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Research Article

Controlled Frame for Operator in Hilbert c^*-Modules

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In this study, we will introduce a new concept, which is a controlled K-operator frame for the space of all adjointable operators on a Hilbert $\mathscr{A}$-module $\mathscr{H}$ which denoted $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$, where $\mathscr{A}$ is a C^*-algebra. Also, we establish some results of the controlled K-operator frame in $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. The presented results are new and of interest for people working in this area. Some illustrative examples are provided to advocate the usability of our results.

1. Introduction

Frame theory has a great revolution for recent years; this theory has several properties applicable in many fields of mathematics and engineering and plays a significant role in signal and image processing, which leads to many applications in informatics, medicine, and probability. Frame theory has been extended from Hilbert spaces to Hilbert C^*-modules and has begun to be study widely and deeply. The basic idea was to consider module over C^*-algebra instead of linear spaces and to allow the inner product to take values in the C^*-algebra.

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [2] by Gabor, frame theory began to be widely used, particularly, in the more specialized context of wavelet frames and Gabor frames [3]. Frames have been used in signal processing, image processing, data compression, and sampling theory.

In 2000, Frank and Larson [4] have extended the theory for the elements of C^*-algebras and Hilbert C^*-modules. Eventually, frames with C^*-valued bounds in Hilbert C^*-modules have been considered in [5].

The theory of frames has been generalized rapidly and there are various generalizations of frames in Hilbert spaces and Hilbert C^*-modules.

In 2012, Gavruta [6] introduced the notion of K-frames in Hilbert space to study the atomic systems with respect to a bounded linear operator K.

Controlled frames in Hilbert spaces have been introduced in 2010 by Balazs et al. [7] to improve the numerical efficiency of iterative algorithms for inverting the frame operator, while the notion of controlled K-frames in Hilbert spaces was introduced in 2015 by Nouri et al. [8].

Controlled frames in C^*-modules were introduced by Kouchi and Rahimi [9], and the authors showed that they share many useful properties with their corresponding notions in a Hilbert space.

The notion of K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$, which is a generalization of K-frames in Hilbert C^*-modules, is studied by Rossafi and Kabbaj [10].

Motivated by the above literature, we introduce the notion of controlled K-operator frame for Hilbert C^*-modules which is a generalization of K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

2. Preliminaries

In the following, we briefly recall the definitions and basic properties of C^*-algebra and Hilbert $\mathscr{A}$-modules. Our references for C^*-algebras are [11, 12].

For a C^*-algebra $\mathscr{A}$, if $a \in \mathscr{A}$ is positive, we write $a \geq 0$ and $\mathscr{A}^+$ denotes the set of all positive elements of $\mathscr{A}$.

Definition 1 (see [13]). Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{H}$ be a left $\mathscr{A}$-module, such that the linear structures of $\mathscr{A}$ and $\mathscr{H}$ are compatible. $\mathscr{H}$ is a pre-Hilbert $\mathscr{A}$-module if $\mathscr{H}$ is equipped with an $\mathscr{A}$-valued inner product $\langle .,. \rangle_{\mathscr{A}}$: $\mathscr{H} \times \mathscr{H} \longrightarrow \mathscr{A}$, such that is it is sesquilinear and positive definite and respects the module action. In other words,

(i) $\langle x, x \rangle_{\mathscr{A}} \geq 0$, for all $x \in \mathscr{H}$, and $\langle x, x \rangle_{\mathscr{A}} = 0$ if and only if $x = 0$

(ii) $\langle ax + y, z \rangle_{\mathscr{A}} = a\langle x, z \rangle_{\mathscr{A}} + \langle y, z \rangle_{\mathscr{A}}$, for all $a \in \mathscr{A}$ and $x, y, z \in \mathscr{H}$

(iii) $\langle x, y \rangle_{\mathscr{A}} = \langle y, x \rangle_{\mathscr{A}}^*$, for all $x, y \in \mathscr{H}$

For $x \in \mathscr{H}$, we define the norm of x by $\|x\| = \|\langle x, x \rangle_{\mathscr{A}}\|_{\mathscr{A}}^{1/2}$. If $\mathscr{H}$ is complete with $\|.\|$, it is called a Hilbert $\mathscr{A}$-module or a Hilbert C^*-modules over $\mathscr{A}$.

For every a in C^*-algebra $\mathscr{A}$, we have $|a|_{\mathscr{A}} = (a^*a)^{1/2}$ and the $\mathscr{A}$-valued norm on $\mathscr{H}$ is defined by $|x| = \langle x, x \rangle_{\mathscr{A}}^{1/2}$, for all $x \in \mathscr{H}$.

Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules; a map T: $\mathscr{H} \longrightarrow \mathscr{K}$ is said to be adjointable if there exists a map T^*: $\mathscr{K} \longrightarrow \mathscr{H}$ such that $\langle \mathrm{Tx}, y \rangle_{\mathscr{A}} = \langle x, T^* y \rangle_{\mathscr{A}}$ for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$.

We reserve the notation $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K})$ for the set of all adjointable operators from $\mathscr{H}$ to $\mathscr{K}$ and $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H}, \mathscr{H})$ is abbreviated to $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$.

$R(T)$ denotes the range of the operator $T \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H}, \mathscr{H})$, and the set of all adjointable, positive, and invertible operators with bounded inverse will be denoted by $\mathrm{GL}^+(\mathscr{H})$.

The following lemmas will be used to prove our main results.

Lemma 1 (see [14]). *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert $\mathscr{A}$-modules and $T \in End_{\mathscr{A}}^*(\mathscr{H}, \mathscr{K})$. The following statements are equivalent:*

(i) T is surjective.

(ii) T^* is bounded below with respect to norm, i.e., there is $m > 0$ such that

$$\|T^* x\| \geq m\|x\| \quad x \in \mathscr{K}. \tag{1}$$

(iii) T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that

$$\langle T^* x, T^* x \rangle_{\mathscr{A}} \geq m' \langle x, x \rangle_{\mathscr{A}} \quad x \in \mathscr{K}. \tag{2}$$

Lemma 2 (see [13]). *Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module and $T \in End_{\mathscr{A}}^*(\mathscr{H})$; then, we have, for all $x \in \mathscr{H}$,*

$$\langle \mathrm{Tx}, \mathrm{Tx} \rangle_{\mathscr{A}} \leq \|T\|^2 \langle x, x \rangle_{\mathscr{A}}. \tag{3}$$

Theorem 1 (see [15]). *Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module over a C^*-algebra $\mathscr{A}$ and let $T, S \in End_{\mathscr{A}}^*(\mathscr{H})$. If $R(S)$ is closed, then the following statements are equivalent:*

(1) $R(T) \subseteq R(S)$

(2) $TT^* \leq \lambda^2 SS^*$, for some $\lambda \geq 0$

(3) There exists $Q \in End_{\mathscr{A}}^*(\mathscr{H})$ such that $T = SQ$

Lemma 3 (see [16]). *Let $\mathscr{H}$ be a Hilbert $\mathscr{A}$-module. If $Q \in End_{\mathscr{A}}^*(\mathscr{H})$ is an invertible $\mathscr{A}$-linear map, then, for all $z \in \mathscr{H} \otimes \mathscr{K}$, we have*

$$\left\| (Q^*)^{-1} \right\|^{-1} . |z| \leq \left| (Q^* \otimes I) z \right| \leq \|Q\| . |z|. \tag{4}$$

Lemma 4 (see [5]). *If φ: $\mathscr{A} \longrightarrow \mathscr{B}$ is a $*$-homomorphism between C^*-algebras, then φ is increasing, that is, if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.*

3. Controlled K-Operator Frame for $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$

We begin this section with the following definitions.

Definition 2 (see [10]). Let $K \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$. A sequence of adjointable operators $\{T_i\}_{i \in I}$ on a Hilbert $\mathscr{A}$-module $\mathscr{H}$ over a unital C^*-algebra is called a K-operator frame for $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$ if there exist two positive constants $A, B > 0$ such that

$$A\langle K^* x, K^* x \rangle_{\mathscr{A}} \leq \sum_{i \in I} \langle T_i x, T_i x \rangle_{\mathscr{A}} \leq B\langle x, x \rangle_{\mathscr{A}}, \quad x \in \mathscr{H}. \tag{5}$$

The numbers A and B are called lower and upper bound of the operator frame, respectively.

Definition 3 (see [17]). Let $C, C' \in \mathrm{GL}^+(\mathscr{H})$. A sequence of adjointable operators $\{T_i\}_{i \in I}$ on a Hilbert $\mathscr{A}$-module $\mathscr{H}$ over a unital C^*-algebra is said to be a (C, C')-controlled operator frame for $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$ if there exist two positive constants $A, B > 0$ such that, for all $x \in \mathscr{H}$, we have

$$A\langle x, x \rangle_{\mathscr{A}} \leq \sum_{i \in I} \langle T_i \mathrm{Cx}, T_i C' x \rangle_{\mathscr{A}} \leq B\langle x, x \rangle_{\mathscr{A}}. \tag{6}$$

The numbers A and B are called lower and upper bounds of the (C, C')-controlled operator frame, respectively.

Now, we define the notion of controlled K-operator frame for $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$.

Definition 4. Let $K \in \mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$ and $C, C' \in \mathrm{GL}^+(\mathscr{H})$. A sequence of adjointable operators $\{T_i\}_{i \in I}$ on a Hilbert $\mathscr{A}$-module $\mathscr{H}$ over a unital C^*-algebra $\mathscr{A}$ is said to be a (C, C')-controlled K-operator frame for $\mathrm{End}_{\mathscr{A}}^*(\mathscr{H})$ if there exist two positive constants $A, B > 0$ such that

$$A\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle T_i Cx, T_i C' x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, \quad x \in \mathscr{H}. \tag{7}$$

The numbers A and B are called lower and upper bound of the (C, C')-controlled K-operator frame, respectively.

If $A = B = \lambda$, the (C, C')-controlled K-operator frame is called λ-tight.

If $A = B = 1$, it is called a Parseval (C, C')-controlled K-operator frame.

If only upper inequality of (7) holds, then $\{T_i\}_{i\in i}$ is called a (C, C')-controlled K-operator Bessel sequence for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

Example 3.4. Let $C \in \mathrm{GL}^+(\mathscr{H})$, $K \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$, and $\{x_i\}_{i\in I}$ be a C-controlled K-frame for $\mathscr{H}$.

Let $(\Gamma_i)_{i\in I} \subset \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ such that

$$\Gamma_i(x) = \langle x, x_i\rangle_{\mathscr{A}} e_i, \text{for all } i \in I \text{ and } x \in \mathscr{H}, \tag{8}$$

where $\langle e_i, e_j\rangle_{\mathscr{A}} = \delta_{ij} 1_{\mathscr{A}}$.

From the definition of $\{x_i\}_{i\in I}$, there exist two constants $A, B > 0$ such that, for all $x \in \mathscr{H}$, we have

$$\begin{aligned} A\langle C^{1/2}K^*x, C^{1/2}K^*x\rangle_{\mathscr{A}} &\le \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle Cx_i, x\rangle_{\mathscr{A}} \\ &\le B\langle x, x\rangle_{\mathscr{A}}. \end{aligned} \tag{9}$$

Hence,

$$\begin{aligned} A\langle C^{1/2}K^*x, C^{1/2}K^*x\rangle_{\mathscr{A}} &\le \sum_{i\in I} \langle x, x_i\rangle_{\mathscr{A}} \langle e_i, e_i\rangle_{\mathscr{A}} \langle Cx_i, x\rangle_{\mathscr{A}} \\ &\le B\langle x, x\rangle_{\mathscr{A}}, \end{aligned} \tag{10}$$

so

$$\begin{aligned} A\langle C^{1/2}K^*x, C^{1/2}K^*x\rangle_{\mathscr{A}} &\le \sum_{i\in I} \langle \langle x, x_i\rangle_{\mathscr{A}} e_i, \langle x, Cx_i\rangle_{\mathscr{A}} e_i\rangle_{\mathscr{A}} \\ &\le B\langle x, x\rangle_{\mathscr{A}}. \end{aligned} \tag{11}$$

Then,

$$\begin{aligned} A\langle C^{1/2}K^*x, C^{1/2}K^*x\rangle_{\mathscr{A}} &\le \sum_{i\in I} \langle \langle x, x_i\rangle_{\mathscr{A}} e_i, \langle Cx, x_i\rangle_{\mathscr{A}} e_i\rangle_{\mathscr{A}} \\ &\le B\langle x, x\rangle_{\mathscr{A}}. \end{aligned} \tag{12}$$

Therefore,

$$A\langle C^{1/2}K^*x, C^{1/2}K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle \Gamma_i x, \Gamma_i Cx\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}. \tag{13}$$

Since C is a surjective operator, then, from Lemma 1, there exists $m > 0$, such that

$$m\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \langle C^{1/2}K^*x, C^{1/2}K^*x\rangle_{\mathscr{A}}. \tag{14}$$

Then,

$$\mathrm{Am}\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle \Gamma_i x, \Gamma_i Cx\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, \tag{15}$$

which shows that $(\Gamma_i)_{i\in I}$ is a $(\mathrm{Id}_{\mathscr{H}}, C)$-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

Proposition 1. *Every (C, C')-controlled operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$ is a (C, C')-controlled K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a (C, C')-controlled operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bounds A and B; then, we have, for all $x \in \mathscr{H}$,

$$A\langle x, x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle T_i Cx, T_i C' x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}. \tag{16}$$

From Lemma 2, we have

$$A\|K\|^{-2}\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle T_i Cx, T_i C' x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}. \tag{17}$$

Therefore, $\{T_i\}_{i\in I}$ is a (C, C')-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bounds $A\|K\|^{-2}$ and B. □

Proposition 2. *Let $\{T_i\}_{i\in I}$ be a (C, C')-controlled K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$. If K is a surjective operator, then $\{T_i\}_{i\in I}$ is a (C, C')-controlled operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a (C, C')-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bounds A and B; hence,

$$A\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle T_i Cx, T_i C' x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, x \in \mathscr{H}. \tag{18}$$

Since K is surjective, then, from Lemma 1, there exists $m > 0$ such that

$$m\langle x, x\rangle_{\mathscr{A}} \le \langle K^*x, K^*x\rangle_{\mathscr{A}}, \quad x \in \mathscr{H}. \tag{19}$$

Using (18) and (19), we have

$$\mathrm{Am}\langle x, x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle T_i Cx, T_i C' x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, \quad x \in \mathscr{H}. \tag{20}$$

Therefore, $\{T_i\}_{i\in I}$ is a (C, C)-controlled operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bounds Am and B. □

Proposition 3. *Let $C, C' \in GL^+(\mathscr{H})$ and $\{T_i\}_{i\in I}$ be a K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$. Assume that C and C' commutes between them and commutes with the operators $T_i^*T_i$ and K, for each $i \in I$. Then, $\{T_i\}_{i\in I}$ is a (C, C')-controlled K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$.*

Proof. Let $\{T_i\}_{i\in I}$ be a K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$; then, there exist $0 < A, B$ such that

$$A\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I} \langle T_i x, T_i x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, x \in \mathscr{H}. \tag{21}$$

On the one hand, we have, for all $x \in \mathscr{H}$,

$$\sum_{i\in I}\langle T_i Cx, T_i C'x\rangle_{\mathscr{A}} = \sum_{i\in I}\langle T_i (CC')^{1/2}x, T_i (CC')^{1/2}x\rangle_{\mathscr{A}} \le B\langle (CC')^{1/2}x, (CC')^{1/2}x\rangle_{\mathscr{A}}. \tag{22}$$

Then,

$$\sum_{i\in I}\langle T_i Cx, T_i C'x\rangle_{\mathscr{A}} \le B\left\|(CC')^{1/2}\right\|^2\langle x, x\rangle_{\mathscr{A}}. \tag{23}$$

On the other hand, we have, for all $x \in \mathscr{H}$,

$$\begin{aligned}\sum_{i\in I}\langle T_i (CC')^{1/2}x, T_i (CC')^{1/2}x\rangle_{\mathscr{A}} &\ge A\langle K^*(CC')^{1/2}x, K^*(CC')^{1/2}x\rangle_{\mathscr{A}}\\ &= A\langle (CC')^{1/2}K^*x, (CC')^{1/2}K^*x\rangle_{\mathscr{A}}.\end{aligned} \tag{24}$$

From Lemma 1, there exists $m > 0$ such that

$$\langle (CC')^{1/2}K^*x, (CC')^{1/2}K^*x\rangle_{\mathscr{A}} \ge m\langle K^*x, K^*x\rangle_{\mathscr{A}}. \tag{25}$$

From (23) and (24), we have, for all $x \in \mathscr{H}$,

$$\begin{aligned}Am\langle K^*x, K^*x\rangle_{\mathscr{A}} &\le \sum_{i\in I}\langle T_i Cx, T_i C'x\rangle_{\mathscr{A}}\\ &\le B\left\|(CC')^{1/2}\right\|^2\langle x, x\rangle_{\mathscr{A}}.\end{aligned} \tag{26}$$

Therefore, $\{T_i\}_{i\in I}$ is a (C, C')-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

From now, we suppose that C and C' commutes between them and commutes with the operators $T_i^*T_i$ for each $i \in I$. Let $\{T_i\}_{i\in I}$ be a (C, C')-controlled K-operator Bessel sequence for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. We define the operator $T_{(C,C')}$ by

$$\begin{aligned}T_{(C,C')}&: \mathscr{H} \longrightarrow l^2(\mathscr{H})\\ x &\longrightarrow T_{(C,C')}x = \left\{T_i (CC')^{1/2}x\right\}_{i\in I}.\end{aligned} \tag{27}$$

The operator $T_{(C,C')}$, called the analysis operator, is well defined and bounded. The adjoint operator of $T_{(C,C')}$, called the synthesis operator, is defined by

$$\begin{aligned}T^*_{(C,C')}&: l^2(\mathscr{H}) \longrightarrow \mathscr{H}\\ \{a_i\}_{i\in I} &\longrightarrow T^*_{(C,C')}(\{a_i\}_{i\in I}) = \sum_{i\in I}(C, C')^{1/2}T_i^*a_i.\end{aligned} \tag{28}$$

We define the frame operator for $\{T_i\}_{i\in I}$ by

$$\begin{aligned}S_{(C,C')}&: \mathscr{H} \longrightarrow \mathscr{H}\\ x &\longrightarrow S_{(C,C')}x = T^*_{(C,C')}T_{(C,C')}x = \sum_{i\in I}C'T_i^*T_iCx.\end{aligned} \tag{29}$$

It is clear to see that $S_{(C,C)}$ is a positive, bounded, and self-adjoint operator. □

Theorem 2. *Let $\{T_i\}_{i\in I}$ be a (C, C')-controlled operator Bessel sequence for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. The following statements are equivalent:*

(1) $\{T_i\}_{i\in I}$ is a (C, C')-controlled K-operator frame

(2) There exists $A > 0$ such that $AKK^* \le S_{(C,C')}$

(3) $K = S^{1/2}_{(C,C')}Q$, for some $Q \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$

Proof. (1)⇒(2): let $\{T_i\}_{i\in I}$ be a (C, C)-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bounds A and B and frame operator $S_{(C,C)}$; then, we have, for all $x \in \mathscr{H}$,

$$A\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_iCx, T_iC'x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}. \tag{30}$$

Therefore,

$$A\langle KK^*x, x\rangle_{\mathscr{A}} \le \langle \sum_{i\in I}C'T_i^*T_iCx, x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}, \tag{31}$$

so

$$\langle AKK^*x, x\rangle_{\mathscr{A}} \le \langle S_{(C,C')}x, x\rangle_{\mathscr{A}} \le B\langle x, x\rangle_{\mathscr{A}}. \tag{32}$$

Hence,

$$AKK^* \le S_{(C,C')}. \tag{33}$$

(2)⇒(3): let $A > 0$ such that

$$AKK^* \le S_{(C,C')}. \tag{34}$$

This gives

$$AKK^* \le \left(S^{1/2}_{(C,C')}\right)\left(S^{1/2}_{(C,C')}\right)^*. \tag{35}$$

From Theorem 1, we have

$$K = S^{1/2}_{(C,C')}Q, \tag{36}$$

for some $Q \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

(3)⇒(1): suppose that

$$K = S^{1/2}_{(C,C')}Q, \tag{37}$$

for some $Q \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.

From Theorem 1, there exists $0 < A$ such that

$$AKK^* \le \left(S^{1/2}_{(C,C')}\right)\left(S^{1/2}_{(C,C')}\right)^*. \tag{38}$$

Hence,

$$AKK^* \le S_{(C,C')}. \tag{39}$$

Therefore, $\{T_i\}_{i\in I}$ is a (C, C')–controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. □

Proposition 4. *Let $K, Q \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ and $\{T_i\}_{i\in I}$ be a (C, C')-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. Suppose that Q commutes with C, C', and K. Then, $\{T_iQ\}_{i\in I}$ is a (C, C')-controlled Q^*K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$.*

Proof. Suppose that $\{T_i\}_{i\in I}$ is a (C,C')-controlled K-operator frame with bounds A and B; then, we have, for all $x \in \mathscr{H}$,

$$A\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i Cx, T_i C'x\rangle_{\mathscr{A}} \le B\langle x,x\rangle_{\mathscr{A}}. \tag{40}$$

Hence,

$$A\langle K^*Qx, K^*Qx\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i CQx, T_i C'Qx\rangle_{\mathscr{A}} \le B\langle Qx, Qx\rangle_{\mathscr{A}}, \tag{41}$$

so

$$\begin{aligned} A\langle (Q^*K)^*x, (Q^*K)^*x\rangle_{\mathscr{A}} &\le \sum_{i\in I}\langle T_i QCx, T_i QC'x\rangle_{\mathscr{A}} \\ &\le B\|Q\|^2\langle x,x\rangle_{\mathscr{A}}. \end{aligned} \tag{42}$$

Therefore, $\{T_iQ\}_{i\in I}$ is a (C,C')-controlled Q^*K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bounds A and $B\|Q\|^2$. □

Theorem 3. *Let $K \in End^*_{\mathscr{A}}(\mathscr{H})$ and $\{T_i\}_{i\in I}$ be a (C,C')-controlled tight K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$ with bound A_1. Then, $\{T_i\}_{i\in I}$ is a (C,C')-controlled tight operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$ with bound A_2 if and only if $K_r^{-1} = A_1/A_2K^*$.*

Proof. Let $\{T_i\}_{i\in I}$ be a (C,C')-controlled tight K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bound A_1; then, we have, for all $x \in \mathscr{H}$,

$$A_1\langle K^*x, K^*x\rangle_{\mathscr{A}} = \sum_{i\in I}\langle T_i Cx, T_i C'x\rangle_{\mathscr{A}}. \tag{43}$$

Assume that $\{T_i\}_{i\in I}$ is a (C,C')-controlled tight operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bound A_2; then, we have, for all $x \in \mathscr{H}$,

$$\sum_{i\in I}\langle T_i Cx, T_i C'x\rangle_{\mathscr{A}} = A_2\langle x,x\rangle_{\mathscr{A}}. \tag{44}$$

Hence,

$$A_1\langle K^*x, K^*x\rangle_{\mathscr{A}} = A_2\langle x,x\rangle_{\mathscr{A}}, \tag{45}$$

so

$$\langle KK^*x, x\rangle_{\mathscr{A}} = \langle \frac{A_2}{A_1}x, x\rangle_{\mathscr{A}}. \tag{46}$$

Then,

$$KK^* = \frac{A_2}{A_1}\mathrm{Id}_{\mathscr{A}}. \tag{47}$$

Therefore,

$$K_r^{-1} = \frac{A_1}{A_2}K^*. \tag{48}$$

For the converse, assume that

$$K_r^{-1} = \frac{A_1}{A_2}K^*. \tag{49}$$

Then,

$$KK^* = \frac{A_2}{A_1}\mathrm{Id}_{\mathscr{A}}. \tag{50}$$

This gives

$$\langle KK^*x, x\rangle = \langle \frac{A_2}{A_1}x, x\rangle_{\mathscr{A}}, x \in \mathscr{H}. \tag{51}$$

Or $\{T_i\}_{i\in I}$ is a (C,C')-controlled tight K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bound A_1, then we have

$$\sum_{i\in I}\langle T_i Cx, T_i C'x\rangle_{\mathscr{A}} = A_2\langle x,x\rangle_{\mathscr{A}}. \tag{52}$$

Therefore, $\{T_i\}_{i\in I}$ is a (C,C')-controlled tight operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$. □

Corollary 1. *Let $K \in End^*_{\mathscr{A}}(\mathscr{H})$ and $\{T_i\}_{i\in I}$ be a sequence for $End^*_{\mathscr{A}}(\mathscr{H})$. Then, those statements are true:*

If $\{T_i\}_{i\in I}$ is a (C,C')-controlled tight K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$, then $\{T_i(K^n)^*\}_{i\in I}$ is a (C,C')-controlled tight K^{n+1}-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$

If $\{T_i\}_{i\in I}$ is a (C,C')-controlled tight operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$, then $\{T_iK^*\}_{i\in I}$ is a (C,C')-controlled tight K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$

Theorem 4. *Let $\{T_i\}_{i\in I}$ be a (C,C')-controlled K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$ with best frame bounds A and B. If Q: $\mathscr{H} \longrightarrow \mathscr{H}$ is an adjointable and invertible operator such that Q^{-1} commutes with K^*, then $\{T_iQ\}_{i\in I}$ is a (C,C')-controlled K-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$ with best frame bounds M and N satisfying the inequalities:*

$$A\|Q^{-1}\|^{-2} \le M \le A\|Q\|^2 \text{ and } B\|Q^{-1}\|^{-2} \le N \le B\|Q\|^2. \tag{53}$$

Proof. Let $\{T_i\}_{i\in I}$ be a (C,C')-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with best frame bounds A and B; then, for all $x \in \mathscr{H}$, we have

$$\sum_{i\in I}\langle T_i CQx, T_i C'Qx\rangle_{\mathscr{A}} \le B\langle Qx, Qx\rangle_{\mathscr{A}} \le B\|Q\|^2\langle x,x\rangle_{\mathscr{A}}. \tag{54}$$

Also, we have

$$\begin{aligned} A\langle K^*x, K^*x\rangle_{\mathscr{A}} &= A\langle K^*Q^{-1}Qx, K^*Q^{-1}Qx\rangle_{\mathscr{A}} \\ &= A\langle Q^{-1}K^*Qx, Q^{-1}K^*Qx\rangle_{\mathscr{A}} \\ &\le \|Q^{-1}\|^2\sum_{i\in I}\langle T_i CQx, T_i C'Qx\rangle_{\mathscr{A}} \\ &= \|Q^{-1}\|^2\sum_{i\in I}\langle T_i QCx, T_i QC'x\rangle_{\mathscr{A}}. \end{aligned} \tag{55}$$

Hence,

$$A\|Q^{-1}\|^{-2}\langle K^*x, K^*x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_i QCx, T_i QC'x\rangle_{\mathscr{A}} \le B\|Q\|^2\langle x,x\rangle_{\mathscr{A}}. \tag{56}$$

Therefore, $\{T_iQ\}_{i\in I}$ is a (C,C')-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with bounds $A\|Q^{-1}\|^{-2}$ and $B\|Q\|^2$.

Now, let M and N be the best bounds of the (C,C')-controlled K-operator frame $\{T_iQ\}_{i\in I}$. Then,

$$A\left\|Q^{-1}\right\|^{-2} \le M \text{ and } N \le B\|Q\|^2. \tag{57}$$

Since $\{T_iQ\}_{i\in I}$ is a (C,C')-controlled K-operator frame for $\mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ with frame bounds M and N, we have, for all $x \in \mathscr{H}$,

$$\begin{aligned}\langle K^*x, K^*x\rangle_{\mathscr{A}} &= \langle QQ^{-1}K^*x, QQ^{-1}K^*x\rangle_{\mathscr{A}}\\ &\le \|Q\|^2\langle K^*Q^{-1}x, K^*Q^{-1}x\rangle_{\mathscr{A}}.\end{aligned} \tag{58}$$

Hence,

$$\begin{aligned}M\|Q\|^{-2}\langle K^*x, K^*x\rangle_{\mathscr{A}} &\le M\langle K^*Q^{-1}x, K^*Q^{-1}x\rangle_{\mathscr{A}}\\ &\le \sum_{i\in\mathbb{J}}\langle T_iQCQ^{-1}x, T_iQC'Q^{-1}x\rangle_{\mathscr{A}}\\ &= \sum_{i\in\mathbb{J}}\langle T_iQQ^{-1}Cx, T_iQQ^{-1}C'x\rangle_{\mathscr{A}}\\ &= \sum_{i\in I}\langle T_iCx, T_iC'x\rangle_{\mathscr{A}}\\ &\le N\left\|Q^{-1}\right\|^2\langle x, x\rangle_{\mathscr{A}}.\end{aligned} \tag{59}$$

Since A and B are the best frame bounds of (C,C')-controlled K-operator frame $\{T_i\}_{i\in I}$, we have

$$M\|Q\|^{-2} \le A \text{ and } B \le N\left\|Q^{-1}\right\|^2. \tag{60}$$

Therefore, inequality (53) follows from (57) and (60). □

4. Tensor Product of Controlled K-Operator Frame

Theorem 5. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert C^*-modules over a unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Let $C_1, C_2 \in GL^+(\mathscr{H})$ and $C_1', C_2' \in GL^+(\mathscr{K})$, and let $\{T_i\}_{i\in I}$ be a (C_1, C_2)-controlled K-operator frame for $\mathscr{H}$ and $\{\Gamma_j\}_{j\in J}$ be a (C_1', C_2')-controlled L-operator frame for $\mathscr{K}$ with frame operators S_T and S_Γ and bounds (A_1, B_1) and (A_2, B_2), respectively, where $K \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$ and $L \in \mathrm{End}^*_{\mathscr{B}}(\mathscr{K})$. Then, $\{T_i\otimes\Gamma_j\}_{i\in I, j\in J}$ is a $(C_1\otimes C_1', C_2\otimes C_2')$-controlled $K\otimes L$-operator frame for Hilbert $\mathscr{A}\otimes\mathscr{B}$-module $\mathscr{H}\otimes\mathscr{K}$, with frame operator $S_T\otimes S_\Gamma$ and bounds $A_1\otimes A_2$ and $B_1\otimes B_2$.*

Proof. According to the definitions of $\{T_i\}_{i\in I}$ and $\{\Gamma_j\}_{j\in J}$ contained in the previous theorem, we have, for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$,

$$\begin{aligned}A_1\langle K^*x, K^*x\rangle_{\mathscr{A}} &\le \sum_{i\in I}\langle T_iC_1x, T_iC_2x\rangle_{\mathscr{A}} \le B_1\langle x, x\rangle_{\mathscr{A}}\\ A_2\langle L^*y, L^*y\rangle_{\mathscr{B}} &\le \sum_{j\in J}\langle \Gamma_jC_1'y, \Gamma_jC_2'y\rangle_{\mathscr{B}} \le B_2\langle y, y\rangle_{\mathscr{B}}.\end{aligned} \tag{61}$$

Therefore,

$$\begin{aligned}&\left(A_1\langle K^*x, K^*x\rangle_{\mathscr{A}}\right)\otimes\left(A_2\langle L^*y, L^*y\rangle_{\mathscr{B}}\right)\\ &\quad\le\left(\sum_{i\in I}\langle T_iC_1x, T_iC_2x\rangle_{\mathscr{A}}\right)\otimes\left(\sum_{j\in J}\langle \Gamma_jC_1'y, \Gamma_jC_2'y\rangle_{\mathscr{B}}\right)\\ &\quad\le\left(B_1\langle x, x\rangle_{\mathscr{A}}\right)\otimes\left(B_2\langle y, y\rangle_{\mathscr{B}}\right),\end{aligned} \tag{62}$$

which gives

$$\begin{aligned}&\left(A_1\otimes A_2\right)\left(\langle K^*x, K^*x\rangle_{\mathscr{A}}\right)\otimes\left(\langle L^*y, L^*y\rangle_{\mathscr{B}}\right)\\ &\quad\le\sum_{i\in I, j\in J}\langle T_iC_1x, T_iC_2x\rangle_{\mathscr{A}}\otimes\langle \Gamma_jC_1'y, \Gamma_jC_2'y\rangle_{\mathscr{A}})\\ &\quad\le\left(B_1\otimes B_2\right)\left(\langle x, x\rangle_{\mathscr{A}}\right)\otimes\left(\langle y, y\rangle_{\mathscr{B}}\right).\end{aligned} \tag{63}$$

Then,

$$\begin{aligned}&\left(A_1\otimes A_2\right)\left(\langle K^*x\otimes L^*y, K^*x\otimes L^*y\rangle_{\mathscr{A}\otimes\mathscr{B}}\right)\\ &\quad\le\sum_{i\in I, j\in J}\langle T_iC_1x\otimes\Gamma_jC_1'y, T_iC_2x\otimes\Gamma_jC_2'y\rangle_{\mathscr{A}\otimes\mathscr{B}}\\ &\quad\le\left(B_1\otimes B_2\right)\langle x\otimes y, x\otimes y\rangle_{\mathscr{A}\otimes\mathscr{B}}),\end{aligned} \tag{64}$$

so

$$\begin{aligned}&\left(A_1\otimes A_2\right)\left(\langle(K^*\otimes L^*)(x\otimes y), (K^*\otimes L^*)(x\otimes y)\rangle_{\mathscr{A}\otimes\mathscr{B}}\right)\\ &\quad\le\sum_{i\in I, j\in J}\langle\left(T_i\otimes\Gamma_j\right)(C_1x\otimes C_1'y), \left(T_i\otimes\Gamma_j\right)(C_2x\otimes C_2'y)\rangle_{\mathscr{A}\otimes\mathscr{B}}\\ &\quad\le\left(B_1\otimes B_2\right)\langle x\otimes y, x\otimes y\rangle_{\mathscr{A}\otimes\mathscr{B}}).\end{aligned} \tag{65}$$

Then,

$$\begin{aligned}&\left(A_1\otimes A_2\right)\left(\langle(K\otimes L)^*(x\otimes y), (K\otimes L)^*(x\otimes y)\rangle_{\mathscr{A}\otimes\mathscr{B}}\right)\\ &\quad\le\sum_{i\in I, j\in J}\langle\left(T_i\otimes\Gamma_j\right)(C_1\otimes C_1')(x\otimes y), \left(T_i\otimes\Gamma_j\right)(C_2\otimes C_2')(x\otimes y)\rangle_{\mathscr{A}\otimes\mathscr{B}}\\ &\quad\le\left(B_1\otimes B_2\right)\langle x\otimes y, x\otimes y\rangle_{\mathscr{A}\otimes\mathscr{B}}).\end{aligned} \tag{66}$$

Therefore, $\{T_i\otimes\Gamma_j\}_{i\in I, j\in J}$ is a $(C_1\otimes C_1', C_2\otimes C_2')$-controlled $K\otimes L$-operator frame for Hilbert $\mathscr{A}\otimes\mathscr{B}$-module $\mathscr{H}\otimes\mathscr{K}$.

Moreover, we have, for all $x \in \mathscr{H}$ and $y \in \mathscr{K}$,

$$\begin{aligned}(S_T\otimes S_\Gamma)(x\otimes y) &= S_Tx\otimes S_\Gamma y\\ &= \sum_{i\in I}C_2T_i^*T_iC_1x\otimes\sum_{j\in J}C_2'\Gamma_j^*\Gamma_jC_1'y\\ &= \sum_{i\in I, j\in J}C_2T_i^*T_iC_1x\otimes C_2'\Gamma_j^*\Gamma_jC_1'y\\ &= \sum_{i\in I, j\in J}(C_2\otimes C_2')\left(T_i^*\otimes\Gamma_j^*\right)\left(T_iC_1x\otimes\Gamma_jC_1'y\right)\\ &= \sum_{i\in I, j\in J}(C_2\otimes C_2')\left(T_i\otimes\Gamma_j\right)^*\left(T_i\otimes\Gamma_j\right)(C_1x\otimes C_1'y)\\ &= \sum_{i\in I, j\in J}(C_2\otimes C_2')\left(T_i\otimes\Gamma_j\right)^*\left(T_i\otimes\Gamma_j\right)(C_1\otimes C_1')(x\otimes y).\end{aligned} \tag{67}$$

By uniqueness of frame operator, the last equality is equal to $S_{T\otimes\Gamma}(x\otimes y)$, which ends the proof. □

Theorem 6. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert C^*-modules over a unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Let $Q \in End^*_{\mathscr{A}}(\mathscr{H})$ and L, C, and C' in $End^*_{\mathscr{A}\otimes\mathscr{B}}(\mathscr{H}\otimes\mathscr{K})$, where Q is an invertible operator. Let $\{T_i\}_{i\in I}$ a sequence of adjointable operators in $End^*_{\mathscr{A}\otimes\mathscr{B}}(\mathscr{H}\otimes\mathscr{K})$ be a (C,C')-controlled L-operator frame for $\mathscr{H}\otimes\mathscr{K}$ with lower and upper bounds A and B, respectively, and frame operator S_T. If $Q\otimes I$ commutes with L, C, and C', then $\{T_i(Q^*\otimes I)\}_{i\in I}$ is a (C,C')-controlled L-operator frame for $\mathscr{H}\otimes\mathscr{K}$ with bounds $\|(Q^*)^{-1}\|^{-2}A$ and $\|Q\|^2B$ and frame operator $(Q\otimes I)S(Q^*\otimes I)$.*

Proof. Let $Q \in \mathrm{End}^*_{\mathscr{A}}(\mathscr{H})$; then, $Q\otimes I \in \mathrm{End}^*_{\mathscr{A}\otimes\mathscr{B}}(\mathscr{H}\otimes\mathscr{K})$ with inverse $Q^{-1}\otimes I$. Also, it is easy to show that the adjoint of $Q\otimes I$ is $Q^*\otimes I$, and we have, for all $x\otimes y \in \mathscr{H}\otimes\mathscr{K}$,

$$\|(Q\otimes I)(x\otimes y)\| \le \|Q\|^2\|x\otimes y\|, \tag{68}$$

which show that $(Q\otimes I)$ is bounded. Similarly, for $Q^*\otimes I$, hence, $Q\otimes I$ is $\mathscr{A}\otimes\mathscr{B}$-linear, adjointable with adjoint $Q^*\otimes I$.

So, for all $z \in \mathscr{H}\otimes\mathscr{K}$, by Lemma 3, we have

$$\left\|(Q^*)^{-1}\right\|^{-1}.|z| \le \left|(Q^*\otimes I)z\right| \le \|Q\|.|z|. \tag{69}$$

From the definition of $\{T_i\}_{i\in I}$, we have, for all $z \in \mathscr{H}\otimes\mathscr{K}$,

$$A\langle L^*z, L^*z\rangle_{\mathscr{A}\otimes\mathscr{B}} \le \sum_{i\in I}\langle T_iCz, T_iC'z\rangle_{\mathscr{A}\otimes\mathscr{B}} \le B\langle z,z\rangle_{\mathscr{A}\otimes\mathscr{B}}. \tag{70}$$

Then,

$$\begin{aligned} &A\langle L^*(Q^*\otimes I)z, L^*(Q^*\otimes I)z\rangle_{\mathscr{A}\otimes\mathscr{B}} \\ &\quad\le \sum_{i\in I}\langle T_iC(Q^*\otimes I)z, T_iC'(Q^*\otimes I)z\rangle_{\mathscr{A}\otimes\mathscr{B}} \\ &\quad\le B\langle (Q^*\otimes I)z, (Q^*\otimes I)z\rangle_{\mathscr{A}\otimes\mathscr{B}}. \end{aligned} \tag{71}$$

On the one hand, we have

$$B\langle (Q^*\otimes I)z, (Q^*\otimes I)z\rangle_{\mathscr{A}\otimes\mathscr{B}} \le \|Q\|^2B\langle z,z\rangle_{\mathscr{A}\otimes\mathscr{B}}. \tag{72}$$

On the other hand, we have

$$\begin{aligned} &A\langle L^*(Q^*\otimes I)z, L^*(Q^*\otimes I)z\rangle_{\mathscr{A}\otimes\mathscr{B}} \\ &\quad= A\langle (Q^*\otimes I)L^*z, (Q^*\otimes I)L^*z\rangle_{\mathscr{A}\otimes\mathscr{B}} \\ &\quad\ge \left\|(Q^*)^{-1}\right\|^{-2}A\langle L^*z, L^*z\rangle_{\mathscr{A}\otimes\mathscr{B}}. \end{aligned} \tag{73}$$

Finally, we obtain

$$\begin{aligned} &\left\|(Q^*)^{-1}\right\|^{-2}A\langle L^*z, L^*z\rangle_{\mathscr{A}\otimes\mathscr{B}} \\ &\quad\le \sum_{i\in I}\langle T_i(Q^*\otimes I)Cz, T_i(Q^*\otimes I)C'z\rangle_{\mathscr{A}\otimes\mathscr{B}} \\ &\quad\le \|Q\|^2B\langle z,z\rangle_{\mathscr{A}\otimes\mathscr{B}}. \end{aligned} \tag{74}$$

Concerning the frame operator, we have

$$\begin{aligned} (Q\otimes I)S(Q^*\otimes I) &= (Q\otimes I)\left(\sum_{i\in I}C'T_i^*T_iC\right)(Q^*\otimes I) \\ &= \sum_{i\in I}(Q\otimes I)(C'T_i^*T_iC)(Q^*\otimes I) \\ &= \sum_{i\in I}C'(Q\otimes I)(T_i^*T_i)(Q^*\otimes I)C \\ &= \sum_{i\in I}C'(T_i(Q^*\otimes I))_i^*(T_i(Q^*\otimes I))C, \end{aligned} \tag{75}$$

which allows us to find the requested result. □

Theorem 7. *Let $\mathscr{H}$ and $\mathscr{K}$ be two Hilbert C^*-modules over a unital C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively. Let $Q \in End^*_{\mathscr{A}}(\mathscr{H})$ and L, C, and C' be in $End^*_{\mathscr{A}\otimes\mathscr{B}}(\mathscr{H}\otimes\mathscr{K})$, where Q is an invertible operator. Let $\{T_i\}_{i\in I}$ a sequence of adjointable operators in $End^*_{\mathscr{A}\otimes\mathscr{B}}(\mathscr{H}\otimes\mathscr{K})$ be a (C,C')-controlled L-operator frame for $\mathscr{H}\otimes\mathscr{K}$ with lower and upper bounds A and B, respectively, and frame operator S_T. If $I\otimes Q$ commutes with L, C, and C', then $\{T_i(I\otimes Q^*)\}_{i\in I}$ is a (C,C')-controlled L-operator frame for $\mathscr{H}\otimes\mathscr{K}$ with bounds $\|(Q^*)^{-1}\|^{-2}A$ and $\|Q\|^2B$ and frame operator $(Q\otimes I)S(Q^*\otimes I)$.*

Proof. It is similar to the proof of the previous theorem. □

Theorem 8. *Let $(\mathscr{H},\mathscr{A},\langle .,.\rangle_{\mathscr{A}})$ and $(\mathscr{H},\mathscr{B},\langle .,.\rangle_{\mathscr{B}})$ be two Hilbert C^*-modules and let φ: $\mathscr{A}\longrightarrow\mathscr{B}$ between C^*-algebras be a $*$-homomorphism and θ be a map on $\mathscr{H}$ such that $\langle\theta x,\theta y\rangle_{\mathscr{B}} = \varphi(\langle x,y\rangle_{\mathscr{A}})$, for all $x, y \in \mathscr{H}$. Also, suppose that $\{T_i\}_{i\in I}$ is a (C,C')-controlled L-operator frame for $End^*_{\mathscr{A}}(\mathscr{H})$ with frame operator $S_{\mathscr{A}}$. If θ is surjective and commutes with L^*, T_i, C, and C', then $\{T_i\}_{i\in I}$ is a (C,C')-controlled L-operator frame for $End^*_{\mathscr{B}}(\mathscr{H})$ with frame operator $S_{\mathscr{B}}$ and $\langle S_{\mathscr{B}}\theta x,\theta y\rangle_{\mathscr{B}} = \varphi(\langle S_{\mathscr{A}}x, y\rangle_{\mathscr{A}})$.*

Proof. Let $z \in \mathscr{H}$; since θ is surjective, then there exist $z \in \mathscr{H}$ such that $\theta(x) = z$. By the definition of (C,C')-controlled L-operator frame $\{T_i\}_{i\in I}$ for $\mathscr{H}$, there exist $0 < A, B$ such that

$$A\langle L^*x, L^*x\rangle_{\mathscr{A}} \le \sum_{i\in I}\langle T_iCx, T_iC'x\rangle_{\mathscr{A}} \le B\langle x,x\rangle_{\mathscr{A}}. \tag{76}$$

By Lemma 4, we have

$$\begin{aligned}
\varphi\left(A\langle L^*x, L^*x\rangle_{\mathscr{A}}\right) &\le \varphi\left(\sum_{i\in I}\langle T_i\mathrm{C}x, T_iC'x\rangle_{\mathscr{A}}\right) \le \varphi\left(B\langle x, x\rangle_{\mathscr{A}}\right)\\
&\Rightarrow A\varphi\left(\langle L^*x, L^*x\rangle_{\mathscr{A}}\right) \le \sum_{i\in I}\varphi\left(\langle T_i\mathrm{C}x, T_iC'x\rangle_{\mathscr{A}}\right) \le B\varphi\left(\langle x, x\rangle_{\mathscr{A}}\right)\\
&\Rightarrow A\langle \theta L^*x, \theta L^*x\rangle_{\mathscr{B}} \le \sum_{i\in I}\langle \theta T_i\mathrm{C}x, \theta T_iC'x\rangle_{\mathscr{A}} \le B\langle \theta x, \theta x\rangle_{\mathscr{B}}\\
&\Rightarrow A\langle L^*\theta x, L^*\theta x\rangle_{\mathscr{B}} \le \sum_{i\in I}\langle T_iC\theta x, T_iC'\theta x\rangle_{\mathscr{A}} \le B\langle \theta x, \theta x\rangle_{\mathscr{B}}\\
&\Rightarrow A\langle L^*z, L^*z\rangle_{\mathscr{B}} \le \sum_{i\in I}\langle T_iCz, T_iC'z\rangle_{\mathscr{A}} \le B\langle z, z\rangle_{\mathscr{B}},
\end{aligned} \tag{77}$$

which shows that $\{T_i\}_{i\in I}$ is a (C, C')-controlled L-operator frame for $\mathrm{End}^*_{\mathscr{B}}(\mathscr{H})$.

Moreover, let $S_{\mathscr{A}}$ and $S_{\mathscr{B}}$ be the frame operators for $\{T_i\}_{i\in I}$ and $\{\theta T_i\}_{i\in I}$, respectively; then, we have

$$\begin{aligned}
\varphi\left(\langle S_{\mathscr{A}}x, y\rangle_{\mathscr{A}}\right) &= \varphi\left(\langle \sum_{i\in I} C'T_i^*T_i\mathrm{C}x, y\rangle_{\mathscr{A}}\right)\\
&= \sum_{i\in I}\varphi\left(\langle C'T_i^*T_i\mathrm{C}x, y\rangle_{\mathscr{A}}\right)\\
&= \sum_{i\in I}\varphi\left(\langle T_i\mathrm{C}x, T_iC'y\rangle_{\mathscr{A}}\right)\\
&= \sum_{i\in I}\langle \theta T_i\mathrm{C}x, \theta T_iC'y\rangle_{\mathscr{A}})\\
&= \sum_{i\in I}\langle T_iC\theta x, T_iC'\theta y\rangle\rangle_{\mathscr{A}})\\
&= \langle \sum_{i\in I} C'T_i^*T_iC\theta x, \theta y\rangle_{\mathscr{A}}\\
&= \langle S_{\mathscr{B}}\theta x, \theta y\rangle_{\mathscr{B}},
\end{aligned} \tag{78}$$

which ends the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there are no conflicts of interest.

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