# Numerical Study of an Euler-Bernoulli Beam Stabilization Controlled in Displacement and Velocity 

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Our model is based on an Euler-Bernoulli beam clamped at one end and subjected at its free end to a control in displacement and velocity. In this work, relative of the control parameters, we study the stabilization in displacement and then in energy of the model using a stable numerical scheme that we implement. This numerical scheme results from the Crank-Nicolson algorithm for the discretization in time and from the finite element method based on the approximation by Hermith cubic functions for the discretization in space. The study shows that, compared to the velocity control, the displacement control has an almost negligible effect on the stabilization of the beam. This result is confirmed later by a sensitivity study on the control parameters involved in our model.

## 1. Introduction

We consider an Euler-Bernoulli beam with variable coefficients, clamped at one end and free at the other end. We subject the free end of the beam whose length is assumed to be equal to unity, to a control resulting from the linear
combination of its transverse displacement and its velocity $v(t,$.$) . We note, respectively, m(x)$ and $E I(x)$ the linear mass and the bending stiffness of the beam at any point of abscissa $x$. The model resulting from the study of the dynamics of the beam is written at any time $t$.

$$
\left\{\begin{array}{l}
m(x) \frac{\partial^{2} u(t, x)}{\partial t^{2}}+\frac{\partial^{2}}{\partial x^{2}}\left(E I(x) \frac{\partial^{2} u(t, x)}{\partial x^{2}}\right)=0 ; \quad 0<x<1 ; t>0,  \tag{1}\\
u(t, 0)=\frac{\partial u(t, 0)}{\partial x}=\frac{\partial^{2} u(t, 1)}{\partial x^{2}}=0 ; \quad \frac{\partial}{\partial x}\left(E I(1) \frac{\partial^{2} u(t, 1)}{\partial x^{2}}\right)=\alpha u(t, 1)+\beta \frac{\partial u(t, 1)}{\partial t} ; t>0, \\
u(0, x)=u_{0}(x) ; \quad \frac{\partial u(0, x)}{\partial t}=u_{1}(x),
\end{array}\right.
$$

where $\alpha$ and $\beta$ are positive constants.
Previous studies [1,2] have focused on the exponential stabilization of the Euler-Bernoulli beam, using the Riezs basis approach. This approach generally makes it possible to
establish the spectrum of the operator associated with the original problem to analyze its properties. Authors My Driss and Abderrahman El , in [3], go further by proposing a numerical study of the operator spectrum to establish the
influence of the control parameters on the speed of convergence of the energy associated with the model.

In this work, we assume problem (1) is well posed in the sense of contraction semigroups according to [1]. Starting from this problem, we implemented an equivalent numerical model and we established its stability. This numerical scheme allows us to analyze graphically the influence of the control parameters on the stabilization of the beam and on the asymptotic behavior of the system energy. Results deduced from these graphical analysis are then confirmed by a study of the relative sensitivities of these control parameters to damping time of the vibrations of the beam.

## 2. Numerical Approximations of the Model

2.1. Implementation of the Numerical Scheme. Since $v(t,)=.\partial u(t,.) / \partial t$, the first equation of system (1) can still be written as follows:

$$
\begin{equation*}
\frac{\partial v}{\partial t}=-\frac{1}{m(x)}\left[\frac{\partial^{2}}{\partial x^{2}}\left(E I(x) \frac{\partial^{2} u(t, .)}{\partial x^{2}}\right)\right] ; \quad 0<x<1 ; t>0 \tag{2}
\end{equation*}
$$

If we set $Y(t)=(u(t,.) ; v(t,)$.$) , then we introduce the$ functional space

$$
\begin{equation*}
H=H_{E}^{2}([0 ; 1]) \times L^{2}([0 ; 1]) \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{E}^{2}([0 ; 1])=\left\{\varphi \in H^{2}([0 ; 1]) \left\lvert\, \varphi(0)=\frac{\partial \varphi(0)}{\partial x}=0\right.\right\} \tag{4}
\end{equation*}
$$

Then, the problem (1) is written as follows:

$$
\begin{equation*}
\frac{\mathrm{d} Y}{\mathrm{~d} t}=A_{0} Y ; Y(0)=Y_{0}=\left(u_{0} ; v_{0}\right) \tag{5}
\end{equation*}
$$

where $A_{0}$ is a linear and unbounded operator defined on the following equation:

$$
\begin{align*}
D\left(A_{0}\right) & =\left\{(f, g) \in\left(H^{4}[0 ; 1] \cap H_{E}^{2}[0 ; 1]\right) \times H_{E}^{2}[0 ; 1] \mid f^{\prime \prime}(1)\right. \\
& \left.=0 \text { and }\left(E I(1) f^{\prime \prime}(1)\right)^{\prime}=\alpha f(1)+\beta g(1)\right\} . \tag{6}
\end{align*}
$$

For a fixed real $T>0$, we subdivide the interval $[0 ; T]$ into $N_{t}$ intervals of the same length $\Delta t$. We then set at any point $t_{n}=n \Delta t\left(0 \leq n \leq N_{t}\right)$.

$$
\begin{equation*}
u^{n}=u\left(t_{n}, x\right), v^{n}=v\left(t_{n}, x\right), Y^{n}=Y\left(t_{n}\right)=\left(u^{n} ; v^{n}\right) . \tag{7}
\end{equation*}
$$

The time discretization of system (5) by the Crank-Nicolson scheme gives the following equations:

$$
\frac{u^{n+1}-u^{n}}{\Delta t}=\frac{v^{n+1}+v^{n}}{2} ; m(x) \frac{v^{n+1}-v^{n}}{\Delta t}+\frac{1}{2}\left[\frac { \partial ^ { 2 } } { \partial x ^ { 2 } } \left(E I(x) \frac{\partial^{2}}{\partial x^{2}}\right.\right.
$$

$$
\begin{equation*}
\left.\left.\left(u^{n+1}+u^{n}\right)\right)\right]=0, \quad \text { with } u^{0}=u_{0} \text { and } v^{0}=v_{0} . \tag{8}
\end{equation*}
$$

For $\psi \in L^{2}[0 ; 1]$ and $\varphi \in H_{E}^{2}[0 ; 1]$, problem (8) is written in the following variational form:

$$
\left\{\begin{array}{l}
\int_{0}^{1} \frac{u^{n+1}-u^{n}}{\Delta t} \psi(x) d x=\int_{0}^{1} \frac{v^{n+1}+v^{n}}{2} \psi(x) d x  \tag{9}\\
\int_{0}^{1} m(x) \frac{v^{n+1}-v^{n}}{\Delta t} \varphi(x) d x+\frac{1}{2} \int_{0}^{1} E I(x)\left(\frac{\partial^{2} u^{n+1}}{\partial x^{2}}+\frac{\partial^{2} u^{n}}{\partial x^{2}}\right) \frac{\partial^{2} \varphi(x)}{\partial x^{2}} d x+\left[\alpha \cdot \frac{u^{n+1}(1)+u^{n}(1)}{2}+\beta \cdot \frac{v^{n+1}(1)+v^{n}(1)}{2}\right] \varphi(1)=0 .
\end{array}\right.
$$

We assume that the Euler-Bernoulli beam is the segment [ $0 ; 1$ ] which we subdivide into $N+1$ fixed length intervals $\left[x_{i} ; x_{i+1}\right](i=0 ; \ldots ; N)$. For the discretization in the space of problem (9), relatively to study [4], we use the finite element method with the cubic Hermite polynomials $\left(\varphi_{k}\right)_{1 \leq k \leq 4}$ as functions of references from which we build on $[0 ; 1]$ polynomials $\phi_{1}^{i}(x)$ and $\phi_{2}^{i}(x)$ defined for $i=1,2, \ldots, N-1$ by the following equation:

$$
\begin{align*}
& \phi_{1}^{i}(x)= \begin{cases}\varphi_{2}^{i-1}(x), & \text { if } x \in\left[x_{i-1} ; x_{i}\right], \\
\varphi_{1}^{i}(x), & \text { if } x \in\left[x_{i} ; x_{i+1}\right],\end{cases}  \tag{10}\\
& \phi_{2}^{i}(x)= \begin{cases}\varphi_{4}^{i-1}(x), & \text { if } x \in\left[x_{i-1} ; x_{i}\right], \\
\varphi_{3}^{i}(x), & \text { if } x \in\left[x_{i} ; x_{i+1}\right],\end{cases}
\end{align*}
$$

then

$$
\begin{align*}
\phi_{1}^{N}(x) & =\varphi_{2}^{N-1}(x) \text { if } x \in\left[x_{N-1} ; 1\right] \text { with } \phi_{2}^{N}(x) \\
& =\varphi_{4}^{N-1}(x) \text { if } x \in\left[x_{N-1} ; 1\right] \tag{11}
\end{align*}
$$

where

$$
\begin{equation*}
\phi_{1}^{i}(0)=\phi_{2}^{i}(0)=\left(\phi_{1}^{i}\right)^{\prime}(0)=\left(\phi_{2}^{i}\right)^{\prime}(0)=0 \tag{12}
\end{equation*}
$$

We set, respectively, $u_{i}^{n}$ and $v_{i}^{n}$ the approximate values of $u^{n}\left(x_{i}\right)$ and $v^{n}\left(x_{i}\right)$, and we set, for $n \in \mathbb{N}^{*}$ the following equation:

$$
\begin{align*}
& U^{n}=\left(u_{1}^{n}, \ldots, u_{N}^{n}, \frac{\partial u_{1}^{n}}{\partial x}, \ldots, \frac{\partial u_{N}^{n}}{\partial x}\right)^{t}, \\
& V^{n}=\left(v_{1}^{n}, \ldots, v_{N}^{n}, \frac{\partial v_{1}^{n}}{\partial x}, \ldots, \frac{\partial v_{N}^{n}}{\partial x}\right)^{t} . \tag{13}
\end{align*}
$$

Then, we get

$$
\begin{align*}
U^{n} & =\sum_{i=1}^{N} \phi_{1}^{i}(x) u_{i}^{n}+\sum_{i=1}^{N} \phi_{2}^{i}(x) \frac{\partial u_{i}^{n}}{\partial x},  \tag{14}\\
V^{n} & =\sum_{i=1}^{N} \phi_{1}^{i}(x) v_{i}^{n}+\sum_{i=1}^{N} \phi_{2}^{i}(x) \frac{\partial v_{i}^{n}}{\partial x},
\end{align*}
$$

where the $2 N-$ tuple $\left(\phi_{1}^{1}, \phi_{2}^{1}, \phi_{1}^{2}, \phi_{2}^{2}, \phi_{1}^{3}, \phi_{2}^{3}, \ldots, \phi_{1}^{N}, \phi_{2}^{N}\right)$ which we simply denote in the following $\left(\Phi_{i}\right)_{1 \leq i \leq 2 N}$ constitutes a basis of dimension 2 N of the interpolation space,

$$
\begin{equation*}
V_{h}=\left\{v \in \mathscr{C}^{0}([0,1]) / v_{\mid\left[x_{i} ; x_{i+1}\right]} \in P_{3}, 1 \leq i \leq N, v(0)=v^{\prime}(0)=v^{\prime \prime}(1)=0\right\} . \tag{15}
\end{equation*}
$$

Equalities (14) are then written as follows:

$$
\begin{align*}
U^{n} & =\sum_{i=1}^{2 N} \Phi_{i}(x) u_{i}^{n+1}  \tag{16}\\
V^{n} & =\sum_{i=1}^{2 N} \Phi_{i}(x) v_{i}^{n+1}
\end{align*}
$$

and variational problem (9) becomes, for all $j=1,2, \ldots, 2 N$

$$
\left\{\begin{array}{l}
\sum_{i=1}^{2 N}\left(2 u_{i}^{n+1}-\Delta t v_{i}^{n+1}\right) \int_{0}^{1} \Phi_{i}(x) \Phi_{j}(x) d x=\sum_{i=1}^{2 N}\left(2 u_{i}^{n}+\Delta t v_{i}^{n}\right) \int_{0}^{1} \Phi_{i}(x) \Phi_{j}(x) d x  \tag{17}\\
\sum_{i=1}^{2 N}\left[\int_{0}^{1}\left(\Delta t \cdot E I(x) \partial_{x x} \Phi_{i}(x) \partial_{x x} \Phi_{j}(x) u_{i}^{n+1}+2 m(x) \Phi_{i}(x) \Phi_{j}(x) v_{i}^{n+1}\right) d x+\Phi_{i}(1) \Phi_{j}(1)\left(\alpha u_{i}^{n+1}+\beta v_{i}^{n+1}\right)\right] \\
\sum_{i=1}^{2 N}\left[\int_{0}^{1}\left(-\Delta t \cdot E I(x) \partial_{x x} \Phi_{i}(x) \partial_{x x} \Phi_{j}(x) u_{i}^{n}+2 m(x) \Phi_{i}(x) \Phi_{j}(x) v_{i}^{n}\right) d x+\Phi_{i}(1) \Phi_{j}(1)\left(-\alpha u_{i}^{n}+\beta v_{i}^{n}\right)\right]
\end{array}\right.
$$

2.2. Stability of the Scheme. Assuming the linear mass $m=$ $m(x)$ and the bending stiffness of the beam (EI) are constants, system (17) is written at any fixed point $x_{j}(1 \leq j \leq 2 N)$,

$$
\left(\begin{array}{cc}
\frac{2}{\Delta t} & -1  \tag{18}\\
1 & \frac{2 a / \Delta t+\beta c}{b+\alpha c}
\end{array}\right)\binom{U_{j}^{n+1}}{V_{j}^{n+1}}=\left(\begin{array}{cc}
\frac{2}{\Delta t} & 1 \\
-1 \frac{2 a / \Delta t-\beta c}{b+\alpha c}
\end{array}\right)\binom{U_{j}^{n}}{V_{j}^{n}}
$$

where

$$
\begin{align*}
& a=\int_{0}^{1}\left(\partial_{x x} \phi_{j}(x)\right)^{2} d x \\
& b=\int_{0}^{1}\left(\phi_{j}(x)\right)^{2} d x  \tag{19}\\
& c=\phi_{j}^{2}(1) \tag{22}
\end{align*}
$$

$$
\binom{U_{j}^{n+1}}{V_{j}^{n+1}}=\frac{1}{\delta}\left(\begin{array}{cc}
\delta & \frac{2 \beta c}{b+\alpha c}  \tag{20}\\
& 1+\frac{2 / \Delta t(2 a / \Delta t-\beta c)}{b+\alpha c}
\end{array}\right)\binom{U_{j}^{n}}{V_{j}^{n}}
$$

with

$$
\begin{equation*}
\delta=1+\frac{2 / \Delta t(2 a / \Delta t+\beta c)}{b+\alpha c} \tag{21}
\end{equation*}
$$

To determine the associated Von-Neumann stability condition, we set $\left(U_{j}^{n} ; V_{j}^{n}\right)=\left(e^{i k x_{j}} \hat{U}^{n} ; e^{i k x_{j}} \hat{V}^{n}\right)$. System (20) then becomes as follows:

$$
\binom{\hat{U}^{n+1}}{\hat{V}^{n+1}}=\binom{1 \frac{2 \beta c}{b+\alpha c+2 / \Delta t(2 a / \Delta t+\beta c)}}{0 \frac{b+\alpha c+2 / \Delta t(2 a / \Delta t-\beta c)}{b+\alpha c+2 / \Delta t(2 a / \Delta t+\beta c)}}\binom{\hat{U}^{n}}{\hat{V}^{n}}
$$

We deduce

The eigenvalues $\lambda_{1}=1$ and $\lambda_{2}=b+\alpha c+2 / \Delta t(2 a / \Delta t$ $-\beta c) / b+\alpha c+2 / \Delta t(2 a / \Delta t+\beta c)$ of the amplification matrix associated with (22) satisfy the Von-Neumann stability condition,

$$
\begin{equation*}
\left|\lambda_{i}\right| \leq 1, i=1,2 . \tag{23}
\end{equation*}
$$

We then deduce the stability of the numerical scheme (17).

## 3. Influence of Control Parameters on the Beam Stabilization

3.1. Stabilization in Motion. If we set

$$
\begin{align*}
M_{m} & =\left(\int_{0}^{1} m \Phi_{i}(x) \Phi_{j}(x) d x\right)_{1 \leq i, j \leq 2 N} \\
S & =\left(\Phi_{i}(1), \Phi_{j}(1)\right)_{1 \leq i, j \leq 2 N}  \tag{24}\\
K & =\left(\int_{0}^{1} E I(x) \partial_{x x} \Phi_{i}(x) \partial_{x x} \Phi_{j}(x) d x\right)_{1 \leq i, j \leq 2 N}
\end{align*}
$$

numerical scheme (17) is then written as follows:

$$
\begin{align*}
P Y^{n+1} & =Q Y^{n}, \\
Y^{0} & =Y_{0}, \tag{25}
\end{align*}
$$

where

$$
\begin{align*}
& P=\left(\begin{array}{cc}
2 I_{2 N} & -\Delta t I_{2 N} \\
\Delta t(K+\alpha S) & 2 M_{m}+\beta \Delta t S
\end{array}\right),  \tag{26}\\
& Q=\left(\begin{array}{cc}
2 I_{2 N} & \Delta t I_{2 N} \\
-\Delta t(K+\alpha S) & 2 M_{m}-\beta \Delta t S
\end{array}\right),
\end{align*}
$$

$I_{2 N}$ denotes the unit matrix of order $2 N$.
For the simulation, we set the following numerical values as follows:

$$
\begin{align*}
E I & =0,004725  \tag{27}\\
m & =0,78
\end{align*}
$$

and

$$
\begin{align*}
N & =10 \\
T & =1,5  \tag{28}\\
\Delta t & =0,001
\end{align*}
$$

for the model parameters. In addition, we note for the initial conditions as follows:

$$
\begin{align*}
& u_{0}(x)=x^{3}-3 x^{2} \\
& v_{0}(x)=0 \tag{29}
\end{align*}
$$

We also choose different values for each parameters $\alpha$ and $\beta$ involved in problem (1), in order to highlight their influence on the system. Thus, the following results are depicted on MATLAB [5].
3.2. Stabilization in Energy. If we multiply the first equation of system (1) by the test function $\varphi \in H_{E}^{2}([0 ; 1])$, we get, after two integrations by parts, the following variational formulation:

$$
\begin{align*}
& \int_{0}^{1} m(x) \frac{\partial^{2} u(t, x)}{\partial t^{2}} \varphi(x) d x+\int_{0}^{1} E I(x) \frac{\partial^{2} u(t, x)}{\partial x^{2}} \frac{\partial^{2} \varphi(x)}{\partial x^{2}} d x \\
& \quad+\left(\alpha u(t, 1)+\beta \frac{\partial u(t, 1)}{\partial t}\right) \varphi(1)=0 . \tag{30}
\end{align*}
$$

By replacing in (30) the test function $\varphi$ by $\partial u(t, x) / \partial x$, we get the following equation:

$$
\begin{equation*}
\frac{\mathrm{d} E(t)}{\mathrm{d} t}+\beta\left(\frac{\partial u(t, 1)}{\partial t}\right)^{2}=0 \tag{31}
\end{equation*}
$$

where the total energy of the system is written as follows:

$$
\begin{align*}
E(t)= & \frac{1}{2} \int_{0}^{1} m(x)\left(\frac{\partial u(t, x)}{\partial x}\right)^{2} d x+\frac{1}{2} \int_{0}^{1} E I(x)\left(\frac{\partial^{2} u(t, x)}{\partial x^{2}}\right)^{2} d x \\
& +\frac{\alpha}{2}(u(t, 1))^{2} . \tag{32}
\end{align*}
$$

Then, we deduce from relations (31) and (32) that

$$
\begin{equation*}
\frac{\mathrm{d} E(t)}{\mathrm{d} t} \leq 0 \tag{33}
\end{equation*}
$$

Therefore, the energy $E(t)$ defines a Lyapunov function and satisfies

$$
\begin{equation*}
0 \leq E(t) \leq E(0)<\infty, \quad \forall t \in \mathbb{R}_{+} . \tag{34}
\end{equation*}
$$

Relative to the Crank-Nicolson scheme used in system (8), we set, for $0 \leq n \leq N_{t}$,

$$
\begin{equation*}
E^{n}=\frac{1}{2} \int_{0}^{1} m(x)\left(v^{n}\right)^{2} d x+\frac{1}{2} \int_{0}^{1} E I(x)\left(\frac{\partial^{2} u^{n}}{\partial x^{2}}\right)^{2} d x+\frac{\alpha}{2}\left(u^{n}(1)\right)^{2}, \tag{35}
\end{equation*}
$$

the associate discrete energy. Then we get the following result.

Proposition 1. For any integer $n$, there is a positive real $\lambda_{n}$ such that

$$
\begin{equation*}
\frac{E^{n+1}-E^{n}}{\Delta t}+\lambda_{n}=0 \tag{36}
\end{equation*}
$$

Proof. According to (34), we get

$$
\begin{align*}
E^{n+1}-E^{n}= & \frac{1}{2} \int_{0}^{1} m(x)\left(\left(v^{n+1}\right)^{2}-\left(v^{n}\right)^{2}\right) d x \\
& +\frac{1}{2} \int_{0}^{1} E I(x)\left[\left(\frac{\partial^{2} u^{n+1}}{\partial x^{2}}\right)^{2}-\left(\frac{\partial^{2} u^{n}}{\partial x^{2}}\right)^{2}\right] d x  \tag{37}\\
& +\frac{\alpha}{2}\left[u^{n+1}(1)^{2}-u^{n}(1)^{2}\right] .
\end{align*}
$$




Figure 1: Influence of the parameter $\beta$ on the beam vibrations in $x=0,5$.


Figure 2: Influence of the parameter $\alpha$ on the beam vibrations in $x=0,5$.


Figure 3: Influence of the parameter $\alpha$ on the energy.

For $x \in[0 ; 1]$, we set in the first equation of the variational problem (9) the following equation:

$$
\begin{equation*}
\psi(x)=m(x)\left(v^{n+1}-v^{n}\right) \tag{38}
\end{equation*}
$$

Then, we get

$$
\begin{align*}
& \int_{0}^{1} m(x) \frac{u^{n+1}-u^{n}}{\Delta t}\left(v^{n+1}-v^{n}\right) d x  \tag{39}\\
& \quad=\int_{0}^{1} m(x) \frac{\left(v^{n+1}\right)^{2}-\left(v^{n}\right)^{2}}{2} d x .
\end{align*}
$$

Moreover, if we set

$$
\begin{equation*}
\varphi(x)=u^{n+1} \tag{40}
\end{equation*}
$$

as a test function in the second equation of problem (9), we get the following equation:

$$
\begin{align*}
& \int_{0}^{1} m(x) \frac{v^{n+1}-v^{n}}{\Delta t} u^{n+1} d x+\frac{1}{2} \int_{0}^{1} E I(x)\left(\frac{\partial^{2} u^{n+1}}{\partial x^{2}}+\frac{\partial^{2} u^{n}}{\partial x^{2}}\right) \frac{\partial^{2} u^{n+1}}{\partial x^{2}} d x \\
& +\left[\frac{\alpha}{2}\left(u^{n+1}(1)+u^{n}(1)\right)+\frac{\beta}{2}\left(v^{n+1}(1)+v^{n}(1)\right)\right] u^{n+1}(1)=0 . \tag{41}
\end{align*}
$$

Analogously, if we choose as test function $\varphi(x)=u^{n}$, the second equation of system (8) is written as follows:

$$
\begin{align*}
& \int_{0}^{1} m(x) \frac{v^{n+1}-v^{n}}{\Delta t} u^{n} d x+\frac{1}{2} \int_{0}^{1} E I(x)\left(\frac{\partial^{2} u^{n+1}}{\partial x^{2}}+\frac{\partial^{2} u^{n}}{\partial x^{2}}\right) \frac{\partial^{2} u^{n}}{\partial x^{2}} d x \\
& +\left[\frac{\alpha}{2}\left(u^{n+1}(1)+u^{n}(1)\right)+\frac{\beta}{2}\left(v^{n+1}(1)+v^{n}(1)\right)\right] u^{n}(1)=0 . \tag{42}
\end{align*}
$$

By differencing (41) and (42) and relative the fact that

$$
\begin{equation*}
u^{n+1}-u^{n}=\frac{\Delta t}{2}\left(v^{n+1}+v^{n}\right) \tag{43}
\end{equation*}
$$

we get

$$
\begin{align*}
\frac{E^{n+1}-E^{n}}{\Delta t}+\lambda_{n} & =0, \text { for } n \in \mathbb{N} \\
\text { where } \lambda_{n} & =\beta\left(\frac{v^{n+1}(1)+v^{n}(1)}{2}\right)^{2} \tag{44}
\end{align*}
$$

Consequently, we state the following result.
Corollary 1. For any integer $n \geq 1$, we get

$$
\begin{equation*}
E^{n+1} \leq E^{n} \leq E^{0} \tag{45}
\end{equation*}
$$

Finally, we notice that the energy of the discrete system decreases and is controlled by that of the initial data. For the numerical simulation of the energy, we deduce from Proposition 1 the following equality:

$$
\begin{equation*}
E^{n+1}=E^{n}-\beta \frac{\Delta t}{4}\left[v^{n+1}(1)+v^{n}(1)\right]^{2}, \quad 0 \leq n \leq N_{t} \tag{46}
\end{equation*}
$$

where

$$
\begin{equation*}
E^{0}=\frac{E I}{2} \int_{0}^{1}\left(u_{0}^{\prime \prime}(x)\right)^{2} d x+\frac{\alpha}{2}\left(u_{0}(1)\right)^{2} \tag{47}
\end{equation*}
$$

Under initial conditions (29) and relation (35), we set

$$
\begin{equation*}
\Delta t=0,001 \tag{48}
\end{equation*}
$$

Curves above are depicted on MATLAB [5].
3.3. Results and Discussion. By scrutinizing the curves depicted in Figures 1 and 2, we notice that for a fixed value of $\beta$ and for different values of $\alpha$, the vibrations tend to stabilize at almost the same time. Conversely, if $\alpha$ is fixed, we notice that for larger values of $\beta$ the beam stabilizes much more quickly.

Furthermore, the analysis of Figures 3 and 4 shows that contrary to the parameter $\alpha$, for increasingly large values of $\beta$, the energy of the system decreases much faster from its initial value $E^{0}$ towards 0 .

Finally, we deduce from the graphical results that the velocity control has more impact on the stabilization of the beam than the displacement control.


Figure 4: Influence of the parameter $\beta$ on the energy.

Table 1: Vibrations damping times $\delta \tau$ of the beam (in seconds).

| $\alpha$ |  | $\beta$ |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 0.5 | 0.85 | 1.0 | 2.5 | 3.5 | 0.0 |
| 50 | 1.82 | 1.15 | 0.90 | 0.42 | 0.24 | 0.20 |
| 100 | 1.85 | 1.12 | 0.98 | 0.40 | 0.20 | 0.20 |
| 200 | 1.79 | 1.15 | 0.95 | 0.40 | 0.25 | 0.20 |
| 250 | 1.82 | 1.10 | 0.90 | 0.35 | 0.20 |  |
| 350 | 1.80 |  |  | 0.35 | 0.30 | 0 |

## 4. Sensitivity of Parameters $\alpha$ and $\boldsymbol{\beta}$ on the Stability

4.1. Sensitivity Indexes Values of $\alpha$ and $\beta$. In this part, we objectively study the influence of the parameters $\alpha$ and $\beta$ on the stability of the Euler-Bernoulli beam. For this purpose, we note in the table below the damping time $\delta \tau$ of the beam vibrations, for different values of $\alpha$ and $\beta$.

The first-order Sobol sensitivity index as defined by McKay in [6] is written as follows:

$$
\begin{equation*}
S_{i}=\frac{\operatorname{Var}\left[\mathbb{E}\left(\delta \tau \mid Y_{i}\right)\right]}{\operatorname{Var}(\delta \tau)}, \tag{49}
\end{equation*}
$$

where $\operatorname{Var}(\delta \tau)$ and $\mathbb{E}\left(\delta \tau \mid Y_{i}\right)$ denote, respectively, the variance of $\delta \tau$ and the conditional expectation of $\delta \tau$ obtained by fixing $Y_{i}$. From the values of $\delta \tau$ contained in Table1, if we set, respectively, $Y_{i}=\alpha$ and $Y_{i}=\beta$ in (49). Then we get the following indexes of sensitivity of $\alpha$ and $\beta$.

$$
\begin{align*}
& S_{\alpha} \simeq 0,00016  \tag{50}\\
& S_{\beta} \simeq 0,58450 .
\end{align*}
$$

4.2. Discussion. The comparison of the sensitivity indexes obtained in (50) clearly shows that, compared to the parameter $\alpha$, the parameter $\beta$ has more impact on the stabilization time of the Euler-Bernoulli beam.

Then, the conclusion of the analyses deduced from the graphical results obtained above is therefore confirmed.

## 5. Conclusion

In this work, we have studied the stabilization of the EulerBernoulli beam clamped at one extreme and subjected at the other extreme to a linear combination of displacement and velocity. To do this, we have implemented a numerical stable scheme. With this numerical scheme, we have studied the stabilization of our original model in displacement and in energy, respectively. The analysis of the graphical results allowed us to affirm that the control in velocity allows the beam to stabilize faster than when the control is done in displacement. This result, which we confirmed after a statistical study of the sensitivity of the control parameters, corroborates that obtained in the numerical study of the spectrum of the operator associated with the model by Aouragh and Yebari in [7] and later by Kouassi et al. in [2].

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there is no conflicts of interest regarding the publication of this paper.

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