H}) = {} & \frac{\operatorname{sgn}\delta(\mathrm{I}) r_2 r_2'}{360 N(\mathrm{I}) c^3}\{(a+d)^3 - 6(a+d)N(\varepsilon) \\ & -240c^3(\operatorname{sgn}c)S^3(a,c) + 180ac^3(\operatorname{sgn}c)S^2(a,c) \\ & -240c^3(\operatorname{sgn}c)S^3(d,c) + 180dc^3(\operatorname{sgn}c)S^2(d,c)\}, \end{aligned} \tag{5}$$

where $N(\mathrm{I})$ denotes the norm of an ideal I, *$N(\varepsilon)$ is the norm of ε, and $S^i(-,-)$ denotes the generalized Dedekind sum as defined in [23].*

Now, we will introduce some basics about quadratic number field $k=\mathbb{Q}(\sqrt{4n^2+1})$. Let n be a positive integer and let $d=4n^2+1$ be a square-free integer. Clearly $d\equiv 1$ or $5 \pmod 8$. In this case, the fundamental unit of k is $\varepsilon = 2n+\sqrt{d}$ and $N(\varepsilon)=-1$. We also know that if $p \mid n$, where p is an odd prime, then p splits in $k=\mathbb{Q}(\sqrt{d})$ as

$$\langle p\rangle = \left\langle p, \frac{1+\sqrt{d}}{2}\right\rangle\left\langle p, \frac{1-\sqrt{d}}{2}\right\rangle. \tag{6}$$

By ([21], Theorem 2.4), we also know that

$$\zeta_k(-1,A) = \frac{4n^3+14n}{180}, \tag{7}$$

where A will always denote the principal ideal class in k.

By ([13], Proposition 3.1), we also know that

$$\zeta_k(-1,\mathrm{C}) = \frac{2n^3 + n\left(2p^4+5p^2\right)}{90p^2}, \tag{8}$$

where C is the ideal class containing $\langle p, (1+\sqrt{d})/2\rangle$ or $\langle p, (1-\sqrt{d})/2\rangle$.

3. Main Results

In this section, we will prove our main results. As a start, we record the following theorem that we derive from ([13], Proposition 3.1).

Theorem 3. *Let n be a positive integer and let $d=4n^2+1$ be a square-free integer and suppose that*

(i) $\omega(n)\geq 3$

(ii) $c=1$ if $d\equiv 5 \pmod 8$ and $c=2$ if $d\equiv 1 \pmod 8$

Then,

$$h(d)\geq \omega(n)+c. \tag{9}$$

Now, we find necessary and sufficient condition for $k=\mathbb{Q}(\sqrt{4n^2+1})$ to have class number $\omega(n)+c$.

Theorem 4. *By keeping the abovementioned notation, we have*

$h(d)=\omega(n)+c$ if and only if

$$\sum_{m=0}^{n-1}\sigma_1\left(n^2-m(m+1)\right) = \begin{cases} \dfrac{4n^3+14n}{6} + \displaystyle\sum_{i=1}^{\omega(n)}\frac{2n^3+n\left(2p_i^4+5p_i^2\right)}{3p_i^2} \text{ if } c=1, \\ \dfrac{5n^3+40n}{6} + \displaystyle\sum_{i=1}^{\omega(n)}\frac{2n^3+n\left(2p_i^4+5p_i^2\right)}{3p_i^2} \text{ if } c=2, \end{cases} \tag{10}$$

where p_i are odd primes, $p_i|n$, and $\omega(n)\geq 3$.

Proof. Now, by Theorem 1,

$$\begin{aligned}\zeta_k(-1) &= \frac{1}{60} \sum_{\substack{|t|<\sqrt{4n^2+1} \\ t^2 \equiv 4n^2+1 \,(\mathrm{mod}\, 4)}} \sigma_1\left(\frac{4n^2+1-t^2}{4}\right), \\ &= \frac{1}{60} \sum_{\substack{1\le|t|\le 2n-1 \\ t \text{ i sodd}}} \sigma_1\left(\frac{4n^2+1-t^2}{4}\right), \\ &= \frac{1}{30} \sum_{\substack{1\le t\le 2n-1 \\ t \text{ i sodd}}} \sigma_1\left(\frac{4n^2+1-t^2}{4}\right), \\ &= \frac{1}{30} \sum_{m=0}^{n-1} \sigma_1\left(\frac{4n^2+1-(2m+1)^2}{4}\right),\end{aligned} \tag{11}$$

so that

$$\zeta_k(-1) = \frac{1}{30} \sum_{m=0}^{n-1} \sigma_1\left(n^2 - m(m+1)\right). \tag{12}$$

Let C_i $(1 \le i \le \omega(n))$ be the ideal class in k such that $\langle p_i, (1+\sqrt{d})/2\rangle \in C_i$, then by (8), we obtain

$$\zeta_k(-1, C_i) = \frac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{90p_i^2}. \tag{13}$$

If $c = 1$, then 2 remains prime in k. We note $\{C_1, \dots C_{\omega(n)}\}$ are distinct nonprincipal ideal classes in k.

If $c = 2$, then 2 splits in k, that is,

$$\langle 2\rangle = \left\langle 2, \frac{(1+\sqrt{d})}{2}\right\rangle \left\langle 2, \frac{(1-\sqrt{d})}{2}\right\rangle. \tag{14}$$

We assume that B denotes the ideal class in k such that $\langle 2, (1+\sqrt{d})/2\rangle \in B$, then by ([21], Theorem 2.5), we obtain

$$\zeta_k(-1, B) = \frac{n^3 + 26n}{180}. \tag{15}$$

We note $\{B, C_1, \dots C_{\omega(n)}\}$ are distinct nonprincipal ideal classes in k.

Necessary: let $h(d) = \omega(n) + c$. Now, by definition, we have

$$\zeta_k(-1) = \sum_{J\in C_k} \zeta_k(-1, J). \tag{16}$$

If $c = 1$, we obtain $C_k = \{A, C_1, \dots C_{\omega(n)}\}$, then

$$\zeta_k(-1) = \zeta_k(-1, A) + \sum_{i=1}^{\omega(n)} \zeta_k(-1, C_i). \tag{17}$$

This implies

$$\zeta_k(-1) = \frac{4n^3 + 14n}{180} + \sum_{i=1}^{\omega(n)} \frac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{90p_i^2}. \tag{18}$$

Hence, by (12), we find

$$\sum_{m=0}^{n-1} \sigma_1\left(n^2 - m(m+1)\right) = \frac{4n^3 + 14n}{6} + \sum_{i=1}^{\omega(n)} \frac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{3p_i^2}. \tag{19}$$

If $c = 2$, we obtain $C_k = \{A, B, C_1, \dots C_{\omega(n)}\}$, then

$$\zeta_k(-1) = \zeta_k(-1, A) + \zeta_k(-1, B) + \sum_{i=1}^{\omega(n)} \zeta_k(-1, C_i). \tag{20}$$

This implies

$$\zeta_k(-1) = \frac{5n^3 + 40n}{180} + \sum_{i=1}^{\omega(n)} \frac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{90p_i^2}. \tag{21}$$

Hence, by (12), we find

$$\sum_{m=0}^{n-1} \sigma_1\left(n^2 - m(m+1)\right) = \frac{5n^3 + 40n}{6} + \sum_{i=1}^{\omega(n)} \frac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{3p_i^2}. \tag{22}$$

Sufficiency: let

$$\sum_{m=0}^{n-1} \sigma_1\left(n^2 - m(m+1)\right) = \begin{cases} \dfrac{4n^3 + 14n}{6} + \displaystyle\sum_{i=1}^{\omega(n)} \dfrac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{3p_i^2} \text{ if } c = 1, \\[2ex] \dfrac{5n^3 + 40n}{6} + \displaystyle\sum_{i=1}^{\omega(n)} \dfrac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{3p_i^2} \text{ if } c = 2. \end{cases} \tag{23}$$

Then, by (12), we find

$$\zeta_k(-1) = \begin{cases} \dfrac{4n^3 + 14n}{180} + \displaystyle\sum_{i=1}^{\omega(n)} \dfrac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{90p_i^2} \text{if} c = 1, \\ \dfrac{5n^3 + 40n}{180} + \displaystyle\sum_{i=1}^{\omega(n)} \dfrac{2n^3 + n\left(2p_i^4 + 5p_i^2\right)}{90p_i^2} \text{if} c = 2. \end{cases} \tag{24}$$

By Theorem 3, we get $h(d) \geq \omega(n) + c$.

Suppose $h(d) > \omega(n) + c$. Then, there exist at least $\omega(n) + c + 1$ ideal classes in k. Since for any ideal class E, $\zeta_k(-1, \mathrm{E}) > 0$, thus

$$\zeta_k(-1) > \zeta_k(-1, A) + (c-1)\zeta_k(-1, B) + \sum_{i=1}^{\omega(n)} \zeta_k(-1, C_i). \tag{25}$$

It is a contradiction. □

Data Availability

No date were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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