# Extended Newton-type Method for Nonsmooth Generalized Equation under ( $n, \alpha$ )-point-based Approximation 

M. Z. Khaton and M. H. Rashid (<br>Department of Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh and Bangabandhu Government College, Department of Mathematics, Naogaon, Bangladesh<br>Department of Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh

Correspondence should be addressed to M. H. Rashid; harun_math@ru.ac.bd
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#### Abstract

Let $\mathcal{X}$ and $\mathscr{y}$ be Banach spaces and $\Omega \subseteq X$. Let $f: \Omega \longrightarrow \mathscr{Y}$ be a single valued function which is nonsmooth. Suppose that $F: \mathscr{X} \rightrightarrows 2^{\mathscr{Y}}$ is a set-valued mapping which has closed graph. In the present paper, we study the extended Newton-type method for solving the nonsmooth generalized equation $0 \in f(x)+F(x)$ and analyze its semilocal and local convergence under the conditions that $(f+F)^{-1}$ is Lipschitz-like and $f$ admits a certain type of approximation which generalizes the concept of pointbased approximation so-called ( $n, \alpha$ )-point-based approximation. Applications of $(n, \alpha)$-point-based approximation are provided for smooth functions in the cases $n=1$ and $n=2$ as well as for normal maps. In particular, when $0<\alpha<1$ and the derivative of $f$, denoted $\nabla f$, is $(\ell, \alpha)$-Hölder continuous, we have shown that $f$ admits ( $1, \alpha$ )-point-based approximation for $n=1$ while $f$ admits (2, $\alpha$ )-point-based approximation for $n=2$, when $0<\alpha<1$ and the second derivative of $f$, denoted $\nabla^{2} f$, is $(K, \alpha)$-Hölder. Moreover, we have constructed an $(n, \alpha)$-point-based approximation for the normal maps $f_{\mathscr{C}}+F$ when $f$ has an $(n, \alpha)$-pointbased approximation. Finally, a numerical experiment is provided to validate the theoretical result of this study.


## 1. Introduction

Robinson [1, 2] introduced the generalized equation as a general tool for describing, analyzing and solving different problems in a unified manner and it has been studied extensively. Typical examples are systems of inequalities, variational inequalities, linear and nonlinear complementary problems, system of nonlinear equations, equilibrium problems, first-order necessary conditions for nonlinear programming etc. They also have plenty of applications in engineering and economics. For more details on these applications and many other ones that we did not mention here, one can refer to [1-3].

In this study, let $\mathcal{X}$ and $\mathscr{y}$ be Banach spaces, $F: \mathscr{X} \rightrightarrows 2^{\mathscr{y}}$ be a set-valued mapping with closed graph and $f: \Omega \subseteq \mathcal{X} \longrightarrow \mathscr{Y}$ be a nonsmooth single-valued function that admits $(n, \alpha)$-point-based approximation $A$ on $\Omega$ with a constant $L>0$. We are concerned with the problem of
approximating the point $\bar{x} \in \Omega$ (which is called the solution of (1)) of the following nonsmooth generalized equation:

$$
\begin{equation*}
0 \in f(\bar{x})+F(\bar{x}) . \tag{1}
\end{equation*}
$$

The classical Newton method is very well known and extensively used to find solutions of (1) when $F=\{0\}$, where $f$ has Lipschitz continuous Fréchet derivatives. A survey of local and semilocal convergence results for Newton method's are found and mentioned in [4-7]. When $f$ is nonsmooth, such a classical linearization is no longer available and we need to seek a replacement. In other words, if $f$ doesn't possess Fréchet derivatives, it is not so clear how a Newton algorithm should be designed. There are many investigators have worked on this question and the applicants have presented different methods for a few things that are important in certain cases and have proved their justification; see for example [4, 8-14]. Several papers have worked on the Newton-type methods for nonsmooth
equations and variational inequalities; see for example [ $8,9,15]$ for inspiration and advanced works on these topics.

In the case when $F=\{0\}$ and $f$ is a nonsmooth function, Robinson [9, Theorem 3.2] considered point-based approximation with Lipschitzian property to show the convergence of Newtons method under the Newton-Kantorovich-type hypothesis. Argyros [10] presented a semilocal convergence analysis of Newtons method based on a suitable point-based approximation. More explicitly, he has taken weaker assumptions in point-based approximation by considering Hölderian property instead of Lipschitzian property in order to cover a wider range of problems than those discussed in [9] and hence showed the convergence result for Newton's method.

In addition, Kummer [16] presented a necessary and adequate condition for superlinear convergence of the Newton method, which was originally designed for deriv-ative-type approximations of a nonsmooth function around an isolated zero. Relevant results, for solving the nonsmooth generalized (1) are given in $[8,17,18]$.

To solve the nonsmooth generalized (1), Geoffroy and Piétrus in [19] considered the following method

$$
\begin{equation*}
0 \in A\left(x_{k}, x_{k+1}\right)+F\left(x_{k+1}\right) \text { for each } k=0,1,2, \ldots \tag{2}
\end{equation*}
$$

where $A: \mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{Y}$ is an approximation of $f$, and presented a local convergence result by using the assumptions that $f$ admits an ( $n, \alpha$ )-point-based approximation $A$ and the set-valued map $\left(A\left(x^{*}, \cdot\right)+F(\cdot)\right)^{-1}$ is $M$-pseudoLipschitz around ( $0, x^{*}$ ). For the first time, Dontchev [11] introduced the iterative procedure (2) for solving (1) and presented the nonsmooth analogue of the Kantorovich-type theorem for this procedure by assuming the Aubin continuity of the map $\left(A\left(x_{0}, \cdot\right)+F(\cdot)\right)^{-1}$ at $\left(0, x_{1}\right)$ (or, equivalently, $\quad(f+F)^{-1}$ is Aubin continuous at $\left.\left(f\left(x_{1}\right)-A\left(x_{0}, x_{1}\right), x_{1}\right)\right)$, where $x_{1}$ is the first iterate of (2).

Let $x \in \Omega \subseteq \mathscr{X}$. The subset of $\Omega$, denoted by $\mathscr{M}(x)$, is defined by

$$
\begin{equation*}
\mathscr{M}(x):=\{d \in \Omega: 0 \in A(x, x+d)+F(x+d)\} \tag{3}
\end{equation*}
$$

Although the method (2) guarantees the existence of a convergent sequence $\left\{x_{k}\right\}$ for solving (1), the constructed points $x_{1}, x_{2}, \ldots, x_{k}$ are not unique and therefore, for a starting point near to a solution, the sequences generated by the method (2) are not uniquely defined. For example, the convergence result established in [19, Theorem 3.3], guarantees the existence of a convergent sequence. Hence, in view of numerical computation, this kind of Newton-type methods are not convenient in practical application. Based on these ideas, Rashid [8] introduced and studied the following algorithm and presented semilocal and local convergence results under the assumptions that $f$ has a pointbased approximation and $(f+F)^{-1}$ is Lipsctiz-like mapping:

It is noted that, in the case when $A$ is replaced by the classical linearization of $f$, the Algorithm 1 is reduced to the Gauss-Newton-type method introduced by Rashid et al. [20].

Moreover, when the single-valued function involved in (1) is smooth, there has been increased amount of interest on semilocal and local analysis (see, for example, [8, 20-23] and the references therein).

Our approach is somewhat different. In this study, we give a more general approach, namely ( $n, \alpha$ )-point-based approximation, which is an extension of the concept of point-based approximation introduced by Robinson [9] and it can apply to a wide range of particular problems. Because of the presence of Step 3 in Algorithm 1, we have shown in the main proof (Theorem 2) that each of the constructed points $x_{1}, x_{2}, \ldots, x_{k}$ has limit. Therefore, in numerical computational view point, Algorithm 1 gives the more accurate result than the result given by the method (2).

In the present paper, we present semilocal and local convergence of Algorithm 1 under some mild conditions for the function $f$ and the set-valued mapping $f+F$. In fact, the main motivation of this research is to analyze the semilocal and local convergence of the sequence generated by Algorithm 1 for solving the nonsmooth generalized (1) using the notion of ( $n, \alpha$ )-point-based approximation introduced by Geoffroy and Piétrus [19] and Lipschitz-like property. Based on the information around the initial point, the main result is the convergence criterion, developed in the section 3, which provides some sufficient conditions, for a starting point near to the solution, ensuring the convergence to the solution of any sequence generated by Algorithm 1. As a result, local convergence result for the extended Newtontype method is obtained.

This paper is organized as follows: In section 2 , we recall some definitions, notations and preliminarily results that will need afterwards. In Section 3, we show the existence of the sequence generated by Algorithm 1 and then establish the convergence of the extended Newton-type method by using the concept of ( $n, \alpha$ )-point-based approximation as well as Lipchitz-like property. In Section 4, we have given some applications of ( $n, \alpha$ )-point-based approximation for smooth functions in the case when $n=1, n=2$ and $0<\alpha<1$ and for normal maps $f_{\mathscr{C}}+F$ which is reformulated by Rashid [8]. In the last section, a numerical experiment is provided to justify the theoretical result of this study.

## 2. Preliminaries

Throughout this paper, we assume that $\mathcal{X}$ and $\mathscr{Y}$ are two real or complex Banach spaces and $\mathbb{N}$ is the set of all Natural numbers and $\mathbb{N}^{*}=\mathbb{N}-\{0\}$. Suppose that $f: \mathscr{X} \longrightarrow \mathscr{Y}$ is a Fréchet differentiable function and $F: \mathscr{X} \rightrightarrows 2^{\mathscr{Y}}$ is a set-valued mapping with closed graph. Let $x \in \mathscr{X}$ and $r>0$. The closed ball centered at $x$ with radius $r$ is denoted by $\mathbb{B}_{r}(x)$.

All the norms are denoted by $\|\cdot\|$. The domain $\operatorname{dom} F$ and the inverse $F^{-1}$ are respectively defined by

$$
\begin{align*}
\operatorname{dom} F & :=\{x \in \mathscr{X}: F(x) \neq \varnothing\} \\
F^{-1}(y) & :=\{x \in \mathscr{X}: y \in F(x)\} \text { for each } y \in \mathscr{Y} . \tag{4}
\end{align*}
$$

Let $D \subseteq \mathscr{X}$. The distance from a point $x$ to a set $D$ is defined by

$$
\begin{equation*}
\operatorname{dist}(x, D):=\inf \{\|x-a\|: a \in D\} \text { for each } x \in \mathcal{X} \tag{5}
\end{equation*}
$$

while the excess from the set $D$ to the set $C \subseteq \mathscr{X}$ is defined by

$$
\begin{equation*}
e(C, D)=\sup \{\operatorname{dist}(x, D): x \in C\} \tag{6}
\end{equation*}
$$

Definition 1. Consider the set-valued mapping $F: X X \nexists 2^{\mathscr{Y}}$. Then the graph of $F$ is defined by

$$
\begin{equation*}
\operatorname{gph} F:=\{(x, y) \in \mathscr{X} \times \mathscr{Y}: y \in F(x)\} \tag{7}
\end{equation*}
$$

Definition 2. A set-valued function $F: \mathscr{X} \rightrightarrows 2^{\mathscr{Y}}$ is said to be a closed graph if the set $\{(x, y): y \in F(x)\}$ is a closed subset of $\mathscr{X} \times \mathscr{Y}$ in the product topology i.e. for all sequences $\left\{x_{k}\right\}_{k \in \mathbb{N}}$ and $\left\{y_{k}\right\}_{k \in \mathbb{N}}$ such that $x_{k} \longrightarrow x$ and $y_{k} \longrightarrow y$ and $y_{k} \in F\left(x_{k}\right)$ for all $n$, we have $y \in F(x)$.

The notions of pseudo-Lipschitz and Lipchitz-like setvalued mappings were introduced by Aubin in [24] and have been studied extensively; see for example [25, 26]. We recall these notions from [20].

Definition 3. Let $\Gamma: y \rightrightarrows 2^{x}$ be a set-valued mapping and let $(\bar{y}, \bar{x}) \in \operatorname{gph} \Gamma$. Let $r_{\bar{x}}>0, r_{\bar{y}}>0$ and $M>0$. $\Gamma$ is said to be
(a) Lipchitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $M$ if the following inequality holds:

$$
\begin{equation*}
e\left(\Gamma\left(y_{1}\right) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \Gamma\left(y_{2}\right)\right) \leq M\left\|y_{1}-y_{2}\right\| \text { for any } y_{1}, y_{2} \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}) . \tag{8}
\end{equation*}
$$

(b) pseudo-Lipschitz around ( $\bar{y}, \bar{x}$ ) if there exist constants $r_{\bar{y}}^{\prime}>0, r_{\bar{x}}^{\prime}>0$ and $M^{\prime}>0$ such that $\Gamma$ is Lipchitzlike on $\mathbb{B}_{r_{\bar{y}}^{\prime}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}^{\prime}}(\bar{x})$ with constant $M^{\prime}$.

Remark 1. The pseudo-Lipschitz property of a set-valued mapping $\Gamma$ is equivalent to the openness with linear rate of $\Gamma^{-1}$ (the covering property) and to the metric regularity of $\Gamma^{-1}$ (a basic well-posedness property in optimization) (see [23, 24, 27, 28] for more details).

Remark 2. Equivalently for the property (a) we can say that $\Gamma$ is Lipschitz-like at $\left(y_{0}, x_{0}\right) \in \operatorname{gph} \Gamma$ on $\mathbb{B}_{r_{y_{0}}}\left(y_{0}\right) \times$ $\mathbb{B}_{r_{x_{0}}}\left(x_{0}\right)$ with constant $M$ if for every $x_{1}, x_{2} \in \mathbb{B}_{r_{x_{0}}}\left(x_{0}\right)$ and for every $x_{1} \in \Gamma\left(y_{1}\right) \cap \mathbb{B}_{r_{x_{0}}}\left(x_{0}\right)$, there exists $x_{2} \in \Gamma\left(y_{2}\right)$ such that

$$
\begin{equation*}
\left\|x_{1}-x_{2}\right\| \leq M\left\|y_{1}-y_{2}\right\|, \text { for every } y_{1}, y_{2} \in \mathbb{B}_{r_{y_{0}}}\left(y_{0}\right) \tag{9}
\end{equation*}
$$

The following lemma is useful and it was proven by Rashid et al. in [20, Lemma 2.1].

Lemma 1. Let $\Gamma: y \rightrightarrows 2^{x}$ be a set-valued mapping and let $(\bar{y}, \bar{x}) \in g p h \Gamma$. Assume that $\Gamma$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $M$. Then

$$
\begin{equation*}
\operatorname{dist}(x, \Gamma(y)) \leq M \operatorname{dist}\left(y, \Gamma^{-1}(x)\right) \tag{10}
\end{equation*}
$$

holds for every $x \in \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $y \in \mathbb{B}_{r_{\bar{y}} / 3}(\bar{y})$ satisfying $\operatorname{dist}\left(y, \Gamma^{-1}(x)\right) \leq r_{\bar{y}} / 3$.

The following concept of $(n, \alpha)$-point-based approximation is extracted from Geoffroy and Piétrus [19].

Definition 4. Let $f: \Omega \subseteq \mathscr{X} \longrightarrow \mathscr{Y}$ be a function and $n \in \mathbb{N}^{*}, \alpha>0$. Then a function $A: \Omega \times \Omega \longrightarrow \mathscr{Y}$ is said to be a $(n, \alpha)$-point-based approximation ( $(n, \alpha)$-PBA in brief) on $\Omega$ for $f$ with modulus $\kappa$ if there exists a scalar $\kappa$ such that, for each $u, v \in \Omega$, both of the following assertions hold:
(a) $\|f(v)-A(u, v)\| \leq \kappa / \pi_{n, \alpha}\|u-v\|^{n+\alpha}$, where

$$
\begin{equation*}
\pi_{n, \alpha}=\prod_{i=1}^{n}(\alpha+i) \tag{11}
\end{equation*}
$$

(b) The function $A(u, \cdot)-A(v, \cdot)$ is Lipschitz continuous on $\Omega$ with modulus $\kappa\|u-\nu\|^{\alpha}$.
It is clear that when $n=1$ and $\alpha=1$, Definition 4 agrees with Robinson's definition of point-based approximation introduced in [9].

Recall the following definition of strict differentiability, which has been taken from [11].

Definition 5. A function $f: \mathscr{X} \longrightarrow \mathscr{Y}$ is said to be strictly differentiable at $x^{*}$ with strict derivative $\nabla f\left(x^{*}\right)$ if for every $\varepsilon>0$ there exists $\delta>0$ such that

$$
\begin{align*}
& \left\|f\left(x^{\prime}\right)-f\left(x^{\prime \prime}\right)-\nabla f\left(x^{*}\right)\left(x^{\prime}-x^{\prime \prime}\right)\right\|  \tag{12}\\
& \leq \varepsilon\left\|x^{\prime}-x^{\prime \prime}\right\| \text {, for every } x^{\prime}, x^{\prime \prime} \in \mathbb{B}_{\delta}\left(x^{*}\right) .
\end{align*}
$$

The following result is a version of [11, Lemma 2]. This result establishes the connection between the strict differentiability of $f$ and ( $n, \alpha$ )-PBA of a function $f$.

Lemma 2. Let A be a $(n, \alpha)$-point-based approximation of a function $f$ in $\Omega$ with a constant $\kappa$ and let $x^{*} \in \Omega$. Then the function $A\left(x^{*}, \cdot\right)-f(\cdot)$ is strictly differentiable at the point $x^{*}$ and its strict derivative at $x^{*}$ is zero.

The following lemma is taken from [25, Corollary 2].
Lemma 3. Let $F: X X 2^{\mathscr{Y}}$ be a set-valued mapping with closed graph and let $f, g: X \longrightarrow \mathscr{Y}$ be two continuous functions. Let $\left(x^{*}, y^{*}\right) \in \operatorname{gphF}$, let $f\left(x^{*}\right)=g\left(x^{*}\right)=0$ and let the difference $f-g$ be strictly differentiable at the point $x^{*} \in \mathcal{X}$ with $\nabla(f-g)\left(x^{*}\right)=0$. Let $L$ be a positive constant. Then the following are equivalent:
(i) The map $(f+F)^{-1}$ is Lipschitz-like at $\left(y^{*}, x^{*}\right)$ with modulus $<L$;
(ii) The map $(g+F)^{-1}$ is Lipschitz-like at $\left(y^{*}, x^{*}\right)$ with modulus $<L$.

Remark 3. Combining Lemma 2 and Lemma 3 we conclude that if $A$ is a $(n, \alpha)$-PBA of a function $f$ in an open neighborhood of some $x^{*} \in(f+F)^{-1}\left(y^{*}\right)$, then $(f+F)^{-1}$
is Lipschitz-like at $\left(y^{*}, x^{*}\right)$ if and only if the map $\left(A\left(x^{*}, \cdot\right)+F(\cdot)\right)^{-1}$ possesses the same property.

The following theorem on the convergence of the nonsmooth function using ( $n, \alpha$ )-point-based approximation is due to Geoffroy and Piétrus; see [19, Theorem 3.3]:

Theorem 1. Let $x^{*}$ be a solution of (1). Fix $n \in \mathbb{N}^{*}$ and $\alpha>0$. Suppose that $F$ has closed graph and $f$ admits a $(n, \alpha)$-pointbased approximation with modulus $k$, denoted by $A$, on some open neighborhood $\Omega$ of $x^{*}$; The set-valued map $\left[A\left(x^{*}, \cdot\right)+F(\cdot)\right]^{-1}$ is M-pseudo-Lipschitz around $\left(0, x^{*}\right)$. Then for every $c>M k / \pi_{n, \alpha}$, one can find $\delta>0$ such that for every starting point $x_{0} \in \mathbb{B}_{\delta}\left(x^{*}\right)$, there exists a sequence $\left\{x_{k}\right\}$ generated by (2), which satisfies

$$
\begin{equation*}
\left\|x_{k+1}-x^{*}\right\| \leq c\left\|x_{k}-x^{*}\right\|^{n+\alpha} \tag{13}
\end{equation*}
$$

Dontchev and Hager [25] proved Banach fixed point theorem, which has been employing the standard iterative concept for contracting mapping. To prove the existence of the sequence generated by Algorithm 1, the following lemma will be played an important rule in this study.

Lemma 4. Let $\Phi: \mathscr{X} \rightrightarrows 2^{\mathscr{X}}$ be a set-valued mapping. Let $x^{*} \in \mathscr{X}, 0<\lambda<1$ and $r>0$ be such that

$$
\begin{equation*}
\text { (a) } \operatorname{dist}\left(x^{*}, \Phi\left(x^{*}\right)\right)<r(1-\lambda) \tag{14}
\end{equation*}
$$

$$
\begin{align*}
& \text { (b) } e\left(\Phi\left(x_{1}\right) \cap \mathbb{B}\left(x^{*}, r\right), \Phi\left(x_{2}\right)\right)  \tag{15}\\
& \quad \leq \lambda\left\|x_{1}-x_{2}\right\| \text { for all } x_{1}, x_{2} \in \mathbb{B}\left(x^{*}, r\right) .
\end{align*}
$$

Then $\Phi$ has a fixed point in $\mathbb{B}\left(x^{*}, r\right)$, that is, there exists $x \in \mathbb{B}\left(x^{*}, r\right)$ such that $x \in \Phi(x)$. Furthermore, if $\Phi$ is singlevalued, then there exists a fixed point $x \in \mathbb{B}\left(x^{*}, r\right)$ such that $x=\Phi(x)$.

## 3. Convergence Analysis

Throughout the whole study we assume that $\mathscr{X}$ and $\mathscr{Y}$ are real or complex Banach spaces. Let $n \in \mathbb{N}^{*}, \alpha>0$ and $F: X \rightrightarrows 2^{\mathscr{Y}}$ be a set-valued mapping with closed graph. Suppose that $f: \Omega \subseteq \mathscr{X} \longrightarrow \mathscr{Y}$ is a nonsmooth function that admits ( $n, \alpha$ )-point-based approximation $A$ on $\Omega$ with a constant $L>0$, where $\Omega$ is an open neighborhood of a point $\bar{x} \in \mathscr{X}$. Let $x \in \mathscr{X}$ and define the mapping $R_{x}$ by

$$
\begin{equation*}
R_{x}(\cdot):=A(x, \cdot)+F(\cdot) . \tag{16}
\end{equation*}
$$

Then
$\mathscr{M}(x)=\left\{d \in \mathscr{X}: 0 \in R_{x}(x+d)\right\}=\left\{d \in \mathscr{X}: x+d \in R_{x}^{-1}(0)\right\}$.
Furthermore, the following equivalence is clear:
$z \in R_{x}^{-1}(y) \Leftrightarrow y \in A(x, z)+F(z)$ for any $z \in \mathscr{X}$ and $y \in \mathscr{Y}$.

In particular,

$$
\begin{equation*}
\bar{x} \in R_{\bar{x}}^{-1}(\bar{y}) \text { for each }(\bar{x}, \bar{y}) \in \operatorname{gph}(f+F) \tag{19}
\end{equation*}
$$

Let $(\bar{x}, \bar{y}) \in \operatorname{gph}(f+F)$ and let $r_{\bar{x}}>0, r_{\bar{y}}>0$. Furthermore, throughout in this section we assume that $\mathbb{B}_{r_{\bar{x}}}(\bar{x}) \subseteq \Omega \cap$ dom $F$. Suppose that $\pi_{n, \alpha}$ is defined in Definition 4.

Define
$\bar{r}:=\min \left\{r_{\bar{y}}-\frac{L r_{\bar{x}}^{n+\alpha}\left(3^{n+\alpha}+2^{n+\alpha}\right)}{\pi_{n, 2} 2^{n+\alpha}}, \frac{r_{\bar{x}}\left(2^{\alpha}-M L r_{\bar{x}}^{\alpha}\right)}{4.2^{\alpha} M}\right\}$.
Then

$$
\begin{equation*}
\bar{r}>0 \Leftrightarrow L<\min \left\{\frac{r_{\bar{y}} \pi_{n, 2} 2^{n+\alpha}}{r_{\bar{x}}^{n+\alpha}\left(3^{n+\alpha}+2^{n+\alpha}\right)}, \frac{2^{\alpha}}{M L r_{\bar{x}}^{\alpha}}\right\} \tag{21}
\end{equation*}
$$

Let us recall that (1) is an abstract model for various problems. From now on, we make the following assumptions.
(i) $F$ has closed graph;
(ii) $f$ admits an $(n, \alpha)$-point-based approximation with modulus $L$, denoted by $A$, on some open neighborhood $\Omega$ of $\bar{x}$
(iii) The set valued map $(f+F)^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $M$.
The following lemma plays an important role to the convergence analysis of the extended Newton-type method which is defined by Algorithm 1. The proof is a refinement of the one for [11, Lemma 1].

Lemma 5. Suppose the assumptions (i)-(iii) hold and let $\bar{r}$ be defined in (10), so that (11) is satisfied. Let $x \in \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$. Then $R_{x}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ with constant $2^{\alpha} M / 2^{\alpha}-M L r_{\bar{x}}^{\alpha}$, that is,

$$
\begin{equation*}
e\left(R_{x}^{-1}\left(y_{1}\right) \cap \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x}), R_{x}^{-1}\left(y_{2}\right)\right) \leq \frac{2^{\alpha} M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}}\left\|y_{1}-y_{2}\right\| \text { for any } y_{1}, y_{2} \in \mathbb{B}_{\bar{r}}(\bar{y}) \tag{22}
\end{equation*}
$$

Proof. Since $f$ has a ( $n, \alpha$ )-point-based approximation $A$ on an open neighbourhood of $\bar{x} \in(f+F)^{-1}(\bar{y})$ with a constant $L$ and the map $(f+F)^{-1}$ is Lipschitz-like around $(\bar{y}, \bar{x})$ with a constant $M$, then by Remark 3 we have that $R_{x}^{-1}(\cdot)$ is Lipschitz-like around $(\bar{y}, \bar{x})$ with a constant $M<L$, that is, there exist constants $r_{\bar{x}}>0, r_{\bar{y}}>0$ and $M$ such that

$$
\begin{equation*}
e\left(R_{\bar{x}}^{-1}\left(y_{1}\right) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), R_{\bar{x}}^{-1}\left(y_{2}\right)\right) \leq M\left\|y_{1}-y_{2}\right\| \text { for all } y_{1}, y_{2} \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}) . \tag{23}
\end{equation*}
$$

Note, by (20 and 21), that $\bar{r}>0$. Now let

$$
\begin{equation*}
y_{1}, y_{2} \in \mathbb{B}_{\bar{r}}(\bar{y}) \text { and } x^{\prime} \in R_{x}^{-1}\left(y_{1}\right) \cap \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x}) \tag{24}
\end{equation*}
$$

It is sufficent to show that there exist $x^{\prime \prime} \in R_{x}^{-1}\left(y_{2}\right)$ such that

$$
\begin{equation*}
\left\|x^{\prime}-x^{\prime \prime}\right\| \leq \frac{2^{\alpha} M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}}\left\|y_{1}-y_{2}\right\| . \tag{25}
\end{equation*}
$$

To this end, we shall verify that there exists a sequence $\left\{x_{k}\right\} \subset \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ such that

$$
\begin{equation*}
y_{2} \in A\left(x, x_{k-1}\right)-A\left(\bar{x}, x_{k-1}\right)+A\left(\bar{x}, x_{k}\right)+F\left(x_{k}\right) \tag{26}
\end{equation*}
$$

$$
\begin{equation*}
\left\|x_{k}-x_{k-1}\right\| \leq M\left\|y_{1}-y_{2}\right\|\left(\frac{M L r_{\bar{\alpha}}^{\alpha}}{2^{\alpha}}\right)^{k-2} \tag{27}
\end{equation*}
$$

hold for each $k=2,3,4, \ldots$ We proceed by mathematical induction. Denote

$$
\begin{equation*}
z_{i}:=y_{i}-A\left(x, x^{\prime}\right)+A\left(\bar{x}, x^{\prime}\right) \text { for each } i=1,2 . \tag{28}
\end{equation*}
$$

Note by (24) that

$$
\begin{equation*}
\left\|x-x^{\prime}\right\| \leq\|x-\bar{x}\|+\left\|\bar{x}-x^{\prime}\right\| \leq \frac{r_{\bar{x}}}{2}+\frac{r_{\bar{x}}}{2} \leq r_{\bar{x}} \tag{29}
\end{equation*}
$$

It follows, from (13) and the relation $\bar{r} \leq r_{\bar{y}}-\operatorname{Lr}_{\bar{x}}^{n+\alpha}\left(3^{n+\alpha}+2^{n+\alpha}\right) / \pi_{n, \alpha} 2^{n+\alpha}$ by (20) that

$$
\begin{align*}
\left\|z_{i}-\bar{y}\right\| & \leq\left\|y_{i}-\bar{y}\right\|+\left\|A\left(x, x^{\prime}\right)-A\left(\bar{x}, x^{\prime}\right)\right\| \\
& \leq \bar{r}+\left\|f\left(x^{\prime}\right)-A\left(x, x^{\prime}\right)\right\|+\left\|f\left(x^{\prime}\right)-A\left(\bar{x}, x^{\prime}\right)\right\| \\
& \leq \bar{r}+\frac{L}{\pi_{n, \alpha}}\left(\left\|x-x^{\prime}\right\|^{n+\alpha}+\left\|\bar{x}-x^{\prime}\right\|^{n+\alpha}\right) \\
& \leq \bar{r}+\frac{L}{\pi_{n, \alpha}}\left(r_{\bar{x}}^{n+\alpha}+\left(\frac{r_{\bar{x}}}{2}\right)^{n+\alpha}\right) \\
& =\bar{r}+\frac{L r_{\bar{x}}^{n+\alpha}\left(2^{n+\alpha}+1\right)}{\pi_{n, \alpha}} \leq r_{\bar{y}} \tag{30}
\end{align*}
$$

This implies that $z_{i} \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ for each $i=1$, 2. Letting $x_{1}:=x^{\prime}$. Then $x_{1} \in R_{x}^{-1}\left(y_{1}\right)$ by (13) and it follows from (18) that

$$
\begin{equation*}
y_{1} \in A\left(x, x_{1}\right)+F\left(x_{1}\right) \tag{31}
\end{equation*}
$$

which can be rewritten as

$$
\begin{equation*}
y_{1}-A\left(x, x_{1}\right)+A\left(\bar{x}, x_{1}\right) \in A\left(\bar{x}, x_{1}\right)+F\left(x_{1}\right) \tag{32}
\end{equation*}
$$

This, by the definition of $z_{1}$, means that $z_{1} \in A\left(\bar{x}, x_{1}\right)+F\left(x_{1}\right)$. Hence $x_{1} \in R_{\bar{x}}^{-1}\left(z_{1}\right)$ by (18). This together with (24) implies that

$$
\begin{equation*}
x_{1} \in R_{\bar{x}}^{-1}\left(z_{1}\right) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}) . \tag{33}
\end{equation*}
$$

According to the concept of Lipschitz-like property of $R_{\bar{x}}^{-1}(\cdot)$ and noting that $z_{1}, z_{2} \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, it follows from (23) that there exists $x_{2} \in R_{\bar{x}}^{-1}\left(z_{2}\right)$ such that

$$
\begin{equation*}
\left\|x_{2}-x_{1}\right\| \leq M\left\|z_{1}-z_{2}\right\|=M\left\|y_{1}-y_{2}\right\| . \tag{34}
\end{equation*}
$$

Moreover, by the definition of $z_{2}$ and noting $x_{1}=x 1$, we have

$$
\begin{equation*}
x_{2} \in R_{\bar{x}}^{-1}\left(z_{2}\right)=R_{\bar{x}}^{-1}\left(y_{2}-A\left(x, x_{1}\right)+A\left(\bar{x}, x_{1}\right)\right) \tag{35}
\end{equation*}
$$

which together with (18) implies that

$$
\begin{equation*}
y_{2} \in A\left(x, x_{1}\right)-A\left(\bar{x}, x_{1}\right)+A\left(\bar{x}, x_{2}\right)+F\left(x_{2}\right) \tag{36}
\end{equation*}
$$

This shows that 26 and 27 are true with constructed points $x_{1}$ and $x_{2}$.

Suppose that the points $x_{1}, x_{2}, \ldots, x_{m}$ have constructed so that 26 and 27 are true for $k=2,3, \ldots, m$. We need to construct $x_{m+1}$ such that (26 and 27) are also true for $k=m+1$. To do this, setting

$$
\begin{equation*}
z_{i}^{m}:=y_{2}-A\left(x, x_{m+i-1}\right)+A\left(\bar{x}, x_{m+i-1}\right) \text { for each } i=0,1 \tag{37}
\end{equation*}
$$

Then, by the inductional assumption together with the concept of ( $n, \alpha$ ) -point-based approximation of A, we obtain that

$$
\begin{align*}
\left\|z_{0}^{m}-z_{1}^{m}\right\| & =\left\|\left[A\left(x, x_{m-1}\right)-A\left(\bar{x}, x_{m-1}\right)\right]-\left[A\left(x, x_{m}\right)-A\left(\bar{x}, x_{m}\right)\right]\right\| \\
& \leq L\|x-\bar{x}\|^{\alpha}\left\|x_{m}-x_{m-1}\right\| \leq \frac{L r_{\bar{x}}^{\alpha}}{2^{\alpha}}\left\|x_{m}-x_{m-1}\right\| \\
& \leq\left\|y_{1}-y_{2}\right\|\left(\frac{M L r_{\bar{x}}^{\alpha}}{2^{\alpha}}\right)^{m-1} . \tag{38}
\end{align*}
$$

We have $\left\|x_{1}-\bar{x}\right\| \leq r_{\bar{x}} / 2$ and $\left\|y_{1}-y_{2}\right\| \leq 2 \bar{r}$ from (24) and using (27) we get

$$
\begin{align*}
\left\|x_{m}-\bar{x}\right\| & \leq \sum_{k=2}^{m}\left\|x_{k}-x_{k-1}\right\|+\left\|x_{1}-\bar{x}\right\| \\
& \leq 2 M \bar{r} \sum_{k=2}^{m}\left(\frac{M L r_{\bar{x}}^{\alpha}}{2^{\alpha}}\right)^{k-2}+\frac{r_{\bar{x}}}{2}  \tag{39}\\
& \leq \frac{2.2^{\alpha} M \bar{r}}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}}+\frac{r_{\bar{x}}}{2} .
\end{align*}
$$

By (20), we have $4.2^{\alpha} M \bar{r} \leq r_{\bar{x}}\left(2^{\alpha}-M L r_{\bar{x}}^{\alpha}\right)$ and then (39) becomes

$$
\begin{equation*}
\left\|x_{m}-\bar{x}\right\| \leq r_{\bar{x}} \tag{40}
\end{equation*}
$$

Consequently,

$$
\begin{equation*}
\left\|x_{m}-x\right\| \leq\left\|x_{m}-\bar{x}\right\|+\|\bar{x}-x\| \leq \frac{3}{2} r_{\bar{x}} \tag{41}
\end{equation*}
$$

Furthermore, using 13 and 20, we get that, for each $i=0,1$

$$
\begin{align*}
\left\|z_{i}^{m}-\bar{y}\right\| \leq & \left\|y_{2}-\bar{y}\right\|+\left\|A\left(x, x_{m+i-1}\right)-A\left(\bar{x}, x_{m+i-1}\right)\right\| \\
\leq & \bar{r}+\left\|f\left(x_{m+i-1}\right)-A\left(x, x_{m+i-1}\right)\right\| \\
& +\left\|f\left(x_{m+i-1}\right)-A\left(\bar{x}, x_{m+i-1}\right)\right\| \\
\leq & \bar{r}+\frac{L}{\pi_{n, \alpha}}\left(\left\|x-x_{m+i-1}\right\|^{n+\alpha}+\left\|\bar{x}-x_{m+i-1}\right\|^{n+\alpha}\right) \\
\leq & \bar{r}+\frac{L}{\pi_{n, \alpha}}\left(\left(\frac{3}{2} r_{\bar{x}}\right)^{n+\alpha}+r_{\bar{x}}^{n+\alpha}\right) \\
= & \bar{r}+\frac{L\left(3^{n+\alpha}+2^{n+\alpha}\right) r_{\bar{x}}^{n+\alpha}}{\pi_{n, \alpha} 2^{n+\alpha}} \leq r_{\bar{y}} . \tag{42}
\end{align*}
$$

It follows that $z_{i}^{m} \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ for each $i=0,1$. Since assumption (14) holds for $k=m$, we have

$$
\begin{equation*}
y_{2} \in A\left(x, x_{m-1}\right)-A\left(\bar{x}, x_{m-1}\right)+A\left(\bar{x}, x_{m}\right)+F\left(x_{m}\right) \tag{43}
\end{equation*}
$$

which can be written as

$$
\begin{equation*}
y_{2}-A\left(x, x_{m-1}\right)+A\left(\bar{x}, x_{m-1}\right) \in A\left(\bar{x}, x_{m}\right)+F\left(x_{m}\right) . \tag{44}
\end{equation*}
$$

Then by definition of $z_{0}^{m}$, it follows that $z_{0}^{m} \in A\left(\bar{x}, x_{m}\right)+F\left(x_{m}\right)$. This, together with 18 and 40 , yields that

$$
\begin{equation*}
x_{m} \in R_{\bar{x}}^{-1}\left(z_{0}^{m}\right) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}) . \tag{45}
\end{equation*}
$$

Using (23) again, inasmuch as $z_{0}^{m}, z_{1}^{m} \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, there exists an element $x_{m+1} \in R_{\bar{x}}^{-1}\left(z_{1}^{m}\right)$ such that

$$
\begin{equation*}
\left\|x_{m+1}-x_{m}\right\| \leq M\left\|z_{0}^{m}-z_{1}^{m}\right\| \leq M\left\|y_{1}-y_{2}\right\|\left(\frac{M L r_{\bar{\alpha}}^{\alpha}}{2^{\alpha}}\right)^{m-1} \tag{46}
\end{equation*}
$$

where the last inequality holds by (38). By the definition of $z_{1}^{m}$, we have

$$
\begin{equation*}
x_{m+1} \in R_{\bar{x}}^{-1}\left(z_{1}^{m}\right)=R_{\bar{x}}^{-1}\left(y_{2}-A\left(x, x_{m}\right)+A\left(\bar{x}, x_{m}\right)\right) \tag{47}
\end{equation*}
$$

which together with (18) implies

$$
\begin{equation*}
y_{2} \in A\left(x, x_{m}\right)-A\left(\bar{x}, x_{m}\right)+A\left(\bar{x}, x_{m+1}\right)+F\left(x_{m+1}\right) . \tag{48}
\end{equation*}
$$

This together with (46) completes the induction step and the existence of sequence $\left\{x_{k}\right\}$ satisfying (14) and (15).

Since $M L r_{\bar{x}}^{\alpha} / 2^{\alpha}<1$, we see from (27) that $\left\{x_{k}\right\}$ is a Cauchy sequence. Define $x^{\prime \prime}:=\lim _{k \rightarrow \infty} x_{k}$. Note that $F$ has closed graph. Then, taking limit in (26), we get $y_{2} \in A\left(x, x^{\prime \prime}\right)+F(x \prime \prime)$ and so $x^{\prime \prime} \in R_{x}^{-1}\left(y_{2}\right)$. Moreover,

$$
\begin{align*}
\left\|x^{\prime}-x^{\prime \prime}\right\| & \leq \lim _{m \rightarrow \infty} \sup \sum_{k=2}^{m}\left\|x_{k}-x_{k-1}\right\| \\
& \leq \lim _{m \rightarrow \infty} \sup \sum_{k=2}^{m}\left(\frac{M L r_{\bar{x}}^{\alpha}}{2^{\alpha}}\right)^{k-2} M\left\|y_{1}-y_{2}\right\|  \tag{49}\\
& \leq \frac{2^{\alpha} M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}}\left\|y_{1}-y_{2}\right\| .
\end{align*}
$$

This completes the proof of the Lemma 5.

Before going to state the main theorem in this study, for our convenience, we define the map $Z_{x}: \mathscr{X} \longrightarrow \mathscr{Y}$, for each $x \in \mathscr{X}$, by

$$
\begin{equation*}
Z_{x}(\cdot):=A(\bar{x}, \cdot)-A(x, \cdot), \tag{50}
\end{equation*}
$$

and the set-valued map $\Phi_{x}: X \rightrightarrows 2^{\mathscr{X}}$ by

$$
\begin{equation*}
\Phi_{x}(\cdot)=R_{\bar{x}}^{-1}\left[Z_{x}(\cdot)\right] . \tag{51}
\end{equation*}
$$

Then we have that

$$
\begin{align*}
\left\|Z_{x}\left(x^{\prime}\right)-Z_{x}\left(x^{\prime \prime}\right)\right\|= & \|\left[A\left(\bar{x}, x^{\prime}\right)-A\left(x, x^{\prime}\right)\right] \\
& -\left[A\left(\bar{x}, x^{\prime \prime}\right)-A\left(x, x^{\prime \prime}\right)\right] \| \\
\leq & L\|\bar{x}-x\|^{\alpha}\left\|x^{\prime}-x^{\prime \prime}\right\| \text { for any } x^{\prime}, x^{\prime \prime} \in \mathscr{X} . \tag{52}
\end{align*}
$$

The main result of this study read as follows, which provides some sufficient conditions ensuring the convergence of the extended Newton-type method for nonsmooth generalized (1) from starting point $x_{0}$.

Theorem 2. Suppose that $\eta>1$. Let $\bar{x} \in \mathscr{X}, \Omega$ be an open and convex subset of $\mathfrak{X}$ containing $\bar{x}$ and let $f$ be a function which has ( $n, \alpha$ )-point-based approximation $A$ on $\Omega$ with a constant $L>0$. Suppose that the map $F$ has closed graph and the map $R_{\bar{x}}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $M>0$. Let $\bar{r}$ be defined by (10) so that (11)holds. Let $\delta>0$ be such that
(a) $\delta \leq \min \left\{r_{\bar{\alpha}} / 4, \bar{r} . \pi_{n, \alpha} / 4^{n+\alpha}\right.$, $\left.\left(r_{\bar{y}} \pi_{n, \alpha} / L\left(3^{n+\alpha}+2^{n+\alpha}+1\right)\right)^{1 / n+\alpha}, 1\right\}$,
(b) $(M+1) L\left(2^{\alpha+1} \eta \delta^{\alpha}+r_{\bar{\alpha}}^{\alpha}\right) \leq 2^{\alpha}$,
(c) $\|\bar{y}\|<L / \pi_{n, \alpha} \delta^{n+\alpha}$.

Suppose that

$$
\begin{equation*}
\lim _{x \rightarrow \bar{x}} \operatorname{dist}(\bar{y}, A(x, x)+F(x))=0 . \tag{53}
\end{equation*}
$$

Then there exists some $\widehat{\delta}>0$ such that any sequence $\left\{x_{m}\right\}$ generated by Algorithm 1 with starting point $x_{0} \in \mathbb{B}_{\hat{\delta}}(\bar{x})$ converges to a solution $x^{*}$ of nonsmooth generalized (1), that is, $x^{*}$ satisfies $0 \in f\left(x^{*}\right)+F\left(x^{*}\right)$.

## Proof.

By assumption (b), it can be easily written that

$$
\begin{equation*}
M L\left(2^{\alpha+1} \eta \delta^{\alpha}+r_{\bar{x}}^{\alpha}\right) \leq(M+1) L\left(2^{\alpha+1} \eta \delta^{\alpha}+r_{\bar{x}}^{\alpha}\right) \leq 2^{\alpha} . \tag{54}
\end{equation*}
$$

Set

$$
\begin{equation*}
t:=\frac{2^{\alpha} \eta M L \delta^{\alpha}}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}} \tag{55}
\end{equation*}
$$

It follows from (54) that

$$
\begin{equation*}
t \leq \frac{1}{2} . \tag{56}
\end{equation*}
$$

Since $\pi_{n, \alpha}\|\bar{y}\|<L \delta^{n+\alpha}$ by assumption (c) and (26) holds, there exists $0<\delta \leq \delta$ be such that
$\operatorname{dist}\left(0, A\left(x_{0}, x_{0}\right)+F\left(x_{0}\right)\right) \leq \frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha}$ for each $x_{0} \in \mathbb{B}_{\widehat{\delta}}(\bar{x})$.

Let $x_{0} \in \mathbb{B}_{\widehat{\delta}}(\bar{x})$. We will proceed by mathematical induction. We will show that Algorithm 1 generates at least one sequence and any sequence $\left\{x_{m}\right\}$ generated by Algorithm 1 for (1) satisfies the following assertions:

$$
\begin{gather*}
\left\|x_{m}-\bar{x}\right\| \leq 2 \delta,  \tag{58}\\
\left\|x_{m+1}-x_{m}\right\| \leq t\left(\frac{1}{\pi_{n, \alpha}}\right)^{(n+\alpha)^{m}} \delta, \tag{59}
\end{gather*}
$$

for each $m=0,1,2, \ldots$. For this purpose we define

$$
\begin{equation*}
r_{x}:=\frac{3}{2}\left(\frac{M L}{\pi_{n, \alpha}}\|x-\bar{x}\|^{n+\alpha}+M\|\bar{y}\|\right) \text { for each } x \in \mathscr{X} \tag{60}
\end{equation*}
$$

Owing to the fact $4 \delta \leq r_{\bar{x}}$ in assumption (a) and $\eta>1$, by assumption (b) we can write as follows

$$
\begin{align*}
(M+1) L 2^{\alpha} .3 \delta^{\alpha} & \leq(M+1) L \cdot 2^{\alpha}\left(2 \delta^{\alpha}+\delta^{\alpha}\right) \\
& =(M+1) L\left(2^{\alpha+1} \delta^{\alpha}+(2 \delta)^{\alpha}\right) \\
& \leq(M+1) L\left(2^{\alpha+1} \eta \delta^{\alpha}+(4 \delta)^{\alpha}\right)  \tag{61}\\
& \leq(M+1) L\left(2^{\alpha+1} \eta \delta^{\alpha}+r_{\bar{x}}^{\alpha}\right) \leq 2^{\alpha} .
\end{align*}
$$

The above inequality gives either

$$
\begin{equation*}
M L \delta^{\alpha} \leq \frac{2^{\alpha}}{2^{\alpha} \cdot 3}=\frac{1}{3} \text { or } L \delta^{\alpha} \leq \frac{2^{\alpha}}{2^{\alpha} \cdot 3}=\frac{1}{3} . \tag{62}
\end{equation*}
$$

By the facts $\pi_{n, \alpha}\|\bar{y}\|<L \delta^{n+\alpha}$ from condition (c)and (34), the inequality (33) reduces to, for each $x \in \mathbb{B}_{2 \delta}(\bar{x})$

$$
\begin{align*}
r_{x} & =\frac{3}{2}\left(\frac{M L}{\pi_{n, \alpha}}\|x-\bar{x}\|^{n+\alpha}+M\|\bar{y}\|\right) \leq \frac{3}{2}\left(\frac{M L}{\pi_{n, \alpha}}\|x-\bar{x}\|^{n+\alpha}+\frac{M L}{\pi_{n, \alpha}} \delta^{n+\alpha}\right) \\
& \leq \frac{3}{2}\left(\frac{M L}{\pi_{n, \alpha}}(2 \delta)^{n+\alpha}+\frac{M L}{\pi_{n, \alpha}} \delta^{n+\alpha}\right)=\frac{3}{2} \frac{M L}{\pi_{n, \alpha}} \delta^{\alpha}\left(2^{n+\alpha}+1\right) \cdot \delta^{n} \tag{63}
\end{align*}
$$

$$
\text { Since } \delta^{n} \leq \delta \text {, we get that, } \leq \frac{3}{2} \frac{M L}{\pi_{n, \alpha}} \delta^{\alpha}\left(2^{n+\alpha}+1\right) \cdot \delta
$$

$$
\leq \frac{3}{2} \cdot \frac{1}{3 \pi_{n, \alpha}}\left(2^{n+\alpha}+1\right) \cdot \delta \leq \frac{1}{2 \pi_{n, \alpha}}\left(2^{n+\alpha}+1\right) \cdot \delta \leq 2 \delta \text { for each } x \in \mathbb{B}_{2 \delta}(\bar{x}) .
$$

It is trivial that (58) is true for $m=0$. To show, (32) holds for $m=0$, firstly we need to verify that $x_{1}$ exists, that is, we need to show that $\mathscr{M}\left(x_{0}\right) \neq \varnothing$. To do this, we consider the mapping $\Phi_{x_{0}}$ defined by (24) and apply Lemma 4 to the map $\Phi_{x_{0}}$ with $\eta_{0}=\bar{x}$. Let us check that both assumptions (5) and (6) of Lemma 4, with $r:=r_{x_{0}}$ and $\lambda:=1 / 3$ hold. Noting that $\bar{x} \in R_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{x_{g}}}(\bar{x})$ by (3) and according to the definition of the excess $e$ and the map $\Phi_{x_{0}}$, we obtain

$$
\begin{align*}
\operatorname{dist}\left(\bar{x}, \Phi_{x_{0}}(\bar{x})\right) & \leq e\left(R_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{x_{0}}}(\bar{x}), \Phi_{x_{0}}(\bar{x})\right) \\
& \leq e\left(R_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{2 \delta}(\bar{x}), R_{\bar{x}}^{-1}\left[Z_{x_{0}}(\bar{x})\right]\right)  \tag{64}\\
& \leq e\left(R_{\bar{x}}^{-1}(\bar{y}) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), R_{\bar{x}}^{-1}\left[Z_{x_{0}}(\bar{x})\right]\right) \\
& \leq M\left\|\bar{y}-Z_{x_{0}}(\bar{x})\right\| .
\end{align*}
$$

From the notion of ( $n, \alpha$ )-point-based approximation $A$ of $f$ with $L$, we obtain that

$$
\begin{align*}
\left\|Z_{x_{0}}(x)-\bar{y}\right\| & =\left\|A(\bar{x}, x)-A\left(x_{0}, x\right)-\bar{y}\right\| \\
& \leq\left\|A(\bar{x}, x)-A\left(x_{0}, x\right)\right\|+\|\bar{y}\| \\
& \leq\|f(x)-A(\bar{x}, x)\|+\left\|f(x)-A\left(x_{0}, x\right)\right\|+\|\bar{y}\| \\
& \leq \frac{L}{\pi_{n, \alpha}}\|\bar{x}-x\|^{n+\alpha}+\frac{L}{\pi_{n, \alpha}}\left\|x_{0}-x\right\|^{n+\alpha}+\|\bar{y}\| \\
& \leq \frac{L}{\pi_{n, \alpha}}\left(\|\bar{x}-x\|^{n+\alpha}+\left\|x_{0}-x\right\|^{n+\alpha}\right)+\|\bar{y}\| . \tag{65}
\end{align*}
$$

Note that $L \delta^{n+\alpha}\left(2^{n+\alpha}+3^{n+\alpha}+1\right) \leq \pi_{n, \alpha} r_{\bar{y}}$ because of assumption (a), $\pi_{n, \alpha}\|\bar{y}\|<L \delta^{n+\alpha}$ by assumption (c) and $\left\|x_{0}-\bar{x}\right\| \leq \hat{\delta} \leq \delta$. It follows from (65), for each $x \in \mathbb{B}_{r_{x_{0}}}(\bar{x}) \subseteq \mathbb{B}_{2 \delta}(\bar{x})$, that

$$
\begin{align*}
\left\|Z_{x_{0}}(x)-\bar{y}\right\| & \leq \frac{L}{\pi_{n, \alpha}}\left(\|\bar{x}-x\|^{n+\alpha}+\left(\left\|x_{0}-\bar{x}\right\|+\|\bar{x}-x\|\right)^{n+\alpha}\right)+\|\bar{y}\| \\
& \leq \frac{L}{\pi_{n, \alpha}}\left((2 \delta)^{n+\alpha}+(\delta+2 \delta)^{n+\alpha}\right)+\|\bar{y}\|=\frac{L}{\pi_{n, \alpha}}\left((2 \delta)^{n+\alpha}+(3 \delta)^{n+\alpha}\right)+\|\bar{y}\| \\
& \leq \frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha}\left(2^{n+\alpha}+3^{n+\alpha}\right)+\frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha}=\frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha}\left(2^{n+\alpha}+3^{n+\alpha}+1\right) \leq r_{\bar{y}} . \tag{66}
\end{align*}
$$

This implies that

$$
\begin{equation*}
Z_{x_{0}}(x) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}) \text { for each } x \in \mathbb{B}_{r_{x_{0}}}(\bar{x}) \tag{67}
\end{equation*}
$$

In particular, letting $x=\bar{x}$ in (65). Then we have that

$$
\begin{align*}
\left\|Z_{x_{0}}(\bar{x})-\bar{y}\right\| & \leq \frac{L}{\pi_{n, \alpha}}\left\|x_{0}-x\right\|^{n+\alpha}+\|\bar{y}\|  \tag{68}\\
& \leq \frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha}+\frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha} \leq \frac{2 L}{\pi_{n, \alpha}} \delta^{n+\alpha} \leq r_{\bar{y}} \tag{69}
\end{align*}
$$

and hence

$$
\begin{equation*}
Z_{x_{0}}(\bar{x}) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y}) . \tag{70}
\end{equation*}
$$

Hence, by the assumed Lipschitz-like property of $R_{\bar{x}}^{-1}$ and (68), we have from (64) that

$$
\begin{array}{r}
\operatorname{dist}\left(\bar{x}, \Phi_{x_{0}}(\bar{x})\right) \leq M\left\|\bar{y}-Z_{x_{0}}(\bar{x})\right\| \\
\leq \frac{M L}{\pi_{n, \alpha}}\left\|x_{0}-x\right\|^{n+\alpha}+M\|\bar{y}\|=\left(1-\frac{1}{3}\right) r_{x_{0}}=(1-\lambda) r \tag{71}
\end{array}
$$

that is, the assumption (5) of Lemma 4 is satisfied.
Below, we will show that the assumption (6) of Lemma 4 holds. To do this, let $x \prime, x \prime \prime \in \mathbb{B}_{r_{x_{0}}}(\bar{x})$. Then from assumption (a) and (35), we have that $x \prime, x_{x} \prime \prime \in \mathbb{B}_{r_{x_{0}}}(\bar{x}) \subseteq \mathbb{B}_{2 \delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}}}(\bar{x})$ and $Z_{x_{0}}(x \prime), Z_{x_{0}}(x \prime \prime) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$ by (39). This, together with the assumed Lipschitz-like property of $R_{\bar{x}}^{-1}$, implies that

$$
\begin{aligned}
& e\left(\Phi_{x_{0}}\left(x^{\prime}\right) \cap \mathbb{B}_{r_{x_{0}}}(\bar{x}), \Phi_{x_{0}}\left(x^{\prime \prime}\right)\right) \leq e\left(\Phi_{x_{0}}(x \prime) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \Phi_{x_{0}}\left(x^{\prime \prime}\right)\right) \\
& =e\left(R_{\bar{x}}^{-1}\left[Z_{x_{0}}\left(x^{\prime}\right)\right] \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), R_{\bar{x}}^{-1}\left[Z_{x_{0}}\left(x^{\prime \prime}\right)\right]\right) \leq M\left\|Z_{x_{0}}\left(x^{\prime}\right)-Z_{x_{0}}\left(x^{\prime \prime}\right)\right\| \\
& \operatorname{dist}\left(0, \mathscr{M}\left(x_{0}\right)\right)=\operatorname{dist}\left(x_{0}, R_{x_{0}}^{-1}(0)\right) .
\end{aligned}
$$

Applying (52), we get that

$$
\begin{equation*}
\left\|Z_{x_{0}}(x \prime)-Z_{x_{0}}(x \prime \prime)\right\| \leq L\left\|\bar{x}-x_{0}\right\|^{\alpha}\|x \prime-x \prime \prime\| . \tag{73}
\end{equation*}
$$

With the help of first relation in (62) and combining the above two inequalities we get,

$$
\begin{align*}
e\left(\Phi_{x_{0}}\left(x^{\prime}\right) \cap \mathbb{B}_{r_{x_{0}}}(\bar{x}), \Phi_{x_{0}}\left(x^{\prime \prime}\right)\right) & \leq M L\left\|\bar{x}-x_{0}\right\|^{\alpha}\left\|x^{\prime}-x^{\prime \prime}\right\| \\
& \leq M L \delta^{\alpha}\left\|x^{\prime}-x^{\prime \prime}\right\| \\
& \leq \frac{1}{3}\left\|x^{\prime}-x^{\prime \prime}\right\|=\lambda\left\|x^{\prime}-x^{\prime \prime}\right\| \tag{74}
\end{align*}
$$

This means that the assumption (6) of Lemma 4 is also satisfied. Since both assumptions (5) and (6) of Lemma 4 are satisfied, we can say that Lemma 4 is applicable and therefore, we conclude that there exists $\hat{x}_{1} \in \mathbb{B}_{r_{r}}(\bar{x})$ such that $\hat{x}_{1} \in \Phi_{x_{0}}\left(\hat{x}_{1}\right)$, that is, $0 \in A\left(x_{0}, \widehat{x}_{1}\right)+F\left(\hat{x}_{1}\right)$ and so $\hat{x}_{1}-x_{0} \in \mathscr{M}\left(x_{0}\right)$. This fact reflects that $\mathscr{M}\left(x_{0}\right) \neq \varnothing$.

Since $\eta>1$ and $\mathscr{M}\left(x_{0}\right) \neq \varnothing$, we can choose $d_{0} \in \mathscr{M}\left(x_{0}\right)$ such that

$$
\begin{equation*}
\left\|d_{0}\right\| \leq \eta \operatorname{dist}\left(0, \mathscr{M}\left(x_{0}\right)\right) \tag{75}
\end{equation*}
$$

By Algorithm 1, $x_{1}:=x_{0}+d_{0}$ is defined. Hence $x_{1}$ is generated for (1).

Furthermore, by the definition of $\mathscr{M}\left(x_{0}\right)$, we can write

$$
\begin{align*}
\mathscr{M}\left(x_{0}\right) & :=\left\{d_{0} \in \Omega: 0 \in A\left(x_{0}, x_{0}+d_{0}\right)+F\left(x_{0}+d_{0}\right)\right\} \\
& =\left\{d_{0} \in \Omega: x_{0}+d_{0} \in R_{x_{0}}^{-1}(0)\right\}, \tag{76}
\end{align*}
$$

Now we are ready to show that (59) is hold for $m=0$. Note that $\bar{r}>0$ by assumption (a). Then (21) is satisfied by (20). Lemma 5 states us that the mapping $R_{x}^{-1}(\cdot)$ is Lipschitzlike on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ with constant $2^{\alpha} M / 2^{\alpha}-$ $M L r_{\bar{x}}^{\alpha}$ for each $x \in \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ when $R_{\bar{x}}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$. Particularly, $R_{x_{0}}^{-1}(\cdot)$ is Lipschitzlike on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ with constant $2^{\alpha} M / 2^{\alpha}-$ $M L r_{\bar{x}}^{\alpha}$ as $x_{0} \in \underset{\widehat{\delta}}{\mathbb{B}_{-}}(\bar{x}) \subseteq \mathbb{B}_{\delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ by assumption (a) and the choice of $\widehat{\delta}$.

Furthermore, assumptions (a), (c) and the 2nd relation of the inequality (62) imply that

$$
\begin{align*}
\|\bar{y}\| & \leq \frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha}=\frac{L}{\pi_{n, \alpha}} \delta^{\alpha} \cdot \delta^{n} \leq \frac{L}{\pi_{n, \alpha}} \delta^{\alpha} \cdot \delta \leq \frac{1}{3 \pi_{n, \alpha}} \cdot \delta \\
& \leq \frac{1}{3 \pi_{n, \alpha}} \cdot \frac{\bar{r} \pi_{n, \alpha}}{4^{n+\alpha}} \leq \frac{\bar{r}}{3} . \tag{78}
\end{align*}
$$

Now (57) becomes

$$
\begin{align*}
\operatorname{dist}\left(0, R_{x_{0}}\left(x_{0}\right)\right) & =\operatorname{dist}\left(0, A\left(x_{0}, x_{0}\right)+F\left(x_{0}\right)\right) \\
& \leq \frac{L}{\pi_{n, \alpha}} \delta^{n+\alpha} \leq \frac{\bar{r}}{3} . \tag{79}
\end{align*}
$$

Noting that $x_{0} \in \mathbb{B}_{r_{-} / 2}(\bar{x})$ as mentioned earlier and by (78)) we have that $0 \in \mathbb{B}_{\bar{r} / 3}(\bar{y})$.

Thus, by applying Lemma 1, we obtain that

$$
\begin{equation*}
\operatorname{dist}\left(x_{0}, R_{x_{0}}^{-1}(0)\right) \leq \frac{2^{\alpha} M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}} \operatorname{dist}\left(0, R_{x_{0}}\left(x_{0}\right)\right) \tag{80}
\end{equation*}
$$

According to Algorithm 1 and using (77 and 80) we have

$$
\begin{align*}
\left\|x_{1}-x_{0}\right\| & =\left\|d_{0}\right\| \leq \eta \operatorname{dist}\left(0, \mathscr{M}\left(x_{0}\right)\right)=\eta \operatorname{dist}\left(x_{0}, R_{x_{0}}^{-1}(0)\right) \leq \frac{2^{\alpha} \eta M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}} \operatorname{dist}\left(0, R_{x_{0}}\left(x_{0}\right)\right) \\
& \leq \frac{2^{\alpha} \eta M L}{\pi_{n, \alpha}\left(2^{\alpha}-M L r_{\bar{x}}^{\alpha}\right)} \delta^{n+\alpha} \leq \frac{2^{\alpha} \eta M L}{\pi_{n, \alpha}\left(2^{\alpha}-M L r_{\bar{x}}^{\alpha}\right)} \delta^{n} \cdot \delta^{\alpha} \leq \frac{2^{\alpha} \eta M L \delta^{\alpha}}{\pi_{n, \alpha}\left(2^{\alpha}-M L r_{\bar{x}}^{\alpha}\right)} \delta .\left[\text { Since } \delta^{n} \leq \delta\right] . \tag{81}
\end{align*}
$$

From (56 and 81) we get,

$$
\begin{equation*}
\left\|x_{1}-x_{0}\right\|=\left\|d_{0}\right\| \leq \frac{t}{\pi_{n, \alpha}} \delta \leq t\left(\frac{1}{\pi_{n, \alpha}}\right) \delta \tag{82}
\end{equation*}
$$

This shows that (59) is hold for $m=0$.
Suppose that the points $x_{1}, x_{2}, \ldots, x_{k}$ have obtained by Algorithm 1 satisfying (2) such that ( 31 and 32 ) are hold for $m=0,1,2, \ldots, k-1$. We show that assertions (31) and (32) are also hold for $m=k$. Since (31) and (32) are true for each $m \leq k-1$, we have the following inequality
$\left\|x_{k}-\bar{x}\right\| \leq \sum_{i=0}^{k-1}\left\|d_{i}\right\|+\left\|x_{0}-\bar{x}\right\| \leq t \delta \sum_{i=0}^{k-1}\left(\frac{1}{\pi_{n, \alpha}}\right)^{(n+\alpha)^{i}}+\delta \leq 2 \delta$,

$$
\begin{align*}
& \operatorname{dist}\left(0, R_{x_{k}}\left(x_{k}\right)\right)=\operatorname{dist}\left(0, A\left(x_{k}, x_{k}\right)+F\left(x_{k}\right)\right) \leq\left\|A\left(x_{k}, x_{k}\right)-A\left(x_{k-1}, x_{k}\right)\right\| \\
&=\frac{L}{\pi_{n, \alpha}}\left\|f\left(x_{k}\right)-A\left(x_{k-1}, x_{k}\right)\right\|^{n+\alpha} \leq \frac{L}{\pi_{n, \alpha}}\left\|x_{k}-x_{k-1}\right\|^{n+\alpha} \leq \frac{L}{\pi_{n, \alpha}}\left(\left\|x_{k}-\bar{x}\right\|+\left\|\bar{x}-x_{k-1}\right\|\right)^{n+\alpha}  \tag{84}\\
& \leq \frac{L}{\pi_{n, \alpha}}(2 \delta+2 \delta)^{n+\alpha}=\frac{L}{\pi_{n, \alpha}} 4^{n+\alpha} \delta^{n+\alpha} \leq \frac{L}{\pi_{n, \alpha}} \delta^{\alpha} 4^{n+\alpha} \delta=\frac{1}{3 \pi_{n, \alpha}} 4^{n+\alpha} \bar{r} \cdot \pi_{n, \alpha} \\
& 4^{n+\alpha} \bar{r} \\
& 3
\end{align*}
$$

It is noted earlier that $x_{k} \in \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$. Moreover, (78) implies that $0 \in \mathbb{B}_{\bar{r} / 3}(\bar{y})$. This, together with (84), implies that Lemma 1 is applicable for the map $R_{x_{k}}^{-1}(\cdot)$ and hence we have that
and so $x_{k} \in \mathbb{B}_{2 \delta}(\bar{x})$. This shows that (58) holds for $m=k$.
Next we show that the assertion (59) is also hold for $m=k$. Let $x_{k} \in \mathbb{B}_{r_{x_{k}}}(\bar{x})$. If we apply Lemma 4 to the map $\Phi_{x_{k}}$ with $\eta=\bar{x}, r:=r_{x_{k}}$ and $\lambda:=1 / 3$, then by the analogue argument as we did for the case $k=0$ one can find that $\mathscr{M}\left(x_{k}\right) \neq \varnothing$. Because of $x_{k} \in \mathbb{B}_{r_{x_{k}}}(\bar{x}) \subseteq \mathbb{B}_{2 \delta}(\bar{x}) \subseteq \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$, Lemma 5 permit us to say that $R_{x_{k}}^{-1^{x}}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ with constant $2^{\alpha} M / 2^{\alpha}-M L r_{\bar{x}}^{\alpha}$.

Moreover, inasmuch as $-A\left(x_{k-1}, x_{k}\right) \in F\left(x_{k}\right)$, using the idea of ( $n, \alpha$ )-point-based approximation of $f$, the inequality $4^{n+\alpha} \delta \leq \bar{r} \pi_{n, \alpha}$ from assumption (a), we obtain that

$$
\begin{equation*}
\operatorname{dist}\left(x_{k}, R_{x_{k}}^{-1}(0)\right) \leq \frac{2^{\alpha} M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}} \operatorname{dist}\left(0, R_{x_{k}}\left(x_{k}\right)\right) \tag{85}
\end{equation*}
$$

Since $\mathscr{M}\left(x_{k}\right) \neq \varnothing$, Algorithm 1 ensure us the existence of a point $x_{k+1}$ which satisfy the following inequality

$$
\begin{align*}
\left\|x_{k+1}-x_{k}\right\| & =\left\|d_{k}\right\| \leq \eta \operatorname{dist}\left(0, \mathscr{M}\left(x_{k}\right)\right)=\eta \operatorname{dist}\left(x_{k}, R_{x_{k}}^{-1}(0)\right) \leq \frac{2^{\alpha} \eta M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}} \operatorname{dist}\left(0, R_{x_{k}}\left(x_{k}\right)\right) \\
& =\frac{2^{\alpha} \eta M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}} \operatorname{dist}\left(0, A\left(x_{k}, x_{k}\right)+F\left(x_{k}\right)\right) \leq \frac{2^{\alpha} \eta M}{2^{\alpha}-M L r_{\bar{x}}}\left\|A\left(x_{k}, x_{k}\right)-A\left(x_{k-1}, x_{k}\right)\right\| \\
& =\frac{2^{\alpha} \eta M}{2^{\alpha}-M L r_{\bar{x}}^{\alpha}}\left\|f\left(x_{k}\right)-A\left(x_{k-1}, x_{k}\right)\right\| \leq \frac{2^{\alpha} \eta L M}{\pi_{n, \alpha}\left(2^{\alpha}-M L r_{\bar{x}}^{\alpha}\right)}\left\|x_{k}-x_{k-1}\right\|^{n+\alpha} \leq \frac{t}{\delta^{\alpha} \pi_{n, \alpha}}\left(t\left(\frac{1}{\pi_{n, \alpha}}\right)^{(n+\alpha)^{k-1}} \delta\right)^{n+\alpha}  \tag{86}\\
& \leq \frac{t}{\delta^{\alpha} \pi_{n, \alpha}}\left(t\left(\frac{1}{\pi_{n, \alpha}}\right)^{(n+\alpha)^{k-1}}\right)^{n+\alpha} \delta^{n+\alpha} \leq \frac{t}{\pi_{n, \alpha}}\left(t\left(\frac{1}{\pi_{n, \alpha}}\right)^{(n+\alpha)^{k-1}}\right)^{n+\alpha} \delta^{n} \leq \frac{t}{\pi_{n, \alpha}}\left(t\left(\frac{1}{\pi_{n, \alpha}}\right)^{(n+\alpha)^{k-1}}\right)^{n+\alpha} \delta\left[\text { Since } \delta^{n} \leq \delta\right] \\
& \leq t\left(\frac{1}{\pi_{n, \alpha}}\right)^{(n+\alpha)^{k}} \delta .
\end{align*}
$$

This shows that (59) holds for $m=k$. Thus, we can see from (59) that $\left\{x_{m}\right\}$ is a Cauchy sequence and hence convergent to some $x^{*}$. Since the graph of $F$ is closed, we can pass to the limit in $x_{k+1} \in R_{x_{k}}^{-1}(0)$ obtaining that $x^{*}$ is a solution of (1). Therefore, the proof is completed.

In particular, in the case when $\bar{x}$ is a solution of (1), that is, $\bar{y}=0$, Theorem 2 is reduced to the following corollary, which gives the local convergent result of the extended Newton-type method for solving nonsmooth generalized (1).

Corollary 1. Suppose that $\eta>1$ and $\bar{x}$ be a solution of (1). Let $\Omega$ be an open and convex subset of $\mathscr{X}$ containing $\bar{x}$ and $\widetilde{r}>0$ be such that $\mathbb{B}_{\tilde{r}}(\bar{x})$ is an open and convex set. Suppose that the function $f$ is continuous which has an ( $n, \alpha$ )-point-based approximation $A$ on $\mathbb{B}_{\vec{r}}(\bar{x})$ with a constant $L>0$, the map $F$ has closed graph. Assume that the map $R_{\bar{x}}^{-1}(\cdot)$ is Lipschitz-like around $(0, \bar{x})$ with constant $M$. Suppose that

$$
\begin{equation*}
\lim _{x \longrightarrow \bar{x}} \operatorname{dist}(0, A(x, x)+F(x))=0 . \tag{87}
\end{equation*}
$$

Then there exists some $\widehat{\delta}>0$ such that any sequence $\left\{x_{m}\right\}$ generated by Algorithm 1 starting from $x_{0} \in \mathbb{B}_{\widehat{\gamma}}(\bar{x})$ converges to a solution $x^{*}$ of nonsmooth generalized (1), that is, $x^{*}$ satisfies that $0 \in f\left(x^{*}\right)+F\left(x^{*}\right)$.

Proof. By hypothesis $R_{\bar{x}}^{-1}(\cdot)$ is pseudo-Lipschitz around $(0, \bar{x})$. Then there exists constants $r_{0}, \widehat{r}_{\bar{x}}$ and $M$ such that $R_{\bar{x}}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{r_{0}}(\bar{y})$ relative to $\mathbb{B}_{\bar{r}_{\bar{x}}}(\bar{x})$ with constant $M$. Then, for each $0<r \leq \hat{r}_{\bar{x}}$, one has that
$e\left(R_{\bar{x}}^{-1}\left(y_{1}\right) \cap \mathbb{B}_{r}(\bar{x}), R_{\bar{x}}^{-1}\left(y_{2}\right)\right) \leq M\left\|y_{1}-y_{2}\right\|$ for any $y_{1}, y_{2} \in \mathbb{B}_{r_{0}}(0)$,
that is, the map $R_{\bar{x}}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{r_{0}}(0)$ relative to $\mathbb{B}_{r}(\bar{x})$ with constant $M$.

Let $L \in(0,1)$ and choose $r_{\bar{x}} \in\left(0, \widehat{r}_{\bar{x}}\right)$ such that

$$
\begin{equation*}
\frac{r_{\bar{x}}}{2} \leq \widetilde{r}, 2^{n+\alpha} \pi_{n, \alpha} r_{0}-L\left(3^{n+\alpha}+2^{n+\alpha}\right) r_{\bar{x}}^{n+\alpha}>0 \tag{89}
\end{equation*}
$$

and $A$ is a $(n, \alpha)$-point-based approximation of $f$ on $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$. Then, define

$$
\begin{equation*}
\bar{r}=\min \left\{r_{0}-\frac{L r_{\bar{x}}^{n+\alpha}\left(3^{n+\alpha}+2^{n+\alpha}\right)}{\pi_{n, \alpha} 2^{n+\alpha}}, \frac{r_{\bar{x}}\left(2^{\alpha}-M L r_{\bar{x}}^{\alpha}\right)}{4.2^{\alpha} M}\right\}>0, \tag{90}
\end{equation*}
$$

$$
\begin{equation*}
\min \left\{\frac{r_{\bar{x}}}{4}, \frac{\bar{r} \pi_{n, \alpha}}{4^{n+\alpha}},\left(\frac{r_{0} \pi_{n, \alpha}}{L\left(3^{n+\alpha}+2^{n+\alpha}+1\right)}\right)^{1 / n+\alpha}\right\} . \tag{91}
\end{equation*}
$$

Thus we can choose $0<\delta \leq 1$ such that

$$
\begin{align*}
& \delta \leq \min \left\{\frac{r_{\bar{x}}}{4}, \frac{\bar{r} \cdot \pi_{n, \alpha}}{4^{n+\alpha}},\left(\frac{r_{0} \pi_{n, \alpha}}{L\left(3^{n+\alpha}+2^{n+\alpha}+1\right)}\right)^{1 / n+\alpha}\right\} \text {, }  \tag{92}\\
& (M+1) L\left(2^{\alpha+1} \eta \delta^{\alpha}+r_{\bar{x}}^{\alpha}\right) \leq 2^{\alpha} . \tag{93}
\end{align*}
$$

Now it is routine to check that all the conditions of Theorem 2 are hold. Thus, Theorem 2 is applicable to complete the proof of the corollary 1 .

## 4. Application of $(n, \alpha)$-pointbased approximation

This section is devoted to present applications of ( $n, \alpha$ )-point-based approximation. In particular, when the Fréchet derivative of $f$ is $(\ell, \alpha)$-Hölder, the function $A$ is an $(1, \alpha)$ - point-based approximation for $f$. Moreover, when $f$ is a twice Fréchet differentiable function such that $\nabla^{2} f$ is ( $K, \alpha$ )-Hölder, then the function $A$ is an ( $2, \alpha$ )-point-based approximation for $f$. In addition, application of ( $n, \alpha$ )-point-based approximation is provided for normal maps.
4.1. Application of $(n, \alpha)$-PBA for smooth function $f$. Let $0<\alpha<1$ and $\Omega$ be a convex subset of $\mathscr{X}$. Let $u, v \in \Omega$.
(1) Suppose that the Fréchet derivative of $f$ is $(\ell, \alpha)$-Hölder continuous. We show that the function

$$
\begin{equation*}
A:(u, v) \mapsto f(u)+\nabla f(u)(v-u) \tag{94}
\end{equation*}
$$

is an $(1, \alpha)$-point-based approximation for $f$. In this case, by using the Algorithm 1 we can infer that there exists a sequence $\left\{x_{k}\right\}$ which converges superlinearly and this result recovers the convergence result of Geoffroy and Piétrus in [19].

In this regards, define the function $\Lambda(u, v)$ by

$$
\begin{equation*}
\Lambda(u, v)=\|f(v)-A(u, v)\| . \tag{95}
\end{equation*}
$$

It follows that

$$
\begin{align*}
\Lambda(u, v) & =\|f(v)-f(u)-\nabla f(u)(v-u)\|=\left\|\int_{0}^{1}(\nabla f(u+t(v-u))-\nabla f(u))(v-u) d t\right\| \\
& \leq\|v-u\| \int_{0}^{1}\|\nabla f(u+t(v-u))-\nabla f(u)\| d t \leq\|v-u\| \int_{0}^{1} \ell\|t(v-u)\|^{\alpha} d t \leq\|v-u\|^{1+\alpha} \ell \int_{0}^{1} t^{\alpha} d t  \tag{96}\\
& \leq \frac{\ell}{(\alpha+1)}\|v-u\|^{1+\alpha} .
\end{align*}
$$

This yields that $A$ satisfies the first property of ( $1, \alpha$ )-point-based approximation on $\Omega$. To proof the second
property of ( $1, \alpha$ )-point-based approximation, we assume that $y, z \in \Omega$. Then, we have that

$$
\begin{align*}
\Lambda^{\prime}(u, v, y, z) & =\|A(u, y)-A(v, y)-A(u, z)+A(v, z)\|, \\
& =\|f(u)+\nabla f(u)(y-u)-f(v)-\nabla f(v)(y-v)-f(u)-\nabla f(u)(z-u)+f(v)+\nabla f(v)(z-v)\|  \tag{97}\\
& \leq\|(\nabla f(u)-\nabla f(v))(y-z)\| \leq\|\nabla f(u)-\nabla f(v)\|\|y-z\| \leq \ell\|u-v\|^{\alpha}\|y-z\| .
\end{align*}
$$

This shows that the second property of $(1, \alpha)$-PBA for $f$ also holds. Therefore, we say that when the Fréchet derivative of $f$ is $(\ell, \alpha)$-Hölder with exponent $\alpha \in(0,1)$, the function $A:(u, v) \mapsto f(u)+\nabla f(u)(v-u)$ is an $(1, \alpha)$-pointbased approximation.
(2) Let $r_{\bar{x}}>0$ be such that $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x}) \subseteq \mathscr{X}$. Suppose that $f$ is a twice Fréchet differentiable function on $\mathbb{B}_{r_{\overline{\bar{x}}} / 2}(\bar{x})$ such that $\nabla^{2} f$ is $(K, \alpha)$-Hölder on $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ and with exponent $\alpha \in(0,1)$. Choose $\ell>0$ and $L>0$ be such that

$$
\begin{equation*}
L>\ell+K\left(r_{\bar{x}}+1\right) . \tag{98}
\end{equation*}
$$

$A(p, q)=f(p)+\nabla f(p)(q-p)+\frac{1}{2} \nabla^{2} f(p)(q-p)^{2}$.
Then, Theorem 2 ensures the existence of a sequence $\left\{x_{k}\right\}$ which converges super-quadratically and the result of Theorem 2 coincides with the result of [22,29].

To show the first property of $(2, \alpha)$-point-based approximation, denote $\Delta(p, q)=\|f(q)-A(p, q)\|$. Then we have that

$$
\begin{equation*}
\Delta(p, q)=\left\|f(q)-f(p)-\nabla f(p)(q-p)-\frac{1}{2} \nabla^{2} f(p)(q-p)^{2}\right\| . \tag{100}
\end{equation*}
$$

Since, $\quad\left\|\int_{0}^{1}\left((1-t) \nabla^{2} f(p+t \quad(q-p))(q-p)^{2}\right) d t\right\|=$ $\|f(q)-f(p)-\nabla f(p)(q-p)\|$, then (100) reduces to

Let $p, q \in \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ and define the function

$$
\begin{align*}
\Delta(p, q) & =\left\|\int_{0}^{1}\left((1-t) \nabla^{2} f(p+t(q-p))(q-p)^{2}\right) d t-\frac{1}{2} \nabla^{2} f(p)(q-p)^{2}\right\| \\
& =\left\|\int_{0}^{1}\left((1-t) \nabla^{2} f(p+t(q-p))-(1-t) \nabla^{2} f(p)\right)(q-p)^{2} d t\right\| \\
& \leq\|q-p\|^{2} \int_{0}^{1}\left\|(1-t) \nabla^{2} f(p+t(q-p))-(1-t) \nabla^{2} f(p)\right\| d t  \tag{101}\\
& \leq\|q-p\|^{2} \int_{0}^{1}\left((1-t) \nabla^{2} f(p+t(q-p))-\nabla^{2} f(p)\right) d t \\
& \leq K\|q-p\|^{2} \int_{0}^{1}(1-t)\|t(q-p)\|^{\alpha} d t \leq K\|q-p\|^{2+\alpha} \int_{0}^{1}(1-t) t^{\alpha} d t \\
& \leq \frac{K}{(\alpha+1)(\alpha+2)}\|q-p\|^{2+\alpha} \leq \frac{L}{(\alpha+1)(\alpha+2)}\|q-p\|^{2+\alpha}
\end{align*}
$$

Therefore, A satisfies the first property of an (2, $\alpha$ )-pointbased approximation on $\Omega$.

For the proof of second property, we assume that $a, b$ be any elements of $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$, Then, we have that

$$
\begin{align*}
& \Delta^{\prime}(p, q, a, b)=\|A(p, a)-A(q, a)-A(p, b)+A(q, b)\| \\
& =\| f(p)+\nabla f(p)(a-p)+\frac{1}{2} \nabla^{2} f(p)(a-p)^{2}-f(q)-\nabla f(q)(a-q)-\frac{1}{2} \nabla^{2} f(q)(a-q)^{2}-f(p)-\nabla f(p)(b-p) \\
& -\frac{1}{2} \nabla^{2} f(p)(b-p)^{2}+f(q)+\nabla f(q)(b-q)+\frac{1}{2} \nabla^{2} f(q)(b-q)^{2} \| \\
& =\left\|[\nabla f(p)-\nabla f(q)](a-b)+\frac{1}{2}\left[\nabla^{2} f(p)(a-p)^{2}-\nabla^{2} f(q)(a-q)^{2}-\nabla^{2} f(p)(b-p)^{2}+\nabla^{2} f(q)(b-q)^{2}\right]\right\| \\
& =\left\|[\nabla f(p)-\nabla f(q)](a-b)+\frac{1}{2}\left[\begin{array}{l}
\nabla^{2} f(p)(a-q+q-p, a-p)-\nabla^{2} f(p)(b-q+q-p, b-p)+ \\
\nabla^{2} f(q)(b-q, b-p+p-q)-\nabla^{2} f(q)(a-q, a-p+p-q)
\end{array}\right]\right\| \\
& =\left\|[\nabla f(p)-\nabla f(q)](a-b)+\frac{1}{2}\left[\begin{array}{c}
\nabla^{2} f(p)(a-q, a-p)+\nabla^{2} f(p)(q-p, a-p)-\nabla^{2} f(p)(b-q, b-p)-\nabla^{2} f(p)(q-p, b-p)+7| | \\
\nabla^{2} f(q)(b-q, b-p)+\nabla^{2} f(q)(b-q, p-q)-\nabla^{2} f(q)(a-q, a-p)-\nabla^{2} f(q)(a-q, p-q)
\end{array}\right]\right\| \\
& =\|[\nabla f(p)-\nabla f(q)](a-b)\|+\frac{1}{2}\left[\begin{array}{c}
\nabla^{2} f(q)(b-q, b-p)-\nabla^{2} f(p)(b-q, b-p)+\nabla^{2} f(p)(a-q, a-p)-\nabla^{2} f(q)(a-q, a-p) \\
+\nabla^{2} f(p)(q-p, a-p)-\nabla^{2} f(p)(q-p, b-p)+\nabla^{2} f(q)(b-q, p-q)-\nabla^{2} f(q)(a-q, p-q)
\end{array}\right] \| \\
& =\left\|\left[\begin{array}{c}
{[\nabla f(p)-\nabla f(q)](a-b)+\frac{1}{2}\left[\nabla^{2} f(q)-\nabla^{2} f(p)\right](b-q, b-p)} \\
+\frac{1}{2}\left[\nabla^{2} f(p)-\nabla^{2} f(q)\right](a-q, a-p)+\frac{1}{2} \nabla^{2} f(p)(q-p, a-b)+\frac{1}{2} \nabla^{2} f(q)(b-a, p-q)
\end{array}\right]\right\| \\
& =\left\|\left[\begin{array}{c}
{[\nabla f(p)-\nabla f(q)](a-b)+\frac{1}{2}\left[\nabla^{2} f(q)-\nabla^{2} f(p)\right](b-q, b-p)} \\
+\frac{1}{2}\left[\nabla^{2} f(p)-\nabla^{2} f(q)\right](a-b+b-q, a-p)+\frac{1}{2} \nabla^{2} f(p)(q-p, a-b)+\frac{1}{2} \nabla^{2} f(q)(b-a, p-q)
\end{array}\right]\right\| . \tag{102}
\end{align*}
$$

This also can be written as

$$
\begin{align*}
\Delta \prime(p, q, a, b)= & \|[\nabla f(p)-\nabla f(q)](a-b)+\frac{1}{2}\left[\nabla^{2} f(q)-\nabla^{2} f(p)\right](b-q, b-a)+\frac{1}{2}\left[\nabla^{2} f(p)-\nabla^{2} f(q)\right](a-b, a-p) \\
& \left.+\frac{1}{2} \nabla^{2} f(p)(q-p, a-b)+\frac{1}{2} \nabla^{2} f(q)(b-a, p-q)\right] \| \tag{103}
\end{align*}
$$

Since there exist an open subset $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x}) \subseteq \mathscr{X}$ and a positive number $K$ such that $\left\|\nabla^{2} f\right\| \leq K$ on $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$. Let $a, b \in \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$. Then, $\|a-b\| \leq r_{\bar{x}}$. Then, by applying
the notion of $(\ell, \alpha)$-Hölder continuity property of $\nabla f$ and ( $K, \alpha$ )-Hölder continuity property of $\nabla^{2} f$, we get

Step 1 Select $\eta \in[1, t \infty), x_{0} \in \Omega$, and put $i:=0$.
Step 2 If $0 \in \mathscr{M}\left(x_{i}\right)$, then stop; otherwise, go to Step 3 .
Step 3 If $0 \notin \mathscr{M}\left(x_{i}\right)$, choose $d_{i}$ such that $d_{i} \in \mathscr{M}\left(x_{i}\right)$ and $\left\|d_{i}\right\| \leq \eta \operatorname{dist}\left(0, \mathscr{M}\left(x_{i}\right)\right)$.
Step 4 Set $x_{i+1}:=x_{i}+d_{i}$.
Step 5 Replace $i$ by $i+1$ and go to Step 2.

Algorithm 1: (The Extended Newton-type Method)(ENM).

Table 1: Numerical results for Example 1 for the case $s<0$.

| iteration no. | $s_{k}$ | $\Gamma=s^{2}+3 s / 14-1 / 7$ | $s_{k}$ | $\Gamma=s^{2}+8 s / 7+1 / 7$ |
| :--- | :---: | :---: | :---: | :---: |
| 1 | -1.7000 | 2.3829 | -1.5000 | 0.6786 |
| 2 | -0.9520 | 0.5595 | -1.1346 | 0.1335 |
| 3 | -0.6209 | 0.1096 | -1.0161 | 0.0140 |
| 4 | -0.5142 | 0.0114 | -1.0003 | 0.0002 |
| 5 | -0.5002 | 0.0002 | -1.0000 | 0.0000 |
| 6 | -0.5000 | 0.0000 | -1.0000 | 0.0000 |
| 7 | -0.5000 | 0.0000 | -1.0000 | 0.0000 |

Table 2: Numerical results for Example 1 for the case $s \geq 0$.

| iteration no. | $s_{k}$ | $\Gamma=10 s^{2} / 7-27 s / 14-1 / 7$ | $s_{k}$ | $\Gamma=10 s^{2} / 7-s+1 / 7$ |
| :--- | :---: | :---: | :---: | :---: |
| 1 | 1.5000 | 0.1786 | 1.7000 | 2.5714 |
| 2 | 1.4242 | 0.0082 | 1.0333 | 0.6349 |
| 3 | 1.4204 | 0.0000 | 0.7081 | 0.1511 |
| 4 | 1.4204 | 0.0000 | 0.5605 | 0.0311 |
| 5 | 1.4204 | 0.0000 | 0.5087 | 0.0038 |
| 6 | 1.4204 | -0.0000 | 0.5002 | 0.0001 |
| 7 | 1.4204 | 0.0000 | 0.5000 | 0.0000 |
| 8 | 1.4204 | 0.0000 | 0.5000 | 0.0000 |



Figure 1: Superlinear rate of convergence of Algorithm 1 at $-1.0000(-0.5000)$ and 0.5000 (1.4204).

$$
\begin{align*}
\Delta \prime(p, q, a, b) \leq & \|[\nabla f(p)-\nabla f(q)](a-b)\|+\frac{1}{2}\left\|\nabla^{2} f(q)-\nabla^{2} f(p)\right\|\|b-q\|\|b-a\|+\frac{1}{2}\left\|\nabla^{2} f(p)-\nabla^{2} f(q)\right\|\|a-b\|\|a-p\| \\
& +\frac{1}{2}\left\|\nabla^{2} f(p)\right\|\|q-p\|\|a-b\|+\frac{1}{2}\left\|\nabla^{2} f(q)\right\|\|b-a\|\|p-q\| \\
\leq & \ell\|p-q\|^{\alpha}\|a-b\|+\frac{K}{2}\|p-q\|^{\alpha}\|b-q\|\|b-a\|+\frac{K}{2}\|p-q\|^{\alpha}\|b-a\|\|a-p\|+\frac{K}{2}\|q-p\|^{\alpha}\|a-b\|+\frac{K}{2}\|p-q\|^{\alpha}\|b-a\|  \tag{104}\\
\leq & \ell\|p-q\|^{\alpha}\|a-b\|+\frac{K}{2} r_{\bar{x}}\|p-q\|^{\alpha}\|a-b\|+K\|p-q\|^{\alpha}\|a-b\| \\
\leq & \left(\ell+K\left(r_{\bar{x}}+1\right)\right)\|p-q\|^{\alpha}\|a-b\| \\
\leq & L\|p-q\|^{\alpha}\|a-b\|, \text { for all } a, b \in \mathbb{B}_{r_{\bar{x}} / 2}(\bar{x}) .
\end{align*}
$$

This shows that the second property of $(2, \alpha)$-PBA is satisfied. Thus, both of properties for $(n, \alpha)$-PBA hold on $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$ when $n=2$ and $0<\alpha<1$. Hence, $A$ is an $(2, \alpha)$-PBA for $f$ on $\mathbb{B}_{r_{\bar{x}} / 2}(\bar{x})$.
4.2. Application to Normal Maps. In this subsection we deal with a class of nonsmooth functions, i.e. normal maps. Normal maps have been studied by many authors to obtain solutions of variational inequalities and comprehensive accounts on this topic can be found in [ $9,12,13,17,30]$.

A detailed discussion about normal maps is given by Robinson [13]. Recall the following notion of normal maps which was introduced by Robinson [9, 13].

Definition 6. Let $\mathscr{C}$ be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and let $\Pi$ be the metric projector from $\mathscr{X}$ onto $\mathscr{C}$. Let $\Omega$ be an open subset of $\mathscr{X}$ meeting $\mathscr{C}$ and let $f$ be a function from $\Omega$ to $\mathscr{X}$. The normal map $f_{\mathscr{E}}$ is defined from the set $\prod^{-1}(\Omega)$ to $\mathscr{X}$ by

$$
\begin{equation*}
f_{\mathscr{C}}(x)=f\left(\prod(x)\right)+\left(x-\prod(x)\right) \tag{105}
\end{equation*}
$$

Moreover, the following variational problem
find $y_{0} \in \mathscr{C}:\left\langle f\left(y_{0}\right), c-y_{0}\right\rangle \geq 0$, for all $c \in \mathscr{C}$,
is completely equivalent to the normal-map equation $f_{\mathscr{C}}\left(x_{0}\right)=0$ through the transformation $x_{0}=y_{0}-f\left(y_{0}\right)$. Robinson has shown that how the first-order necessary optimality conditions for nonlinear optimization, as well as linear and nonlinear complementarity problems and more general variational inequalities, can all be expressed compactly and conveniently in the form of equations $f_{\mathscr{C}}(x)=0$ involving normal maps.

However, sometimes the use of normal maps enables one to gain insight into special properties of problem classes that might have remained obscure in the formalism of variational inequalities. A particular illustration of this is the characterization of the local and global homeomorphism properties of linear normal maps, given in [13] and improved in [31, 32].

In [8, Proposition 4.1], Rashid proved that for any function $f$ admitting a $P B A$ on a nonempty closed convex
subset $\mathscr{C}$ of a Hilbert space $H$, the normal map associated with $f$ admits a PBA on $H$. In our study we will show that the same result holds when we replace the normal maps $f_{\mathscr{C}}+F$ in lieu of the normal maps $f_{\mathscr{C}}$. Rashid [8, 14] reformulate the normal maps $f_{\mathscr{C}}+F$ by simple modification of the definition of normal maps given by Robinson [13]. In [ 8,14 ] Rashid assumed the concept of point-based approximation and $p$-point-based approximation. Here we extend that concept to ( $n, \alpha$ )-point-based approximation which is reformulated by Rashid [8,14], then we show that if $f$ have a ( $n, \alpha$ )-point-based approximation, then one can easily be constructed a ( $n, \alpha$ )-point-based approximation for $f_{\mathscr{C}}+F$.

The following reformulation of the normal maps $f_{\mathscr{C}}+F$ is due to [14].

Definition 7. Let $\mathscr{C}$ be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and let $\Pi$ be the metric projector from $\mathscr{X}$ onto $\mathscr{C}$. Let $\Omega$ be an open subset of $\mathscr{X}$ meeting $\mathscr{C}$ and let $f: \Omega \longrightarrow \mathcal{X}$ and $F: \Omega \rightrightarrows X$. The normal map $f_{\mathscr{C}}+F$ is defined from the set $\Pi^{-1}(\Omega)$ to $\mathscr{X}$ by

$$
\begin{equation*}
\left(f_{\mathscr{C}}+F\right)(x)=f\left(\prod(x)\right)+F\left(\prod(x)\right)+\left(x-\prod(x)\right) \tag{107}
\end{equation*}
$$

We are now able to construct a $(n, \alpha)$-point-based approximation for the normal map $f_{\mathscr{C}}+F$ provided that a ( $n, \alpha$ )-point-based approximation exists for $f$. The following proposition can be extracted from [14, Proposition 4.3].

Proposition 1. Let $\mathscr{X}$ be a Banach space and $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{X}$ and let $\Pi$ be the metric projector on $\mathscr{C}$ which is nonexpansive. Let $f: \mathscr{C} \longrightarrow \mathscr{X}$, $A: \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{X}$ be functions and let $F: \mathscr{C} \rightrightarrows \mathscr{X}$ be a setvalued map with closed graph. If $A$ is a $(n, \alpha)$-point-based approximation for $f$ on $\mathscr{C}$ with a constant $L$, then the function $\quad H: \mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{X}$ defined by $H(y, x)=\left(A(\Pi(y), \cdot)_{\mathscr{C}}+F(\cdot)\right)(x)$ is a $(n, \alpha)$-point-based approximation for $f_{\mathscr{C}}+F$ on $\mathscr{X}$ with the same constant $L$.

Proof. Let $y, x \in \mathscr{X}$. We note that by the definition of normal map, $\left(f_{\mathscr{C}}+F\right)(x)$ and $H(y, x)$ are respectively defined by
$\left(f_{\mathscr{C}}+F\right)(x)=f\left(\prod(x)\right)+F\left(\prod(x)\right)+\left(x-\prod(x)\right)$,

$$
\begin{equation*}
H(y, x)=A\left(\prod(x), \prod(x)\right)+F\left(\prod(x)\right)+\left(x-\prod(x)\right) \tag{109}
\end{equation*}
$$

By hypothesis we have that $A$ has the two properties for $f$ given in Definition 4 with a constant $L$. We need to show that $H$ also has these same two properties for $f_{\mathscr{C}}+F$ with the constant $L$. Since $A$ is the ( $n, \alpha$ )-point-based approximation for $f$ on $\mathscr{C}$, then using the notion of first property of ( $n, \alpha$ )-point-based approximation and the non expansiveness of the metric projector we have that

$$
\begin{align*}
\left\|\left(f_{\mathscr{C}}+F\right)(x)-H(y, x)\right\| & =\left\|f\left(\prod(x)\right)+F\left(\prod(x)\right)+\left(x-\prod(x)\right)-\left[A\left(\prod(y), \prod(x)\right)+F\left(\prod(x)\right)+\left(x-\prod(x)\right)\right]\right\| \\
& =\left\|f\left(\prod(x)\right)-A\left(\prod(y), \prod(x)\right)\right\| \leq \frac{L}{\pi_{n, \alpha}}\left\|\prod(y)-\prod(x)\right\|^{n+\alpha} \leq \frac{L}{\pi_{n, \alpha}}\|y-x\|^{n+\alpha} \tag{110}
\end{align*}
$$

This implies that $H$ satisfies the first property of ( $n, \alpha$ )-point-based approximation. For proving the second property, we suppose that $x, x i \in \mathscr{X}$. To this end, let $y, z \in \mathscr{X}$. We will show that $H(x, \cdot)-H(x \prime, \cdot)$ is Lipschitz continuous
on $\mathscr{X}$ with lipschitz constant $L\|x-x \prime\|^{\alpha}$. Again using the concept of second property of ( $n, \alpha$ )-point-based approximation and non expansiveness of metric projector, we obtain that

$$
\begin{align*}
\|[ & H(x, y)-H(x \prime, y)]-[H(x, z)-H(x \prime, z)] \\
= & \|\left[A\left(\prod(x), \prod(y)\right)+F\left(\prod(y)\right)+\left(y-\prod(y)\right)-A\left(\prod(x \prime), \prod(y)\right)-F\left(\prod(y)\right)-\left(y-\prod(y)\right)\right] \\
& -\left[A\left(\prod(x), \prod(z)\right)+F\left(\prod(z)\right)+\left(z-\prod(z)\right)-A\left(\prod(x \prime), \prod(z)\right)-F\left(\prod(z)\right)-\left(z-\prod(x)\right)\right] \|  \tag{111}\\
= & \|\left[A\left(\prod(x), \prod(y)\right)-A\left(\prod(x \prime), \prod(y)\right)\right]-\left[A\left(\prod(x), \prod(z)\right)-A\left(\prod(x \prime), \prod(z)\right) \|\right. \\
\leq & L\left\|\prod(x)-\prod(x \prime)\right\|\left\|\prod(x)-\prod(z)\right\| \leq L\|x-x \prime\|^{\alpha}\|y-z\| .
\end{align*}
$$

This shows that the second property of the $(n, \alpha)$-pointbased approximation is satisfied. Since the both properties in Definition 4 are fulfilled for $H$, we can conclude that $H$ is a $(n, \alpha)$-point-based approximation for $f_{\mathscr{C}}+F$ on $\mathscr{X}$. The proof is completed.

## 5. Numerical Experiment

In this section, to present the numerical experiment we recall some necessary notations and notions. Let $s \in \mathbb{R}^{n}$ and $\psi: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{m}$ be a Fréchet differentiable function at $s$. Suppose that the set of all points $s \in \mathbb{R}^{n}$ is denoted by $P_{\psi}$ at which the derivative $\psi \prime(s)$ exists. The B-subdifferential of $\psi$ at $s \in \mathbb{R}^{n}$, denoted by $\partial_{B} \psi(s)$, is the set

$$
\begin{equation*}
\partial_{B} \psi(s)=\left\{J \in \mathbb{R}^{m \times n}: J=\lim _{k \rightarrow+\infty} \psi \prime\left(s_{k}\right) \text { for some }\left\{s_{k}\right\} \subset P_{\psi} \text { such that }\left\{s_{k}\right\} \longrightarrow s\right\} \tag{112}
\end{equation*}
$$

Then, Clarke's generalized Jacobian of $\psi$ at $s \in \mathbb{R}^{n}$ is the set $\partial \psi(s)=\operatorname{conv} \partial_{B} \psi(s)$. If $\psi$ is differentiable near $s$, and $\psi^{\prime}$ is continuous at $s$, then obviously
$\partial \psi(s)=\partial_{B} \psi(s)=\{\psi \prime(s)\}$. Otherwise, $\partial_{B} \psi(s)$ is not necessarily a singleton, even if $\psi$ is differentiable at $s$. In this case, $\psi^{\prime}(s) \in \partial_{B} \psi(s)$ holds. Now, in order to illustrate the
theoretical result of the extended Newton-type method, we consider the following example.

Example 1. Let $\quad \mathscr{X}=\mathscr{Y}=\mathbb{R}, s_{0}=-1.7, \eta=5$, $L=0.5, r_{\bar{s}}=3, n=1, \alpha=0.9$ and $M=1$. Let $\zeta: \mathbb{R} \longrightarrow \mathbb{R}$ and $\xi: \mathbb{R} \rightrightarrows \mathbb{R}$ be defined, respectively, by

$$
\begin{align*}
& \zeta(s)= \begin{cases}\frac{s}{7}+s^{2}, & \text { if } s<0 \\
\frac{10 s^{2}}{7}-2 s, & \text { if } s \geq 0\end{cases}  \tag{113}\\
& \xi(s)=\left\{\frac{s}{14}-\frac{1}{7}, s+\frac{1}{7}\right\}
\end{align*}
$$

Then Algorithm 1 generates a sequence which converges superlinearly to $s^{*}=-0.5000$ and $s^{*}=-1.0000$, respectively, with initial points $s_{0}=-1.7$ and $s_{0}=-1.5$ in the case $s<0$. On the other hand, Algorithm 1 generates a superlinear convergent sequence which converges to $s^{*}=01.4204$ and $s^{*}=0.5000$, respectively, with initial points $s_{0}=1.5$ and $s_{0}=1.7$ in the case $s \geq 0$.

Solution: It is manifest that $\zeta$ is not differentiable at $s=0$ and hence $\zeta$ is nonsmooth function on $\mathbb{R}$. But this function is differentiable on $\mathbb{R}-\{0\}$ and hence $\partial_{B} \zeta(s)=\{\zeta(s)\}$. So, we get

$$
\partial_{B} \zeta(s)=\left\{\zeta_{\prime}(s)\right\}= \begin{cases}\frac{1}{7}+2 s, & \text { if } s<0  \tag{114}\\ \frac{20 s}{7}-2, & \text { if } s \geq 0\end{cases}
$$

We mark that
$\Gamma(s):=(\zeta+\xi)(s) \begin{cases}\left\{s^{2}+\frac{3 s}{14}-\frac{1}{7}, s^{2}+\frac{8 s}{7}+\frac{1}{7}\right\}, & \text { if } s<0, \\ \left\{\frac{10 s^{2}}{7}-\frac{27 s}{14}-\frac{1}{7}, \frac{10 s^{2}}{7}-s+\frac{1}{7}\right\}, & \text { if } s \geq 0 .\end{cases}$

Initially, we study the set-valued mapping $\Gamma(s)=s^{2}+$ $3 s / 14-1 / 7$ for the case $x<0$ and note that $\Gamma$ has a closed graph at $(\bar{s}, \bar{t})$ with $\bar{s}=-1$ and $\bar{t}=0.64$. Thus, $(-1,0.64) \in$ gph $\Gamma$ and $(\zeta+\xi)^{-1}$, that is, $\Gamma^{-1}$ is Lipschitz-like at $(0.64,-1)$. By taking $A(s, \cdot)=\zeta(s)+\partial_{B} \zeta(s)(\cdot-s)$, it is easily shown that $R_{\bar{s}}(\cdot)=\left[\zeta(\bar{s})+\partial_{B} \zeta(\bar{s})(\cdot-\bar{s})+\xi(\cdot)\right]^{-1}$ is Lipschitz-like at $(\bar{t}, \bar{s})$ for $\bar{t}=0.64$ and $\bar{s}=-1$. Therefore, the assumptions of Theorem 2 hold. From the definition of $\mathscr{M}\left(s_{k}\right)$, we get

$$
\begin{align*}
\mathscr{M}\left(s_{k}\right) & =\left\{d_{k} \in \mathbb{R}: 0 \in \zeta\left(s_{k}\right)+\partial_{B} \zeta\left(s_{k}\right) d_{k}+\xi\left(s_{k}+d_{k}\right)\right\} \\
& =\left\{d_{k} \in \mathbb{R}: d_{k}=\frac{2-3 s_{k}-14 s_{k}^{2}}{3+28 s_{k}}\right\} . \tag{116}
\end{align*}
$$

Alternatively, if $\mathscr{M}\left(s_{k}\right) \neq \varnothing$ we take

$$
\begin{equation*}
0 \in \zeta\left(s_{k}\right)+\partial_{B} \zeta\left(s_{k}\right)\left(s_{k+1}-s_{k}\right)+\xi\left(s_{k+1}\right) \Rightarrow s_{k+1}=\frac{2+14 s_{k}^{2}}{3+28 s_{k}} \tag{117}
\end{equation*}
$$

Also, from (86) with $0 \leq \alpha \leq 1$ we consume

$$
\begin{equation*}
\left\|d_{k}\right\| \leq \frac{2^{\alpha} \eta L M}{\pi_{n, \alpha}\left(2^{\alpha}-M L r_{\bar{s})}^{\alpha}\right.}\left\|d_{k-1}\right\|^{n+\alpha} \tag{118}
\end{equation*}
$$

Hereafter, for the given values of $L, M, \eta, r_{\bar{s}}, n$ and $\alpha, \mathrm{w}$ get that Algorithm 1 generates a superlinearly convergent sequence with initial point $s_{0}=-1.7$ in a neighborhood of $\bar{s}=-1.9$. The following Tables 1 and 2, obtained by using Matlab code, indicate that the solution of the variational inclusion $\Gamma(s) \ni 0$ has the solutions $s^{*}=-1.0000$ and $s^{*}=$ -0.5000 in the case $s<0$ and $s^{*}=0.5000$ and $s^{*}=1.4202$ in the case $s \geq 0$. The graphs of $\Gamma$ are plotted in Figure 1.

Remark 4. If we set $\alpha=1$ in Example 1, we get the quadratic convergence of Algorithm 1.

## 6. Concluding Remarks

We have established semilocal and local convergence of the extended Newton-type method for solving the nonsmooth generalized (1) under the conditions $\eta>1,(f+F)^{-1}$ is Lipschitz-like and the nonsmooth function $f$ has a ( $n, \alpha$ )-point-based approximation. Moreover, when $0<\alpha<1$ and $\nabla f$ is $(\ell, \alpha)$-Hölder, we have presented an application of ( $n, \alpha$ )-point-based approximation for smooth function with $n=1$, that is, we have shown $A$ is an $(1, \alpha)$-point-based approximation. In this case Theorem 2 provides the superlinear convergent result and this result extends the convergence theorem of Geoffroy and Piétrus [19]. On the other hand, for $n=2$ and $0<\alpha<1$, if $f$ is a twice Fréchet differentiable function and $\nabla^{2} f$ is ( $\left.K, \alpha\right)$-Hölder, we have given an application of $(n, \alpha)$-point-based approximation, that is, we have shown $A$ is an $(2, \alpha)$-point-based approximation. In this case Theorem 2 yields the superquadratic convergent result and this result extends the convergence result of [22, 29]. Furthermore, we have given another application of normal maps for $f_{\mathscr{C}}+F$ which extends the concept of point-based-approximation reformulated by Rashid [8]. That is, we have shown that if $f$ has an ( $n, \alpha$ )-point-based approximations, it is easy to construct an $(n, \alpha)$-point-based approximation for the $f_{\mathscr{C}} C+F$. Finally, we have presented a numerical experiment to validate the theoretical result.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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