Brief reminders and demonstration

We begin our study of the behaviour of the measures $\mathrm{M_{GK}}$, OR and OR_{hn}, in this section by using the 21 properties already identified in the work of [Grissa(2013)Grissa] which are recalled below.

Property 1 (P_1) : *Intelligibility or comprehensibility of the measure*

- If the interpretation of m is difficult, then $P_1(m) = 0$;
- If m is reduced to usual quantities, then $P_1(m) = 1$;
- If m can be explained by a sentence, then $P_1(m) = 2$.

It is important that a measurement is intelligible in order to interpret the results obtained.

1. In fact, we admit that $\mathrm{M_{GK}}(X \to Y)$ is the rate of growth of probability of Y under the presence of X.
2. And, the Odds-Ration (OR) measure and the normalized Odds-Ratio (OR_{hn}) are the eventual odds ratio of Y under the presence of X.

Therefore, we have : $\boxed{\mathrm{P_1(M_{GK})} = P_1(OR) = P_1(OR_{hn}) = 2}$.

Property 2 (P_2) : *Ease of setting a threshold for acceptance of the rule*

- f determining the threshold is problematic, then $P_2(m) = 0$;
- If the determination of the threshold is immediate, then $P_2(m) = 1$.

This property is proposed in order to keep interesting rules without having to classify them. And also intelligible, standardised and statistical measures lend themselves well to the determination of this threshold ([Grissa(2013)Grissa]).

1. Since the decision taken from the $\mathrm{M_{GK}}$ measure is based entirely on a critical value, the determination of the acceptance threshold of a rule by this measure is immediate.
2. Secondly, the Odds-Ratio measure is an intelligible and statistical measure but not standardised, so determining the threshold is problematic.
3. And, the OR_{hn} measure is an intelligible, standardised and statistical measure, so the determination of the acceptance threshold of a rule is immediate.

Therefore, we have : $\boxed{\mathrm{P_2(M_{GK})} = P_2(OR_{hn}) = 1 \textit{ and } P_2(OR) = 0}$.

Property 3 (P_3) : *Measure non-symmetrical*

- If m is symmetrical, i.e. if $\forall(X \to Y), m(X \to Y) = m(Y \to X)$, then $P_3(m) = 0$;
- If m is non-symmetrical, i.e. if $\exists(X \to Y)$ such that $m(X \to Y) \neq m(Y \to X)$ then $P_3(m) = 1$.

It is preferable for a measure to evaluate the rules X and Y differently since the premise and conclusion have distinct rationale. Nevertheless, in some cases, the orientation of the link between X and Y may not provide additional information to the user, i.e., whether the applied measures are symmetrical or not, it will not change anything in the obtained results ([Grissa(2013)Grissa]).
The propositions below show the measures OR, OR_{nh} are symmetrical

Proposition 1. *Let X and Y be two patterns, we have the relation :*

$$OR(X \to Y) = OR(Y \to X). \tag{1}$$

Proposition 2. *Let X and Y be two patterns, we have the relation :*

$$OR_{nh}(X \to Y) = OR_{nh}(Y \to X) \tag{2}$$

According to [Feno(2007)Feno], the proposition below shows that the measure M_{GK}^d are symmetric, while M_{GK}^f is non-symmetric.

Proposition 3. *(a) If X favours Y, we have the relationship :*

$$M_{GK}^f(X \to Y) = \frac{1-P(Y)}{1-P(Y)} \frac{P(X)}{P(Y)} M_{GK}^f(Y \to X) \tag{3}$$

(b) If X disfavours Y, we have the relationship :

$$M_{GK}^d(X \to Y) = M_{GK}^d(Y \to X) \tag{4}$$

Demonstrations :
Let A and B be two patterns,

1. Pour $\mathrm{M_{GK}}$:
   In the work of [Feno(2007)Feno], pages: 73, we have well shown that $M_{GK}^f(A \to B) \neq M_{GK}^f(B \to A)$ and $M_{GK}^d(A \to B) = M_{GK}^d(B \to A)$
   Hence the favouring $\mathrm{M_{GK}}$ measure is non-symmetric and the unfavouring $\mathrm{M_{GK}}$ measure is symmetric.

2. For Odd-Ratio :

$$\begin{aligned} OR(A \to B) - OR(B \to A) &= \frac{\mathrm{P}(A \cap B)\mathrm{P}(\overline{A} \cap \overline{B})}{\mathrm{P}(\overline{A} \cap B)\mathrm{P}(A \cap \overline{B})} - \frac{\mathrm{P}(B \cap A)\mathrm{P}(\overline{B} \cap \overline{A})}{\mathrm{P}(\overline{B} \cap A)\mathrm{P}(B \cap \overline{A})} \\ &= \frac{\mathrm{P}(A \cap B)\mathrm{P}(\overline{A} \cap \overline{B})}{\mathrm{P}(\overline{A} \cap B)\mathrm{P}(A \cap \overline{B})} - \frac{\mathrm{P}(B \cap A)\mathrm{P}(\overline{A} \cap \overline{B})}{\mathrm{P}(A \cap \overline{B})\mathrm{P}(\overline{A} \cap B)} \\ &= 0. \end{aligned}$$

   Hence the Odd-Ratio measure is symmetrical

3. For OR_{nh} : Indeed

$$\begin{aligned} OR_{nh}^f(A \to B) - OR_{nh}^f(B \to A) &= \frac{\mathrm{P}_A(B) - \mathrm{P}(B)}{\mathrm{P}_A(B)(1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}_A(B)\mathrm{P}(A))} - \\ &\quad \frac{\mathrm{P}_B(A) - \mathrm{P}(A)}{\mathrm{P}_B(A)(1 - \mathrm{P}(B) - \mathrm{P}(A) + \mathrm{P}_B(A)\mathrm{P}(A))} \\ &= \frac{\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} - \mathrm{P}(B)}{\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} - \mathrm{P}(A \cap B) - \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}\mathrm{P}(B) + \frac{\mathrm{P}^2(A \cap B)}{\mathrm{P}(A)}} - \\ &\quad \frac{\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} - \mathrm{P}(A)}{\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} - \mathrm{P}(A \cap B) - \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}\mathrm{P}(A) + \frac{\mathrm{P}^2(A \cup B)}{\mathrm{P}(B)}} \\ &= \frac{\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B)}{\mathrm{P}(A \cap B)(1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A \cap B))} - \\ &\quad \frac{\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B)}{\mathrm{P}(A \cap B)(1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}(A \cap B))} \\ &= 0. \end{aligned}$$

   Therefore OR_{nh}^f is symmetrical.

For

$$
\begin{aligned}
OR^d_{nh}(A \to B) - OR^d_{nh}(B \to A) &= \frac{\mathrm{P}_A(B) - \mathrm{P}(B)}{(1-\mathrm{P}_A(B))(\mathrm{P}(B) - \mathrm{P}_A(B)\mathrm{P}(A))} - \\
&\quad \frac{\mathrm{P}_B(A) - \mathrm{P}(A)}{(1-\mathrm{P}_B(A))(\mathrm{P}(A) - \mathrm{P}_B(A)\mathrm{P}(B))} \\
&= \frac{\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A)} - \mathrm{P}(B)}{(1-\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A)})(\mathrm{P}(B) - \mathrm{P}(A\cap B))} - \\
&\quad \frac{\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} - \mathrm{P}(A)}{(1-\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)})(\mathrm{P}(A) - \mathrm{P}(A\cap B))} \\
&= \frac{\mathrm{P}(A\cap B) - \mathrm{P}(A)\mathrm{P}(B)}{(\mathrm{P}(A) - \mathrm{P}(A\cap B))(\mathrm{P}(B) - \mathrm{P}(A\cap B))} - \\
&\quad \frac{\mathrm{P}(A\cap B) - \mathrm{P}(A)\mathrm{P}(B)}{(\mathrm{P}(A) - \mathrm{P}(A\cap B))(\mathrm{P}(B) - \mathrm{P}(A\cap B))} \\
&= 0.
\end{aligned}
$$

Therefore OR^d_{nh} is symmetrical.

Hence we have : $\boxed{\mathrm{P}_3(M^d_{GK}) = P_3(OR) = P_3(OR^f_{nh}) = P_3(OR^d_{nh}) = 0 \textit{ and } P_3(M^f_{GK}) = 1}$.

Property 4 (P_4) : *non-symmetrical measure in the sense of negation of the conclusion*

- If m is symmetrical in the sense of the negation of the conclusion, i.e. if $\forall(X \to Y), m(X \to Y) = m(X \to \overline{Y})$, then $P_4(m) = 0$;
- If m is non-symmetric in the sense of the negation of the conclusion, i.e. if $\exists(X \to Y)$ such that $m(X \to Y) \neq m(X \to \overline{Y})$ then $P_4(m) = 1$.

The following propositions show that the measures $\mathrm{M_{GK}}$, OR and OR_{hn} are not symmetric measures in the sense of the negation of the conclusion.

Proposition 4. *According to [Feno(2007)Feno], let X and Y be two positive patterns. We have the following equality :*

$$M_{GK}(X \to \overline{Y}) = -M_{GK}(X \to Y). \tag{5}$$

Proposition 5. *Let X and Y be two positive patterns. We have the following equality :*

$$OR(X \to \overline{Y}) = \frac{1}{OR(X \to Y)}. \tag{6}$$

Proposition 6. *Let X and Y be two positive patterns.*

(a) If X favours Y, we have the following equality :

$$OR^f_{hn}(X \to \overline{Y}) = -OR^d_{hn}(X \to Y) \tag{7}$$

(b) If X disfavours Y, we have the following equality :

$$OR^d_{hn}(X \to \overline{Y}) = -OR^f_{hn}(X \to Y) \tag{8}$$

Demonstrations :

1. For the measure $\mathrm{M_{GK}}$
   In the work of [Feno(2007)Feno], pages : 75, we have well shown that $M^f_{GK}(X \to \overline{Y}) = -M^f_{GK}(X \to Y) \neq M^f_{GK}(X \to Y)$ and $M^d_{GK}(X \to \overline{Y}) = -M^d_{GK}(X \to Y) \neq M^d_{GK}(X \to Y)$.
   Therefore $\mathrm{M_{GK}}$ is not symmetric in the sense of the negation of the conclusion.

2. For the OR measure
   Indeed

$$
\begin{aligned}
OR(X \to \overline{Y}) &= \frac{\mathrm{P}(X \cap \overline{Y})\mathrm{P}(\overline{X} \cap \overline{\overline{Y}})}{\mathrm{P}(\overline{X} \cap \overline{Y})\mathrm{P}(X \cap \overline{\overline{Y}})} \\
&= \frac{\mathrm{P}(X \cap \overline{Y})\mathrm{P}(\overline{X} \cap Y)}{\mathrm{P}(\overline{X} \cap \overline{Y})\mathrm{P}(X \cap Y)} \\
&= \frac{1}{OR(X \to Y)}.
\end{aligned}
$$

   We obtain : $OR(X \to \overline{Y}) \neq OR(X \to Y)$
   So OR is not symmetric in the sense of the negation of the conclusion.

3. For the measure OR_{hn}
   Indeed

$$
\begin{aligned}
OR^f_{hn}(X \to \overline{Y}) &= \frac{\mathrm{P}_X(\overline{Y}) - \mathrm{P}(\overline{Y})}{\mathrm{P}_X(\overline{Y})(1 - \mathrm{P}(X) - \mathrm{P}(\overline{Y}) + \mathrm{P}_X(\overline{Y})\mathrm{P}(X))} \\
&= \frac{1 - \mathrm{P}_X(Y) - 1 + \mathrm{P}(Y)}{(1 - \mathrm{P}_X(Y))(1 - \mathrm{P}(X) - 1 + \mathrm{P}(Y) + (1 - \mathrm{P}_X(Y))\mathrm{P}(X))} \\
&= -\frac{P(Y'/X') - P(Y')}{(1 - P(Y'/X'))(P(Y') - P(Y'/X')P(X'))} \\
&= -OR^d_{hn}(X \to Y)
\end{aligned}
$$

   So $OR^f_{hn}(X \to \overline{Y}) \neq OR^f_{hn}(X \to Y)$
   And

$$
\begin{aligned}
OR^d_{hn}(X \to \overline{Y}) &= \frac{\mathrm{P}_X(\overline{Y}) - \mathrm{P}(\overline{Y})}{(1 - \mathrm{P}_X(\overline{Y}))(\mathrm{P}(\overline{Y}) - \mathrm{P}_X(\overline{Y})\mathrm{P}(X))} \\
&= \frac{1 - \mathrm{P}_X(Y) - 1 + \mathrm{P}(Y)}{(1 - 1 + \mathrm{P}_X(Y))(1 - \mathrm{P}(Y) - (1 - \mathrm{P}_X(Y))\mathrm{P}(X))} \\
&= -\frac{P(Y'/X') - P(Y')}{P(Y'/X')(1 - P(X') - P(Y') + P(Y'/X')P(X'))} \\
&= -OR^f_{hn}(X \to Y)
\end{aligned}
$$

   So $OR^d_{hn}(X \to \overline{Y}) \neq OR^d_{hn}(X \to Y)$
   Hence OR_{hn} is not symmetric in the sense of the negation of the conclusion.
   Hence we have : $\boxed{\mathrm{P}_4(M_{GK}) = P_4(OR) = P_4(OR_{nh}) = 1}$.

Property 5 (P_5) : *MMeasure evaluating $(X \to Y)$ and $\overline{Y} \to \overline{X}$ in the same way in the case of logical implication*

- If $\exists (X \to Y)$ such that $\mathrm{P}_X(Y) = 1$ and $m(X \to Y) \neq m(\overline{Y} \to \overline{X})$, then $P_5(m) = 0$;
- If $\forall (X \to Y)$, such that $\mathrm{P}_X(Y) = 1 \implies m(X \to Y) = m(\overline{X} \to \overline{Y})$ then $P_5(m) = 1$.

The following propositions show that ${\mathrm{M_{GK}}}^f$, OR and OR_{hn} are implicative measures, while ${\mathrm{M_{GK}}}^d$ is not an implicative measure.

Proposition 7. *(a) According to [Feno(2007)Feno], if X favours Y, we have the equivalence relation of the two contraposed rules :*

$$\mathrm{M_{GK}}(\overline{Y} \to \overline{X}) = \mathrm{M_{GK}}(X \to Y). \tag{9}$$

(b) If X disfavours Y, we have the following relationship :

$$\mathrm{M_{GK}}(\overline{Y} \to \overline{X}) = \frac{P(X)P(Y)}{(1 - P(X))(1 - P(Y))}\mathrm{M_{GK}}(X \to Y). \tag{10}$$

Proposition 8. *Let X and Y be two patterns, we have the equivalence relation of the two contraposed rules :*

$$OR(\overline{Y} \to \overline{X}) = OR(X \to Y). \tag{11}$$

Proposition 9. *Let X and Y be two patterns, we have the equivalence relation of the two contraposed rules :*

$$OR_{hn}(\overline{Y} \rightarrow \overline{X}) = OR_{hn}(X \rightarrow Y). \tag{12}$$

Demonstrations :

Let A and B be two patterns and $\forall A \rightarrow B \in \mathbb{K}(P; C, R)$,

1. For the measure $\mathrm{M_{GK}}$
   In the work of [Feno(2007)Feno], pages: 74, we have well shown that $M^f_{GK}(A \rightarrow B) = M^f_{GK}(\overline{B} \rightarrow \overline{A})$ and $M^d_{GK}(A \rightarrow B) \neq M^d_{GK}(\overline{B} \rightarrow \overline{A})$
   Therefore M^f_{GK} is an implicative measure and M^d_{GK} is not an implicative measure. If $\mathrm{P}_A(B) = 1$, we have $M^f_{GK}(A \rightarrow B) = M^f_{GK}(\overline{B} \rightarrow \overline{A}) = \dfrac{1 - \mathrm{P}(B)}{1 - \mathrm{P}(B)} = 1$.
   And for the case of M^d_{GK}
   If $\mathrm{P}_A(B) = 1$, we have $M^d_{GK}(\overline{B} \rightarrow \overline{A}) = \dfrac{1}{M^d_{GK}(A \rightarrow B)} \neq M^d_{GK}(A \rightarrow B)$

2. For the OR measure
   By definition

$$\begin{aligned}
OR(\overline{B} \rightarrow \overline{A}) &= \frac{\mathrm{P}(\overline{B} \cap \overline{A})\mathrm{P}(\overline{\overline{B}} \cap \overline{\overline{A}})}{\mathrm{P}(\overline{\overline{B}} \cap \overline{A})\mathrm{P}(\overline{B} \cap \overline{\overline{A}})} \\
&= \frac{\mathrm{P}(\overline{B} \cap \overline{A})\mathrm{P}(B \cap A)}{\mathrm{P}(B \cap \overline{A})\mathrm{P}(\overline{B} \cap A)} \\
&= \frac{\mathrm{P}(\overline{A} \cap \overline{B})\mathrm{P}(A \cap B)}{\mathrm{P}(A \cap \overline{B})\mathrm{P}(\overline{A} \cap B)} \\
&= OR(A \rightarrow B).
\end{aligned}$$

   So OR is an implicit measure.
   If $\mathrm{P}_A(B) = 1$, we have $OR(\overline{B} \rightarrow \overline{A}) = OR(A \rightarrow B) = +\infty$

3. For the measure OR_{hn}
   Indeed

$$\begin{aligned}
OR_{{f}nh(\overline{B} \rightarrow \overline{A})} &= \frac{\mathrm{P}_{\overline{B}}(\overline{A}) - \mathrm{P}(\overline{A})}{\mathrm{P}_{\overline{B}}(\overline{A})(1 - \mathrm{P}(\overline{B}) - \mathrm{P}(\overline{A}) + \mathrm{P}_{\overline{B}}(\overline{A})\mathrm{P}(\overline{B}))} \\
&= \frac{\mathrm{P}_{\overline{B}}(\overline{A})(1 - \frac{\mathrm{P}(\overline{A})}{\mathrm{P}_{\overline{B}}(\overline{A})})}{\mathrm{P}_{\overline{B}}(\overline{A})(1 - \mathrm{P}(\overline{B}) - \mathrm{P}(\overline{A}) + \mathrm{P}_{\overline{B}}(\overline{A})\mathrm{P}(\overline{B}))} \\
&= \frac{1 - \frac{1 - \mathrm{P}(A)}{1 - \frac{\mathrm{P}(A \cap \overline{B})}{\mathrm{P}(\overline{B})}}}{\mathrm{P}(A) - \frac{\mathrm{P}(A \cap \overline{B})}{\mathrm{P}(\overline{B})} + \frac{\mathrm{P}(A \cap \overline{B})\mathrm{P}(B)}{\mathrm{P}(\overline{B})}} \\
&= \frac{\frac{\mathrm{P}_A(B)\mathrm{P}(A) - \mathrm{P}(A)\mathrm{P}(B)}{1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}_A(B)\mathrm{P}(A)}}{\frac{\mathrm{P}_A(B)\mathrm{P}(A) - \mathrm{P}_A(B)\mathrm{P}(A)\mathrm{P}(B)}{1 - \mathrm{P}(B)}} \\
&= \frac{\mathrm{P}_A(B) - \mathrm{P}(B)}{\mathrm{P}_A(B)\big(1 - \mathrm{P}(A) - \mathrm{P}(B) + \mathrm{P}_A(B)\mathrm{P}(A)\big)} \\
&= OR^f_{nh}(A \rightarrow B).
\end{aligned}$$

   So OR^f_{hn} is an implicit measure.
   If $\mathrm{P}_A(B) = 1$, we have $OR^f_{nh}(\overline{B} \rightarrow \overline{A}) = OR^f_{nh}(A \rightarrow B) = \dfrac{1 - \mathrm{P}(B)}{1 - \mathrm{P}(B)} = 1$.

At the end

$$\begin{aligned}
OR^d_{hn}(\overline{B} \to \overline{A}) &= \frac{\mathrm{P}_{\overline{B}}(\overline{A}) - \mathrm{P}(\overline{A})}{\big(1-\mathrm{P}_{\overline{B}}(\overline{A})\big)\big(\mathrm{P}(\overline{A}) - \mathrm{P}_{\overline{B}}(\overline{A})\mathrm{P}(\overline{B})\big)} \\
&= \frac{\mathrm{P}(A) - \frac{\mathrm{P}(A\cap\overline{B})}{\mathrm{P}(\overline{B})}}{\left(\frac{\mathrm{P}(A\cap\overline{B})}{\mathrm{P}(\overline{B})}\right)\left(\mathrm{P}(B) - \mathrm{P}(A) + \frac{\mathrm{P}(A\cap\overline{B})}{\mathrm{P}(\overline{B})} - \frac{\mathrm{P}(A\cap\overline{B})\mathrm{P}(B)}{\mathrm{P}(\overline{B})}\right)} \\
&= \frac{\mathrm{P}(A) - \frac{\mathrm{P}(A)-\mathrm{P}_A(B)\mathrm{P}(A)}{1-\mathrm{P}(B)}}{\left(\frac{\mathrm{P}(A)-\mathrm{P}_A(B)\mathrm{P}(A)}{1-\mathrm{P}(B)}\right)\left(\mathrm{P}(B) - \mathrm{P}(A) + \frac{\mathrm{P}(A)-\mathrm{P}_A(B)\mathrm{P}(A)}{1-\mathrm{P}(B)} - \frac{\mathrm{P}(A)\mathrm{P}(B)-\mathrm{P}_A(B)\mathrm{P}(A)\mathrm{P}(B)}{1-\mathrm{P}(B)}\right)} \\
&= \frac{\big(1-\mathrm{P}(A)\big)\big(\mathrm{P}_A(B) - \mathrm{P}(B)\big)}{\big(1-\mathrm{P}_A(B)\mathrm{P}(A)\big)\big(1-\mathrm{P}(B)\big)\big(\mathrm{P}(B) - \mathrm{P}_A(B)\mathrm{P}(A)\big)} \\
&= \frac{\mathrm{P}_A(B) - \mathrm{P}(B)}{\big(1-\mathrm{P}_A(B)\big)\big(\mathrm{P}(B) - \mathrm{P}_A(B)\mathrm{P}(A)\big)} \\
&= OR^d_{nh}(A \to B).
\end{aligned}$$

Therefore OR^d_{hn} is an implicit measure.
If $\mathrm{P}_A(B) = 1$, we have $OR^d_{nh}(\overline{B} \to \overline{A}) = OR^d_{nh}(A \to B) = +\infty$

Hence we have : $\boxed{\mathrm{P}_5(M^f_{GK}) = P_5(OR) = P_5(OR_{nh}) = 1 \textit{ and } P_5(M^d_{GK}) = 0.}$.

Propriété 6 (P_6) : *Increasing measure according to the number of examples*

- If m is not increasing as a function of n_{XY}, then $P_6(m) = 0$;
- If m is increasing as a function of n_{XY}, then $P_6(m) = 1$.

The following propositions show that the measures $\mathrm{M_{GK}}$, OR and OR^d_{hn} are increasing measures as a function of n_{XY}, while OR^f_{hn} is not an increasing measure as a function of n_{XY}.

Proposition 10. *(a) If X favours Y, and n_{XY} the variable, with $a = \dfrac{1}{n_X(n-n_Y)} > 0$ and $b = -\dfrac{n_X n_Y}{n_X(n-n_Y)}$, we have the following function :*

$$M_{GK}(X \to Y) = a.n_{XY} + b. \tag{13}$$

(b) If X disfavours Y, and n_{XY} the variable, with $a = \frac{1}{n_Y.n_Y} > 0$, we have the following function :

$$M_{GK}(X \to Y) = a.n_{XY} - 1. \tag{14}$$

Proposition 11. *If X favours Y, and n_{XY} the variable, with $a = frac1n_Y.n_Y > 0$, we have the following function :*

$$OR(X \to Y) = a.n_{XY}. \tag{15}$$

Proposition 12. *(a) Let X and Y be two patterns of the context $\mathbb{K}$ and the variable n_{XY}, with $a = n - n_X - n_Y$ and $b = n_Y$, we have the following function :*

$$OR_{nh}(X \to Y) = \frac{1}{n_{XY}+a} - \frac{b}{n_{XY}(n_{XY}+a)}. \tag{16}$$

(b) If X disfavours Y, and n_{XY} the variable, with $a = \dfrac{1}{(n.n_X - n_{XY})(n_Y - n_{XY})} > 0$ and $b = -\dfrac{n_X n_Y}{(n.n_X - n_{XY})(n_Y - n_{XY})}$, $n.n_X \neq n_{XY}$ and $n_Y \neq n_{XY}$; we have the following function :

$$OR_{nh}(X \to Y) = a.n_{XY} + b. \tag{17}$$

Demonstrations :

1. Let X and Y be two patterns of the context $\mathbb{K}$, such that $n_X.n_Y \neq 0$, we define : $M^f_{GK}(X \to Y) = \dfrac{n_{XY} - n_X n_Y}{n_X(n-n_Y)}$. By posing $a = \dfrac{1}{n_X(n-n_Y)}$ and $b = -\dfrac{n_X n_Y}{n_X(n-n_Y)}$, we have: $M_{GK}(X \to Y) = a.n_{XY} + b$. As $n_X, n_Y \in \mathbb{N}^*$ and $n > n_Y$, then we have : $n_X(n-n_Y) > 0$ and $a = \dfrac{1}{n_X(n-n_Y)} > 0$.
   Therefore M^f_{GK} is an increasing function of n_{XY}.
   In the end, we have : $M^f_{GK} = -a.n_{X\overline{Y}} + a.n_X + b$. By posing $c = a.n_X + b$, we have : $M^f_{GK}(X \to Y) = -a.n_{X\overline{Y}} + c$ which is a decreasing function of $n_{X\overline{Y}}$.
   Then, $M^d_{GK}(X \to Y) = \dfrac{n_{XY} - n_X n_Y}{n_X.n_Y}$. By fixing the value of n_X and n_Y we obtain : $M^d_{GK}(X \to Y) = a.n_{XY} - 1$. Since n_Y, $n_Y \in \mathbb{N}^*$, then $a = \frac{1}{n_Y.n_Y} > 0$.
   Therefore $M^d_{GK}(X \to Y) = a.n_{XY} - 1$ is increasing by n_{XY}. Moreover, we have : $n_{XY} = n_X - n_{X\overline{Y}}$. Therefore $M^d_{GK}(X \to Y) = -a.n_{X\overline{Y}} + a.n_X - 1$. It comes $M^d_{GK}(X \to Y) = -a.n_{X\overline{Y}} + \lambda$,with $\lambda = a.n_X - 1$, is a decreasing function of $n_{X\overline{Y}}$.

2. Then, $OR(X \to Y) = \dfrac{n_{XY}.n_{\overline{X}.\overline{Y}}}{n_{\overline{X}Y}.n_{X\overline{Y}}}$. As $n_{\overline{X}.\overline{Y}}, n_{\overline{X}Y}, n_{X\overline{Y}} \in \mathbb{N}^*$, as $a = \dfrac{n_{\overline{X}.\overline{Y}}}{n_{\overline{X}Y}.n_{X\overline{Y}}} > 0$.
   Therefore $OR(X \to Y) = a.n_{XY}$ is increasing by n_{XY}. Moreover, we have $OR(X \to Y) = -a.n_{X\overline{Y}} + a.n_X$. We obtain $OR(X \to Y) = -a.n_{X\overline{Y}} + \beta$ with $\beta = a.n_X$ is a decreasing function of $n_{X\overline{Y}}$.

3. After, $OR^f_{nh}(X \to Y) = \dfrac{\frac{n_{XY}}{n_X} - n_Y}{\frac{n_{XY}}{n_X}(n - n_X - n_Y + n_{XY})}$
   Therefore, $OR^f_{nh}(X \to Y) = \dfrac{n_{XY} - n_Y}{n_{XY}(n - n_X - n_Y + n_{XY})}$.
   It comes $OR^f_{nh}(X \to Y) = \dfrac{1}{n_{XY} + n - n_X - n_Y} - \dfrac{n_Y}{n_{XY}(n_{XY} + n - n_X - n_Y)} = \dfrac{1}{n_{XY} + a} - \dfrac{b}{n_{XY}(n_{XY} + a)}$, with $a = n - n_X - n_Y$ and $b = n_Y$ is not increasing as a function of n_{XY}.
   Then, $OR^d_{nh}(X \to Y) = \dfrac{\frac{n_{XY}}{n_X} - n_Y}{\left(n - \frac{n_{XY}}{n_X}\right)\left(n_Y - \frac{n_{XY}}{n_X} n_X\right)})$
   Therefore, $OR^d_{nh}(X \to Y) = \dfrac{n_{XY} - n_X n_Y}{(n.n_X - n_{XY})(n_Y - n_{XY})}$
   We obtain, $OR^d_{nh}(X \to Y) = \dfrac{1}{(n.n_X - n_{XY})(n_Y - n_{XY})} n_{XY} - \dfrac{n_X n_Y}{(n.n_X - n_{XY})(n_Y - n_{XY})} = a.n_{XY} + b$, with $a = \dfrac{1}{(n.n_X - n_{XY})(n_Y - n_{XY})}$ and $b = -\dfrac{n_X n_Y}{(n.n_X - n_{XY})(n_Y - n_{XY})}$ is increasing as a function of n_{XY}.
   Finally, $OR^d_{nh}(X \to Y) = -a.n_{X\overline{Y}} + a.n_X + b$. By posing : $c = a.n_X + b$, we have : $OR^d_{nh}(X \to Y) = -a.n_{X\overline{Y}} + c$ which is a decreasing function of $n_{X\overline{Y}}$.

Hence we have : $\boxed{\mathrm{P}_6(M_{GK}) = P_6(OR) = P_6(OR^d_{nh}) = 1 \textit{ and } P_6(OR^f_{nh}) = 0.}$.

Property 7 (P_7) : *Increasing measure as a function of the size of the learning set*

- $P_7(m) = 0$, sif m is not increasing as a function of n;
- Let $P_7(m) = 1$, if m is increasing as a function of n.

LThe following propositions show that the measures $\mathrm{M_{GK}}$, OR and OR_{hn} are increasing measures as a function of n.

Proposition 13. *(a) If X favours Y, and n is a variable with* $\alpha = \dfrac{n_{XY}}{n_X(n-n_Y)} > 0$, $\beta =$

$\dfrac{n_X.n_Y}{n_X.(n-n_Y)}$ *and* $n \neq n_Y$ *; we have the following function expression :*

$$M_{GK}(X \to Y) = \alpha.n - \beta. \tag{18}$$

(b) If X *disfavours* Y*, and* n *is a variable with* $\Theta = \frac{n_{XY}}{n_X.n_Y} > 0$*, we have the following function expression :*

$$M_{GK}(X \to Y) = \Theta.n - 1. \tag{19}$$

Proposition 14. *Let* X *and* Y *be two patterns, the variable* n *with* $a = \dfrac{n_{XY}}{(n_Y - n_{XY})(n_X - n_{XY})} > 0$ *and* $b = \dfrac{(n_{XY} - n_X - n_Y).n_{XY}}{(n_Y - n_{XY})(n_X - n_{XY})}$*,* $n_Y \neq n_{XY}$*,* $n_X \neq n_{XY}$*, we have the following function :*

$$OR(X \to Y) = a.n + b. \tag{20}$$

Proposition 15. *(a) If* X *favours* Y*, and* n *is a variable with* $\alpha = \frac{n_{XY}}{n_{XY}(n-n_X-n_Y+n_{XY})} > 0$ *and* $\beta = -\dfrac{n_{XY}^2}{n_{XY}(n-n_X-n_Y+n_{XY})}$*, then we have the following function expressions :*

$$OR_{hn}(X \to Y) = \alpha.n + \beta. \tag{21}$$

(b) If X *disfavours* Y*, and* n *is a variable with* $\eta = \dfrac{n_{XY}}{(n_X - n_{XY})(n_Y - n_{XY})} > 0$ *and* $\iota = -\dfrac{n_X^2}{(n_X - n_{XY})(n_Y - n_{XY})}$*,* $n_Y \neq n_{XY}$*,* $n_X \neq n_{XY}$*, we have the following function expression :*

$$OR_{hn}(X \to Y) = \eta.n + \iota. \tag{22}$$

Demonstrations :

1. For $\mathrm{M_{GK}}$, In effect $MG^d_{GK}(X \to Y) = \dfrac{\frac{n_{XY}}{n_X} - \frac{n_Y}{n}}{\frac{n_Y}{n}} = \dfrac{\frac{n_{XY}}{n_X}}{\frac{n_Y}{n}} - 1 = \Theta.n - 1$, with $\Theta = \frac{n_{XY}}{n_X.n_Y} \succ 0$.
   Therefore, M^d_{GK} is an increasing measure as a function of n. Moreover, $M^f_{GK}(X \to Y) = \dfrac{\frac{n_{XY}}{n_X - \frac{n_Y}{n}}}{1 - \frac{n_Y}{n}} = \dfrac{n_{XY}}{n_X(n-n_Y)}n - \dfrac{n_X.n_Y}{n_X.(n-n_Y)} = \alpha.n - \beta$, with $\alpha = \dfrac{n_{XY}}{n_X(n-n_Y)} \succ 0$ and $\beta = \dfrac{n_X.n_Y}{n_X.(n-n_Y)}$.
   Therefore M^d_{GK} is an increasing measure of n.

2. Then, $OR(X \to Y) = \dfrac{\frac{n_{XY}}{n_X}\left(1 - \frac{n_X}{n} - \frac{n_Y}{n} + \frac{n_{XY}}{n_X}\frac{n_X}{n}\right)}{\left(\frac{n_y}{n} - \frac{n_{XY}}{n_X}\frac{n_X}{n}\right)\left(1 - \frac{n_{XY}}{n_X}\right)}$
   Simplifying this expression, we obtain :
   $OR(X \to Y) = \dfrac{n_{XY}}{(n_Y - n_{XY})(n_X - n_{XY})}.n + \dfrac{(n_{XY} - n_X - n_Y).n_{XY}}{(n_Y - n_{XY})(n_X - n_{XY})}$
   So $OR(X \to Y) = a.n + b$ with $a = \frac{n_{XY}}{(n_Y - n_{XY})(n_X - n_{XY})} > 0$ and $b = \frac{(n_{XY} - n_X - n_Y).n_{XY}}{(n_Y - n_{XY})(n_X - n_{XY})}$, which is increasing in function of n.

3. Then, for OR_{hn}, we have :
   $OR^f_{hn}(X \to Y) = \dfrac{\frac{n_{XY}}{n_X} - \frac{n_X}{n}}{\frac{n_{XY}}{n_X}\left(1 - \frac{n_X}{n} - \frac{n_Y}{n} + \frac{n_{XY}}{n_X}.\frac{n_X}{n}\right)}$
   After the simplification, it comes :
   $OR^f_{hn}(X \to Y) = \dfrac{n_{XY}}{n_{XY}(n - n_X - n_Y + n_{XY})}.n - \dfrac{n_{XY}^2}{n_{XY}(n - n_X - n_Y + n_{XY})}$
   Therefore $OR^f_{hn}(X \to Y) = \alpha.n+\beta$ with $\alpha = \frac{n_{XY}}{n_{XY}(n-n_X-n_Y+n_{XY})}$ and $\beta = -\dfrac{n_{XY}^2}{n_{XY}(n - n_X - n_Y + n_{XY})}$,

is an increasing measure as a function of n.

In the end, $OR^d_{hn}(X \rightarrow Y) = \dfrac{\frac{n_{XY}}{n_X} - \frac{n_X}{n}}{(1 - \frac{n_{XY}}{n_X})(\frac{n_Y}{n} - \frac{n_{XY}}{n_X}.\frac{n_X}{n})}$

After simplification, we obtain :

$OR^d_{hn}(X \rightarrow Y) = \dfrac{n_{XY}}{(n_X - n_{XY})(n_Y - n_{XY})}.n - \dfrac{n_X^2}{(n_X - n_{XY})(n_Y - n_{XY})}$

So $OR^d_{hn}(X \rightarrow Y) = \eta.n + \iota$, with $\eta = \dfrac{n_{XY}}{(n_X - n_{XY})(n_Y - n_{XY})}$ and

$\iota = -\dfrac{n_X^2}{(n_X - n_{XY})(n_Y - n_{XY})}$ is an increasing measure of n.

Hence we have : $\boxed{P_7(M_{GK}) = P_7(OR) = P_7(OR_{nh}) = 1}$.

Property 8 (P_8) : *Decreasing measure depending on the size of the consequent or the size of the premise*

- SIf m is not decreasing according to n_Y i.e. if $\exists(X_1 \rightarrow Y_1), \exists(X_2 \rightarrow Y_2)$, such that $n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}$ and $m(X_1 \rightarrow Y_1) < m(X_2 \rightarrow Y_2)$; then $P_8(m) = 0$;

- If m is decreasing as a function of n_Y i.e. if $\forall(X_1 \rightarrow Y_1), \forall(X_2 \rightarrow Y_2)$ $(n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}) \Rightarrow m(X_1 \rightarrow Y_1) \geq m(X_2 \rightarrow Y_2)$ and $\exists(X_1 \rightarrow Y_1), \exists(X_2 \rightarrow Y_2)$, such that $n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}$ and $m(X_1 \rightarrow Y_1) > m(X_2 \rightarrow Y_2)$, $P_8(m) = 1$.

The following proposition shows that M^f_{GK} and OR^f_{hn} are not decreasing measures of the size of the consequent or premise, while M^d_{GK}, OR and OR^d_{hn} are decreasing measures of the size of the consequent or the premise.

Proposition 16. *(a) If X favours Y, $(X_1 \rightarrow Y_1), (X_2 \rightarrow Y_2)$ two rules, such that $n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}$ and the variable n_Y with $\gamma = \dfrac{n.n_{XY}}{n_X} > 0$, we have the following inequality :*

$$M_{GK}(X_1 \rightarrow Y_1) < M_{GK}(X_2 \rightarrow Y_2). \tag{23}$$

(b) If X disfavours Y, $(X_1 \rightarrow Y_1), (X_2 \rightarrow Y_2)$ two rules, such that $n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}$ and the variable n_Y or n_X with $\theta = \frac{n_{XY}.n}{n_X} > 0$ or $\beta = \frac{n_{XY}.n}{n_Y} > 0$, we have the following inequality :

$$M_{GK}(X_1 \rightarrow Y_1) > M_{GK}(X_2 \rightarrow Y_2). \tag{24}$$

Proposition 17. *Let X and Y be two patterns, $(X_1 \rightarrow Y_1), (X_2 \rightarrow Y_2)$ two rules, such that $n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}$ and the variable n_Y or n_X with $\alpha = n_{XY}(n - n_Y + n_{XY}) > 0$ ou $\beta = n_{XY}(n - n_X + n_{XY}) > 0$, we have the following inequality :*

$$OR(X_1 \rightarrow Y_1) > OR(X_2 \rightarrow Y_2). \tag{25}$$

Proposition 18. *(a) If X favours Y, $(X_1 \rightarrow Y_1), (X_2 \rightarrow Y_2)$ two rules, such that $n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}$ and the variable n_Y with $\alpha = n > 0, \beta = n - n_X + n_{XY} > 0, \lambda = \frac{n_X}{n_{XY}} > 0$, we have the following inequality :*

$$OR_{hn}(X_1 \rightarrow Y_1) < OR_{hn}(X_2 \rightarrow Y_2). \tag{26}$$

(b) If X disfavours Y, $(X_1 \rightarrow Y_1), (X_2 \rightarrow Y_2)$ two rules, such that $n_{X_1} = n_{X_2}$ and $n_{X_1Y_1} = n_{X_2Y_2}$ and $n_{Y_1} < n_{Y_2}$ and the variable n_Y or n_X with $\alpha = \frac{n.n_{XY}}{n_X - n_{XY}} > 0 ou \theta = \frac{n.n_{XY}}{n_Y - n_{XY}} > 0, \beta = \frac{n_X}{n_X - n_{XY}} > 0 ou \lambda = \frac{n_Y}{n_Y - n_{XY}} > 0$, we have the following inequality :

$$OR_{hn}(X_1 \rightarrow Y_1) > OR_{hn}(X_2 \rightarrow Y_2). \tag{27}$$

Demonstration :

1. Indeed, $M^f_{GK}(X \to Y) = \dfrac{n.n_{XY}}{n_X(n-n_Y)} - \dfrac{n_X.n_Y}{n_X(n-n_Y)} = \dfrac{\gamma}{n-n_Y} - \dfrac{n_Y}{n-n_Y}$, with $\gamma = \dfrac{n.n_{XY}}{n_X} > 0$. Therefore, as n_Y grows tending to n, the measure M^f_{GK} also grows. Hence M^f_{GK} does not verify property 8.
   Then, $M^d_{GK}(X \to Y) = \dfrac{n_{XY}.n}{n_X.n_Y} - 1 = \dfrac{\theta}{n_X} - 1 = \dfrac{\beta}{n_Y - 1}$, with $\theta = \frac{n_{XY}.n}{n_X}$ ou $\beta = \frac{n_{XY}.n}{n_Y}$.Therefore, as n_X or n_Y increases, the measure M^d_{GK} decreases to the limit -1. Hence M^dGK is decreasing with the size of the consequent or the size of the premise.

2. Then, $OR(X \to Y) = \dfrac{n_{XY}(n-n_X-n_Y+n_{XY})}{(n_Y-n_{XY})(n_X-n_{XY})} = \dfrac{\alpha}{(n_Y-n_{XY})(n_X-n_{XY})} - \dfrac{n_{XY}.n_X}{(n_Y-n_{XY})(n_X-n_{XY})}$ $= \dfrac{\beta}{(n_Y-n_{XY})(n_X-n_{XY})} - \dfrac{n_{XY}.n_Y}{(n_Y-n_{XY})(n_X-n_{XY})}$, with $\alpha = n_{XY}(n-n_Y+n_{XY}) > 0$ or $\beta = n_{XY}(n-n_X+n_{XY}) > 0$. Thus, as n_X or n_Y increases, OR decreases to 0. Hence OR is decreasing with the size of the consequent or the size of the premise.

3. After, $OR^f_{hn}(X \to Y) = \dfrac{\frac{n_{XY}}{n_X} - \frac{n_Y}{n}}{\frac{n_{XY}}{n_X}(1 - \frac{n_X}{n} - \frac{n_Y}{n} + \frac{n_{XY}}{n})} = \dfrac{n.n_{XY} - n_X.n_Y}{n_{XY}(n-n_X-n_Y-n_{XY})}$
   Therefore, $OR^f_{hn}(X \to Y) = \dfrac{n}{n-n_X-n_Y-n_{XY}} - \dfrac{n_X.n_Y}{n_{XY}(n-n_X-n_Y-n_{XY})} = \dfrac{\alpha}{\beta - n_Y} - \dfrac{\lambda.n_Y}{\beta - n_Y}$, with $\alpha = n > 0, \beta = n - n_X + n_{XY} > 0, \lambda = \frac{n_X}{n_{XY}} > 0$. So the more n_Y tends to β, the more OR^f_{hn} grows. Hence OR^f_{hn} does not verify property 8.
   Then, $OR^d_{hn}(X \to Y) = \dfrac{\frac{n_{XY}}{n_X - \frac{n_Y}{n}}}{(1-\frac{n_{XY}}{n_X})(\frac{n_Y}{n} - \frac{n_{XY}}{n})} = \dfrac{n.n_{XY} - n_Y.n_X}{(n_X-n_{XY})(n_Y-n_{XY})}$
   Therefore, $OR^d_{hn}(X \to Y) = \dfrac{n.n_{XY}}{(n_X-n_{XY})(n_Y-n_{XY})} - \dfrac{n_X.n_Y}{(n_X-n_{XY})(n_Y-n_{XY})} = \dfrac{\alpha}{n_Y-n_{XY}} - \dfrac{\beta.n_Y}{n_Y-n_{XY}} = \dfrac{\theta}{n_X-n_{XY}} - \dfrac{\lambda.n_X}{n_X-n_{XY}}$ with $\alpha = \frac{n.n_{XY}}{n_X-n_{XY}} > 0\, or\, \theta = \frac{n.n_{XY}}{n_Y-n_{XY}} > 0, \beta = \frac{n_X}{n_X-n_{XY}} > 0\, or\, \lambda = \frac{n_Y}{n_Y-n_{XY}} > 0$; Therefore, the more n_X or n_Y grows, the more OR^d_{hn} decreases and has limit -1. Hence OR^d_{hn} is decreasing as a function of n_X or n_Y.

Hence we have : $\boxed{P_8(M^d_{GK}) = P_8(OR) = P_8(OR^d_{nh}) = 1\, and\, P_8(M^f_{GK}) = P_8(OR^f_{nh}) = 0}$.

Property 9 (P_9) : *Measure admitting a fixed value in the case of independence*

- If m does not admit a fixed value in the case of independence i.e. if $\forall a \in \mathbb{R}, \exists (X \to Y)$, such that $P_X(Y) = P(Y)$ and $m(X \to Y) \neq a$; then $P_9(m) = 0$;
- If m admits a fixed value in the case of independence i.e. if $\exists a \in \mathbb{R}, \forall (X \to Y)$, such that $P_X(Y) = P(Y) \Rightarrow m(X \to Y) = a$, then $P_9(m) = 1$.

The following propositions show that the measures M_{GK}, OR and OR_{hn} admit a fixed value in the case of independence.

Proposition 19. *Let X and Y be two patterns, $\forall (X \to Y)$, such that $P_X(Y) = P(Y)$, we have the following value*

$$M_{GK}(X \to Y) = 0. \tag{28}$$

Proposition 20. *Let X and Y be two patterns, $\forall (X \to Y)$, such that $P_X(Y) = P(Y)$, we have the following value*

$$OR(X \to Y) = 1. \tag{29}$$

Proposition 21. *Let X and Y be two patterns, $\forall(X \to Y)$, such that $\mathrm{P}_X(Y) = \mathrm{P}(Y)$, we have the following value*

$$OR_{hn}(X \to Y) = 0. \tag{30}$$

Théorème .1. *(a) A probabilistic quality measure mu admits a fixed value at independence if, and only if, for any association rule XtoY, the following condition is verified at the reference situation : $\mu_{ind} \neq \infty$, if $(\mu_{imp}, \mu_{ind}, \mu_{inc}) \in \overline{\mathbb{R}}^3$.*

(b) All normalizable and normalized affine measures and homographies are allowed a fixed value of independence.

Preuve :
(i) if $(\mu_{imp}, \mu_{ind}, \mu_{inc}) \in \overline{\mathbb{R}}^3$, and $\mu_{ind} \neq \infty$, it is indeed sufficient to use property 9 (ii) above ;
(ii) if $(\mu_{imp}, \mu_{ind}, \mu_{inc}) \in \overline{\mathbb{R}}^3$, following the process of normalization of the measure values between [-1;1] and following the definition of normalized measures, it is enough to use on property 9 above. Hence the stated theorem.

Demonstrations :

1. In fact, $\forall(X \to Y) \in \mathbb{K}(P, C, \mathbb{R})$, $M^f_{GK}(X \to Y) = \dfrac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{1 - \mathrm{P}(Y)}$. So, at independence $\mathrm{P}_X(Y) = \mathrm{P}(Y)$, we get $M^f_{GK}(X \to Y) = 0 \in \mathbb{R}$ (fixed value). And for $M^d_{GK}(X \to Y) = \dfrac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{\mathrm{P}(Y)}$. Similarly, at independence, $M^d_{GK}(X \to Y) = 0 \in \mathbb{R}(valeur fixe)$. Hence M_{GK} is a measure admitting a fixed value at independence.

2. Then, $\forall(X \to Y) \in \mathbb{K}(P, C, \mathbb{R})$, $OR(X \to Y) = \dfrac{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))}$. Then, at independence for $\mathrm{P}_X(Y) = \mathrm{P}(Y)$, on a $OR(X \to Y) = 1 \in \mathbb{R}$ (fixed value). Hence OR is a measure admitting a fixed value at independence..

3. After, $\forall(X \to Y) \in \mathbb{K}(P, C, \mathbb{R})$,
$OR^f_{hn}(X \to Y) = \dfrac{P(Y'/X') - P(Y')}{P(Y'/X')(1 - P(X') - P(Y') + P(Y'/X')P(X'))}$.
At independence, $OR^f_{hn}(X \to Y) = 0 \in \mathbb{R}(valeur\ fixe)$.
And for $OR^d_{hn}(X \to Y) = \dfrac{P(Y'/X') - P(Y')}{(1 - P(Y'/X'))(P(Y') - P(Y'/X')P(X'))}$. Similarly, at independence $OR^d_{hn}(X \to Y) = 0$ (fixed value). Hence OR_{hn} is a measure admitting a fixed value at independence.

Hence we have : $\boxed{\mathrm{P}_9(M_{GK}) = P_9(OR) = P_9(OR_{nh}) = 1}$.

Property 10 (P_{10}) : ***Measure admitting a fixed value in the case of logical implication***

- If m does not admit a fixed value in the case of the logical implication i.e. if $\forall b \in \mathbb{R}, \exists(X \to Y)$, such that $\mathrm{P}_X(Y) = 1$ and $m(X \to Y) \neq b$; then $P_{10}(m) = 0$;
- If m admits a fixed value in the case of the logical implication i.e. if $\exists b \in \mathbb{R}, \forall(X \to Y)$, such that $\mathrm{P}_X(Y) = 1 \Rightarrow m(X \to Y) = b$, then $P_{10}(m) = 1$.

The following propositions show that the measures M^f_{GK}, OR^f_{hn} admit a fixed value in the case of logical implication, while M^d_{GK},OR and OR^d_{hn} do not admit a fixed value in the case of logical implication.

Proposition 22. *(a) If X favours Y, for $\mathrm{P}_X(Y) = 1$, we have the following value :*

$$M_{GK}(X \to Y) = 1 \tag{31}$$

(b) *If X disadvantages Y, for $\mathrm{P}_X(Y) = 1$, we have the relation :*

$$M_{GK}(X \to Y) = \frac{1 - \mathrm{P}(Y)}{\mathrm{P}(Y)} \tag{32}$$

Proposition 23. *Let X and Y be two patterns, for $\mathrm{P}_X(Y) = 1$, we have the following value :*

$$OR(X \to Y) = +\infty \tag{33}$$

Proposition 24. *(a) If X favours Y, for $\mathrm{P}_X(Y) = 1$, we have the following value :*

$$OR_{hn}(X \to Y) = 1 \tag{34}$$

(b) SIf X disadvantages Y, for $\mathrm{P}_X(Y) = 1$, we have the following value :

$$OR_{hn}(X \to Y) = +\infty \tag{35}$$

Théorème .2. *All normalisable affine measures and normalisable homographies are allowed a fixed value at the logical implication.*

Proof :
if $(\mu_{imp}, \mu_{ind}, \mu_{inc}) \in \overline{\mathbb{R}}^3$, following the process of normalization to bring back the values of measure between [-1;1] and according to the definition of normalized measures, and to use there on the property 10 above. Hence the stated theorem **Demonstration :**

1. Indeed, we have $M^f_{GK}(X \to Y) = \dfrac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{1 - \mathrm{P}(Y)}$.So, at the logical implication $\mathrm{P}_X(Y) = 1$, we get $M^f_{GK}(X \to Y) = 1 \in \mathbb{R}$ which is a fixed value. And for $M^d_{GK}(X \to Y) = \dfrac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{\mathrm{P}(Y)}$. Similarly, by logical implication, $M^d_{GK}(X \to Y) = \dfrac{1 - \mathrm{P}(Y)}{\mathrm{P}(Y)}$ which is not a fixed value for any pattern Y of a context $\mathbb{K}$.

2. Then, $OR(X \to Y) = \dfrac{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))}$. At the logical implication, we have $\mathrm{P}_X(Y) = 1$, il vient $OR(X \to Y) = +\infty$ which is a nonfixed value.

3. Then, $OR^f_{hn}(X \to Y) = \dfrac{P(Y'/X') - P(Y')}{P(Y'/X')(1 - P(X') - P(Y') + P(Y'/X')P(X'))}$. At the logical implication, it comes a $\mathrm{P}_X(Y) = 1$, we obtain $OR^f_{hn}(X \to Y) = 1$ qwhich is a fixed value.
In the end, $OR^d_{hn}(X \to Y) = \dfrac{P(Y'/X') - P(Y')}{(1 - P(Y'/X'))(P(Y') - P(Y'/X')P(X'))}$. At the logical implication, it comes $OR^d_{hn}(X \to Y) = +\infty$ which is not a fixed value.

Hence we have : $\boxed{\mathrm{P}_{10}(M^d_{GK}) = P_{10}(OR) = P_{10}(OR^d_{nh}) = 0 \, and \, P_{10}(M^f_{GK}) = P_{10}(OR^f_{nh}) = 1}$.

Property 11 (P_{11}) : *Measure admitting a fixed value in the case of equilibrium*

- If m does not admit a fixed value in the case of equilibrium i.e. if $\forall c \in \mathbb{R}, \exists (X \to Y)$, such that $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$ and $m(X \to Y) \neq c$; then $P_{11}(m) = 0$;
- If m has a fixed value in the case of equilibrium, i.e. if $\exists c \in \mathbb{R}, \forall (X \to Y)$, such that $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2} \Rightarrow m(X \to Y) = c$, then $P_{11}(m) = 1$.

The situation of equilibrium is another situation of reference other than independence and logical implication. We have a situation of equilibrium when the rule has as many examples as the counterexamples[Grissa(2013)Grissa].

The following propositions show that the measures $\mathrm{M_{GK}}$, OR and OR_{hn} do not admit a fixed value in the equilibrium case.

Proposition 25. *(a) If X favours Y, and $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, we have the following relationship :*

$$M_{GK}(X \to Y) = \frac{\mathrm{P}(X) - 2\mathrm{P}(Y)}{2(1 - \mathrm{P}(Y))} \tag{36}$$

(b) If X disadvantages Y, and $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$,we have the following relationship :

$$M_{GK}(X \to Y) = \frac{\mathrm{P}(X) - 2\mathrm{P}(Y)}{2\mathrm{P}(Y)} \tag{37}$$

Proposition 26. *Let X and Y be two patterns, if $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, we have the following relationship :*

$$OR(X \to Y) = \frac{\mathrm{P}(X)(2 - 2\mathrm{P}(X) - 2\mathrm{P}(Y) - \mathrm{P}^2(X))}{(2\mathrm{P}(Y) - \mathrm{P}(X))(2 - \mathrm{P}(X))} \tag{38}$$

Proposition 27. *(a) If X favours Y, and $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, we have the following relationship :*

$$R_{hn}(X \to Y) = \frac{2(P(X) - 2P(Y))}{2 - 2P(X) - 2P(Y) + P^2(X)} \tag{39}$$

(b) SIf X disadvantages Y, and $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, we have the following relationship :

$$OR_{hn}(X \to Y) = \frac{2(P(X) - P(Y))}{(2 - P(X))(2P(Y) - P^2(X))} \tag{40}$$

Théorème .3. *All affine and homograph normalisable measures and normalised measures do not admit of a fixed value in the case of equilibrium.*

Proof :
if $(\mu_{imp}, \mu_{ind}, \mu_{inc}) \in \mathbb{R}^3$,following the process of normalization of the measurement values between [-1;1] and following the definition of normalized measurements, all its measurements only admit values that at the reference situation($\mu_{imp} = 1, \mu_{ind} = 0, \mu_{inc} = -1$),of which it is enough to use on property 11 above. Hence the stated theorem.

Demonstration :

1. Indeed, we have $M^f_{GK}(X \to Y) = \dfrac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{1 - \mathrm{P}(Y)}$. At equilibrium, if $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, on obtient $M^f_{GK}(X \to Y) = \dfrac{\frac{\mathrm{P}(X)}{2} - \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} = \dfrac{\mathrm{P}(X) - 2\mathrm{P}(Y)}{2(1 - \mathrm{P}(Y))}$ which is not a fixed value.
   And for, $M^d_{GK}(X \to Y) = \dfrac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{\mathrm{P}(Y)}$. At equilibrium, $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, it comes $M^d_{GK}(X \to Y) = \dfrac{\frac{\mathrm{P}(X)}{2} - \mathrm{P}(Y)}{\mathrm{P}(Y)} = \dfrac{\mathrm{P}(X) - 2\mathrm{P}(Y)}{2\mathrm{P}(Y)}$ which is not a fixed value too. Hence M_{GK} is not a measure admitting a fixed value at the equilibrium situation.

2. Then, $OR(X \to Y) = \dfrac{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))}$. At equilibrium, if $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, we have $OR(X \to Y) = \dfrac{\frac{\mathrm{P}(X)}{2}(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \frac{\mathrm{P}(X)}{2}.\mathrm{P}(X))}{(\mathrm{P}(Y) - \frac{\mathrm{P}(X)}{2}.\mathrm{P}(X))(1 - \frac{\mathrm{P}(X)}{2})}$.
   It comes $OR(X \to Y) = \dfrac{\mathrm{P}(X)(2 - 2\mathrm{P}(X) - 2\mathrm{P}(Y) - \mathrm{P}^2(X))}{(2\mathrm{P}(Y) - \mathrm{P}(X))(2 - \mathrm{P}(X))}$ which is not a fixed value. Hence OR is not a measure admitting a fixed value at the equilibrium situation.

3. Then, $OR^f_{hn}(X \to Y) = \dfrac{P(Y/X) - P(Y)}{P(Y/X)(1 - P(X) - P(Y) + P(Y/X)P(X))}$. At equilibrium, if $\mathrm{P}_X(Y) = \frac{\mathrm{P}(X)}{2}$, on obtient $OR^f_{hn}(X \to Y) = \dfrac{\frac{\mathrm{P}(X)}{2} - P(Y)}{\frac{\mathrm{P}(X)}{2}(1 - P(X) - P(Y) + \frac{\mathrm{P}(X)}{2}P(X))}$

So $OR^f_{hn}(X \to Y) = \dfrac{2(P(X) - 2P(Y))}{2 - 2P(X) - 2P(Y) + P^2(X)}$ which is not a fixed value at equilibrium.
In the end, $OR^d_{hn}(X \to Y) = \dfrac{P(Y/X) - P(Y)}{(1 - P(Y/X))(P(Y) - P(Y/X)P(X))}$. At equilibrium, we have
$OR^d_{hn}(X \to Y) = \dfrac{\frac{P(X)}{2} - P(Y)}{(1 - \frac{P(X)}{2})(P(Y) - \frac{P(X)}{2}P(X))}$.
Therefore $OR^d_{hn}(X \to Y) = \dfrac{2(P(X) - P(Y))}{(2 - P(X))(2P(Y) - P^2(X))}$ which is not a fixed value at equilibrium. Hence OR_{hn} is not a measure admitting a fixed value at the equilibrium situation.

Hence we have : $\boxed{P_{11}(M_{GK}) = P_{11}(OR) = P_{11}(OR_{nh}) = 0}$.

Property 12 (P_{12}) : *Measure admitting identifiable values in case of attraction between X and Y*

- If m does not admit identifiable values in case of attraction between X and Y i.e. if $\forall a \in \mathbb{R}, \exists (X \to Y)$, such that $P_X(Y) > P(Y)$ and $m(X \to Y) \leq a$; then $P_{12}(m) = 0$;
- If m admits identifiable values in case of attraction between X and Y i.e. if $\exists a \in \mathbb{R}, \forall (X \to Y)$, such that $P_X(Y) > P(Y) \Rightarrow m(X \to Y) > a$, then $P_{12}(m) = 1$.

The following propositions show that the measures M_{GK}, OR and OR_{hn} admit identifiable values in case of attraction between X and Y.

Proposition 28. *If X favours Y, and $\forall (X \to Y)$ and $\forall a \in \mathbb{R}^- \subset \mathbb{R}$ for $P_X(Y) > P(Y)$, we have the following inequality :*

$$M_{GK}(X \to Y) > a. \tag{41}$$

Proposition 29. *Let X and Y be two patterns, $\forall (X \to Y)$, $\forall b \in]-\infty, 1] \subset \mathbb{R}$ for $P_X(Y) > P(Y)$, we have the following inequality :*

$$OR(X \to Y) > b. \tag{42}$$

Proposition 30. *If X favours Y, and $\forall (X \to Y)$, $\forall c \in \mathbb{R}^- \subset \mathbb{R}$ for $P_X(Y) > P(Y)$, we have the following inequality :*

$$OR^f_{hn}(X \to Y) > c. \tag{43}$$

Demonstration :

1. Indeed, we have $M^f_{GK}(X \to Y) = \dfrac{P_X(Y) - P(Y)}{1 - P(Y)} \in [0, 1]$. It is obvious that $\forall (X \to Y)$ and $\forall a \in \mathbb{R}^- \subset \mathbb{R}$ such that in case of attraction between X and Y, if $P_X(Y) > P(Y)$, we have $M^f_{GK}(X \to Y) \in]0, 1[$ and $M^f_{GK}(X \to Y) > a$. Hence M_{GK} admits identifiable values in case of attraction between X and Y.

2. Then, $OR(X \to Y) = \dfrac{P(Y/X)(1 - P(X) - P(Y) + P(Y/X).P(X))}{(P(Y) - P(Y/X).P(X))(1 - P(Y/X))} \in [0; +\infty[$. Donc $\forall (X \to Y)$ and $\forall b \in]-\infty, 1] \subset \mathbb{R}$ in case of attraction between X and Y, if $P_X(Y) > P(Y)$, we have $OR(X \to Y) \in]1, +\infty[$ and $OR(X \to Y) > b$. Hence OR admits identifiable values in case of attraction between X and Y.

3. In the end, $OR^f_{hn}(X \to Y) = \dfrac{P(Y/X) - P(Y)}{P(Y/X)(1 - P(X) - P(Y) + P(Y/X)P(X))} \in [0, 1]$. Thus $\forall (X \to Y)$ and $\forall c \in \mathbb{R}^- \subset \mathbb{R}$ such that in case of attraction between X and Y, if $P_X(Y) > P(Y)$, we have $OR^f_{hn}(X \to Y) \in]0, 1[$ and $OR^f_{hn}(X \to Y) > c$. Hence OR_{hn} admits identifiable values in case of attraction between X and Y.

Hence we have : $\boxed{P_{12}(M_{GK}) = P_{12}(OR) = P_{12}(OR_{nh}) = 1}$.

Property 13 (P_{13}) : *Measure admitting identifiable values in case of repulsion between X and Y*

- If m does not admit identifiable values in case of repulsion between X and Y i.e. if $\forall a \in \mathbb{R}, \exists (X \to Y)$, such that $\mathrm{P}_X(Y) < \mathrm{P}(Y)$ and $m(X \to Y) \geq a$; then $P_{13}(m) = 0$;
- If m admits identifiable values in case of repulsion between X and Y i.e. if $\exists a \in \mathbb{R}, \forall (X \to Y)$, such that $\mathrm{P}_X(Y) < \mathrm{P}(Y) \Rightarrow m(X \to Y) < a$, then $P_{13}(m) = 1$.

The following propositions show that $\mathrm{M_{GK}}$, OR and OR_{hn} admit identifiable values in case of repulsion between X and Y.

Proposition 31. *If X disadvantages Y, and $\forall (X \to Y)$ and $\forall a \in \mathbb{R}^+ \subset \mathbb{R}$ for $\mathrm{P}_X(Y) < \mathrm{P}(Y)$, we have the following inequality :*

$$M_{GK}(X \to Y) < a. \tag{44}$$

Proposition 32. *Let X and Y be two patterns, and $\forall (X \to Y)$ and $\forall b \in [1, +\infty[\subset \mathbb{R}$ for $\mathrm{P}_X(Y) < \mathrm{P}(Y)$, we have the following inequality :*

$$OR(X \to Y) < b. \tag{45}$$

Proposition 33. *If X disadvantages Y, and $\forall (X \to Y)$ and $\forall c \in \mathbb{R}^+ \subset \mathbb{R}$ for $\mathrm{P}_X(Y) < \mathrm{P}(Y)$, we have the following inequality :*

$$OR^d_{hn}(X \to Y) < c. \tag{46}$$

Demonstration :

1. Indeed $M^d_{GK}(X \to Y) = \dfrac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{\mathrm{P}(Y)} \in [-1, 0]$. Therefore $\forall (X \to Y)$ and $\forall a \in \mathbb{R}^+ \subset \mathbb{R}$ such that in case of repulsion between X and Y, $\mathrm{P}_X(Y) < \mathrm{P}(Y)$, we have $M^d_{GK}(X \to Y) \in]-1, 0[$ and $M^d_{GK}(X \to Y) < a$. Hence M_{GK} admits identifiable values in case of repulsion between X and Y.

2. Then, $OR(X \to Y) = \dfrac{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))} \in [0; +\infty[$. Thus $\forall (X \to Y)$ and $\forall b \in [1, +\infty[\subset \mathbb{R}$ in case of repulsion between X and Y, if $\mathrm{P}_X(Y) < \mathrm{P}(Y)$, one has $OR(X \to Y) \in]0, 1[$ and $OR(X \to Y) < b$. Hence OR admits identifiable values in the case of the repulsion between X and Y.

3. In the end, $OR^d_{hn}(X \to Y) = \dfrac{P(Y/X) - P(Y)}{(1 - P(Y/X))(P(Y) - P(Y/X)P(X))} \in [-1, 0]$. Thus $\forall (X \to Y)$ and $\forall c \in \mathbb{R}^+ \subset \mathbb{R}$ such that in case of repulsion between X and Y, $\mathrm{P}_X(Y) < \mathrm{P}(Y)$, we have $OR^d_{hn}(X \to Y) \in]-1, 0[$ and $OR^d_{hn}(X \to Y) < c$. Hence OR_{hn} admits identifiable values in case of repulsion between X and Y.

Hence we have : $\boxed{\mathrm{P}_{13}(M_{GK}) = P_{13}(OR) = P_{13}(OR_{nh}) = 1}$.

Property 14 (P_{14}) : *Measure capable of tolerating the first counterexamples*

A measure is able to tolerate the first few counterexamples if it decreases slowly as few counterexamples appear, and then more rapidly until it reaches a minimum or zero value. That is, at fixed n_X and n_Y margins, the evolution function of the measure m, as a function of n_{XY}, has a concave shape.

- If m is a convex function of n_{XY}, i.e. $\exists min_{conf} \in [0;1]$ $\forall X_1 \rightarrow Y_1, \forall X_2 \rightarrow Y_2, \forall \lambda \in [0,1]$; $n_{X_1Y_1} \geq min_{conf} n_{X_1}$ and $n_{X_2Y_2} \geq min_{conf} n_{X_2}$ impliquent $f_{m,n_{XY}}(\lambda n_{X_1Y_1} + (1-\lambda)n_{X_2Y_2}) \leq \lambda f_{m,n_{XY}}(n_{X_1Y_1} + (1-\lambda)n_{X_2Y_2})$; alors $P_{14}(m) = 0$,(rejection) ;

- If m is a linear function o n_{XY} i.e. $P_{14}(m) \neq 0$ and $P_{14}(m) \neq 2$, then $P_{14}(m) = 1$ (indifference) ;

- If m is a concave function of n_{XY} i.e. $\exists min_{conf} \in [0,1], \forall X_1 \rightarrow Y_1, \forall X_2 \rightarrow Y_2, \forall \lambda \in [0,1]$; $n_{X_1Y_1} \geq min_{conf} n_{X_1}$ and $n_{X_2Y_2} \geq min_{conf} n_{X_2}$ imply $f_{m,n_{XY}}(\lambda n_{X_1Y_1} + (1-\lambda)n_{X_2Y_2}) \geq \lambda f_{m,n_{XY}}(n_{X_1Y_1} + (1-\lambda)n_{X_2Y_2})$, then $P_{14}(m) = 2$ (tolerance).

The notation $f_{m,n_{XY}}$ corresponds to the function of evolution of the measure m as a function of n_{XY} when the numbers n_X, n_Y and n remain constant.

We saw in the proof of property 6 that the measures M_{GK}^f, M_{GK}^d, OR and OR_{hn}^d are linear measures as a function of n_{XY}. Therefore, these are measures that are indifferent to the first counterexamples. And the measure OR_{hn}^f is a decreasing measure as a function of n_{XY}. Therefore, it is a convex measure as a function of n_{XY}. Hence OR_{hn}^f is the measure rejected by the first counterexamples.

Hence we have : $\boxed{\mathrm{P}_{14}(M_{GK}) = P_{14}(OR) = P_{14}(OR_{nh}^d) = 1 \, and \, P_{14}(OR_{nh}^f) = 0}$.

Property 15 (P_{15}) : *Invariant measure in case of dilation of certain numbers*

- If $\exists (k_1,k_2) \in \mathbb{N}^{*2}$, $\exists X_1 \rightarrow Y_1$, $\exists X_2 \rightarrow Y_2$, $\big(n_{X_1Y_1} = k_1 n_{X_2Y_2} \, and \, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2} \, and$ $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2} \, and \, n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2} \, and \, m(X_1 \rightarrow Y_1) \neq m(X_2 \rightarrow Y_2)\big)$ or $\big(n_{X_1Y_1} = k_2 n_{X_2Y_2} \, and \, n_{X_1\overline{Y}_1} = k_2 n_{X_2\overline{Y}_2} \, and \, n_{\overline{X}_1Y_1} = k_1 n_{\overline{X}_2Y_2} \, and \, n_{\overline{X}_1\overline{Y}_1} = k_1 n_{\overline{X}_2\overline{Y}_2} \, and$ $m(X_1 \rightarrow Y_1) \neq m(X_2 \rightarrow Y_2)\big)$; then $P_{15}(m) = 0$ (variance) ;

- If $\forall (k_1,k_2) \in \mathbb{N}^{*2}$, $\forall X_1 \rightarrow Y_1$, $\forall X_2 \rightarrow Y_2$, $\big(n_{X_1Y_1} = k_1 n_{X_2Y_2} \, and \, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2} \, and$ $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2} \, and \, n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2} \, and \Rightarrow m(X_1 \rightarrow Y_1) = m(X_2 \rightarrow Y_2)\big)$ and $\big(n_{X_1Y_1} = k_2 n_{X_2Y_2} \, and \, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2} \, and \, n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2} \, and \, n_{\overline{X}_1\overline{Y}_1} = k_1 n_{\overline{X}_2\overline{Y}_2}$ $\Rightarrow m(X_1 \rightarrow Y_1) + m(X_2 \rightarrow Y_2)\big)$; , then $P_{15}(m) = 1$ (invariance).

The following propositions show that OR, OR with the 2^{th} expression, OR_{hn} are not measures that vary with the growth of some numbers, while OR with the 2^{th} expression is a measure that is invariant with the expansion of some numbers.

Proposition 34. *Let X and Y be two patterns, $\forall X_1 \rightarrow Y_1$, $\forall X_2 \rightarrow Y_2$, and$\forall (k_1,k_2) \in \mathbb{N}^{*2}$, such that* $n_{X_1Y_1} = k_1 n_{X_2Y_2} \, and \, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2} \, and$ $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2} \, and \, n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, *we have the following relationship :*

$$M_{GK}(X_1 \rightarrow Y_1) \neq M_{GK}(X_2 \rightarrow Y_2). \tag{47}$$

For $OR(X \rightarrow Y) = \dfrac{\mathrm{P}(X \cap Y)\mathrm{P}(\overline{X} \cap \overline{Y})}{\mathrm{P}(\overline{X} \cap Y)\mathrm{P}(X \cap \overline{Y})} = \dfrac{n_{XY}.n_{\overline{X}.\overline{Y}}}{n_{\overline{X}Y}.n_{X\overline{Y}}}$.

Proposition 35. *Let X and Y be two patterns, $\forall X_1 \rightarrow Y_1$, $\forall X_2 \rightarrow Y_2$, and$\forall (k_1,k_2) \in \mathbb{N}^{*2}$, such that* $n_{X_1Y_1} = k_1 n_{X_2Y_2} \, and \, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2} \, and$ $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2} \, and \, n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, *we have the following relationship :*

$$OR(X_1 \rightarrow Y_1) = OR(X_2 \rightarrow Y_2). \tag{48}$$

For $OR(X \to Y) = \dfrac{\mathrm{P}(Y/X)(1-\mathrm{P}(X)-\mathrm{P}(Y)+\mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y)-\mathrm{P}(Y/X).\mathrm{P}(X))(1-\mathrm{P}(Y/X))}$.

Proposition 36. *Let X and Y be two patterns, $\forall X_1 \to Y_1$, $\forall X_2 \to Y_2$, and $\forall (k_1, k_2) \in \mathbb{N}^{*2}$, such that $n_{X_1Y_1} = k_1 n_{X_2Y_2}$ and $n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}$ and $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}$ and $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, we have the following relationship :*

$$OR(X_1 \to Y_1) \neq OR(X_2 \to Y_2). \tag{49}$$

Proposition 37. *Let X and Y be two patterns, $\forall X_1 \to Y_1$, $\forall X_2 \to Y_2$, and $\forall (k_1, k_2) \in \mathbb{N}^{*2}$, such that $n_{X_1Y_1} = k_1 n_{X_2Y_2}$ and $n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}$ and $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}$ and $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, we have the following relationship :*

$$OR_{hn}(X_1 \to Y_1) \neq OR_{hn}(X_2 \to Y_2). \tag{50}$$

Demonstration :

1. In effect
   Let $n_{XY} = n_X - n_{X\overline{Y}}$ and $n_{XY} = n_Y - n_{\overline{X}Y}$
   So

$$\begin{aligned} M^f_{GK}(X \to Y) &= \frac{\mathrm{P}_X(Y)-\mathrm{P}(Y)}{1-\mathrm{P}(Y)} = \frac{\frac{n_{XY}}{n_X}-\frac{n_Y}{n}}{1-\frac{n_Y}{n}} \\ &= \frac{n.n_{XY}-n_X.n_Y}{n.n_X-n_X.n_Y} = \frac{n.n_{XY}-(n_{XY}+n_{X\overline{Y}})(n_{XY}+n_{\overline{X}Y})}{n.(n_{XY}+n_{X\overline{Y}})-(n_{XY}+n_{X\overline{Y}})(n_{XY}+n_{\overline{X}Y})} \\ &= \frac{n.n_{XY}-n^2_{XY}-n_{XY}.n_{\overline{X}Y}-n_{X\overline{Y}}.n_{\overline{X}Y}}{n.n_{XY}+n.n_{X\overline{Y}}-n^2_{XY}-n_{XY}.n_{\overline{X}Y}-n_{X\overline{Y}}.n_{\overline{X}Y}}. \end{aligned}$$

   Therefore, if $\forall (k_1;k_2) \in \mathbb{N}^{*2}$, $n_{X_1Y_1} = k_1 n_{X_2Y_2}$ and $n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}$ and $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}$ and $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, we have :

$$\begin{aligned} M^f_{GK}(X_1 \to Y_1) &= \frac{k_1.n.n_{X_2Y_2}-k_1^2 n^2_{X_2Y_2}-k_1.k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}-k_1.k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k_1.n.n_{X_2Y_2+k_1.n.n_{X_2\overline{Y}_2}}-k_1^2 n^2_{X_2Y_2}-k_1.k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}-k_1.k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &= \frac{n.n_{X_2Y_2}-k_1 n^2_{X_2Y_2}-k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}-k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{n.n_{X_2Y_2+n.n_{X_2\overline{Y}_2}}-k_1 n^2_{X_2Y_2}-k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}-k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &\neq \frac{n.n_{X_2Y_2}-n^2_{X_2Y_2}-n_{X_2Y_2}.n_{\overline{X}_2Y_2}-n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{n.n_{X_2Y_2+n.n_{X_2\overline{Y}_2}}-n^2_{X_2Y_2}-n_{X_2Y_2}.n_{\overline{X}_2Y_2}-n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} = M^f_{GK}(X_2 \to Y_2) \end{aligned}$$

   Hence M^f_{GK} is a measure that varies with the growth of certain numbers.
   And for

$$\begin{aligned} M^d_{GK}(X \to Y) &= \frac{\mathrm{P}_X(Y)-\mathrm{P}(Y)}{\mathrm{P}(Y)} = \frac{\frac{n_{XY}}{n_X}-\frac{n_Y}{n}}{\frac{n_Y}{n}} \\ &= \frac{n.n_{XY}-n_X.n_Y}{n_X.n_Y} = \frac{n.n_{XY}-(n_{XY}+n_{X\overline{Y}})(n_{XY}+n_{\overline{X}Y})}{(n_{XY}+n_{X\overline{Y}})(n_{XY}+n_{\overline{X}Y})} \\ &= \frac{n.n_{XY}-n^2_{XY}-n_{XY}.n_{\overline{X}Y}-n_{X\overline{Y}}.n_{\overline{X}Y}}{n^2_{XY}+n_{XY}.n_{\overline{X}Y}+n_{X\overline{Y}}.n_{\overline{X}Y}}. \end{aligned}$$

   Therefore, if $\forall (k_1;k_2) \in \mathbb{N}^{*2}$, $n_{X_1Y_1} = k_1 n_{X_2Y_2}$ and $n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}$ and $n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}$ and $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, then we have :

$$\begin{aligned} M^d_{GK}(X_1 \to Y_1) &= \frac{k_1.n.n_{X_2Y_2}-k_1^2.n_{X_2Y_2}-k_1.k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}-k_1.k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k_1^2.n_{X_2Y_2}+k_1.k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}+k_1.k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &= \frac{n.n_{X_2Y_2}-k_1.n_{X_2Y_2}-k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}-k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k_1.n_{X_2Y_2}+k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}+k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &\neq \frac{n.n_{X_2Y_2}-n_{X_2Y_2}-n_{X_2Y_2}.n_{\overline{X}_2Y_2}-n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{n_{X_2Y_2}+n_{X_2Y_2}.n_{\overline{X}_2Y_2}+n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} = M^d_{GK}(X_2 \to Y_2) \end{aligned}$$

   Hence M^d_{GK} is a measure that varies with the growth of certain numbers.

2. Then, the very strange situation on the Odd-Ratio measure, Grissa in his work [Grissa(2013)Grissa], on the $15^{i\grave{e}me}$ property, concerning the measure $m(X \to Y)$ invariant in case of dilation of some numbers is in function $n_{XY}, n_{\overline{X}}, n_{X\overline{Y}} and n_{\overline{XY}}$. This means that on the Odd-Ratio measure, we have a choice of using one of the two expressions of this measure. This leads to another problem :

   - Using the $1^{`ere}$ expression, we have :

$$OR(X \to Y) = \frac{\mathrm{P}(X \cap Y)\mathrm{P}(\overline{X} \cap \overline{Y})}{\mathrm{P}(\overline{X} \cap Y)\mathrm{P}(X \cap \overline{Y})} = \frac{n_{XY}.n_{\overline{X}.\overline{Y}}}{n_{\overline{X}Y}.n_{X\overline{Y}}}$$

     Therefore, if $\forall(k_1;k_2) \in \mathbb{N}^{*2}$, $n_{X_1Y_1} = k_1 n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}\, and$ $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, we have :

$$\begin{aligned} OR(X_1 \to Y_1) &= \frac{k_1.n_{X_2Y_2}.k_2.n_{\overline{X}_2\overline{Y}_2}}{k_2.n_{\overline{X}_2Y_2}.k_1.n_{X_2\overline{Y}_2}} \\ &= \frac{n_{X_2Y_2}.n_{\overline{X}_2\overline{Y}_2}}{n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &= \frac{n_{X_2Y_2}.n_{\overline{X}_2\overline{Y}_2}}{n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} = OR(X_2 \to Y_2) \end{aligned}$$

     From this result, it was concluded that the measure OR is a measure that invariates with the growth of some number of people.

   - Now, using the $2^{i`th}$ expression :

$$\begin{aligned} OR(X \to Y) &= \frac{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))} \\ &= \frac{\frac{n_{XY}}{n_X}(1 - \frac{n_X}{n} - \frac{n_Y}{n} + \frac{n_{XY}}{n})}{(\frac{n_Y}{n} - \frac{n_{XY}}{n})(1 - \frac{}{n}{}_{XY}n_X)} \\ &= \frac{n.n_{XY} - n_X.n_{XY} - n_Y.n_{XY} + n_{XY}^2}{n_X.n_Y - n_Y.n_{XY} - n_X.n_{XY} + n_{XY}^2} \\ &= \frac{n.n_{XY} - (n_{XY}+n_{X\overline{Y}}).n_{XY} - (n_{XY}+n_{\overline{X}Y}).n_{XY} + n_{XY}^2}{(n_{XY}+n_{X\overline{Y}}).(n_{XY}+n_{\overline{X}Y}) - (n_{XY}+n_{\overline{X}Y}).n_{XY} - (n_{XY}+n_{X\overline{Y}}).n_{XY} + n_{XY}^2} \\ &= \frac{n.n_{XY} - n_{XY}^2 - n_{XY}.n_{X\overline{Y}} - n_{XY}.n_{\overline{X}Y}}{n_{X\overline{Y}}.n_{\overline{X}Y}}. \end{aligned}$$

     Therefore, if $\forall(k_1;k_2) \in \mathbb{N}^{*2}$, $n_{X_1Y_1} = k_1 n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}\, and$ $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, we have :

$$\begin{aligned} OR(X_1 \to Y_1) &= \frac{k_1.n.n_{X_2Y_2} - k_1^2.n_{X_2Y_2}^2 - k_1.n_{X_2Y_2}.k_1.n_{X_2\overline{Y}_2} - k_1.n_{X_2Y_2}.k_2.n_{\overline{X}_2Y_2}}{k_1.n_{X_2\overline{Y}_2}.k_2.n_{\overline{X}_2Y_2}} \\ &= \frac{n.n_{X_2Y_2} - k_1.n_{X_2Y_2}^2 - k_1.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}}{k_2.n_{X_2\overline{Y}_2}.n_{\overline{X}_2Y_2}} \\ \neq &= \frac{n.n_{X_2Y_2} - n_{X_2Y_2}^2 - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{X_2Y_2}.n_{\overline{X_2}Y_2}}{n_{X_2\overline{Y_2}}.n_{\overline{X_2}Y_2}} = OR(X_2 \to Y_2). \end{aligned}$$

     From this result, it was concluded that the OR measure is a measure that varies with the growth of certain numbers..
     However, Grissa stated in her work that the Odd-Ratio(OR) measure varies according to the growth of certain numbers. Indeed, she used the $2^{i`th}$ expression of this measure. But the question arises, why did she not use the first expression (fundamental expression of Odd-Ratio) ?

3. Then,

$$\begin{aligned}
OR^f_{hn}(X \to Y) &= \frac{P(Y/X) - P(Y)}{P(Y/X)(1 - P(X) - P(Y) + P(Y/X)P(X))} \\
&= \frac{\frac{n_{XY}}{n_X} - \frac{n_Y}{n}}{\frac{n_{XY}}{n_X}(1 - \frac{n_x}{n} - \frac{n_Y}{n} + \frac{n_{XY}}{n})} \\
&= \frac{n.n_{XY} - n_X.n_Y}{n.n_{XY} - n_X.n_{XY} - n_Y.n_{XY} + n^2_{XY}} \\
&= \frac{n.n_{XY} - (n_{XY} + n_{X\overline{Y}}).(n_{XY} + n_{\overline{X}Y})}{n.n_{XY} - (n_{XY} + n_{X\overline{Y}}).n_{XY} - (n_{XY} + n_{\overline{X}Y}).n_{XY} + n^2_{XY}} \\
&= \frac{n.n_{XY} - n^2_{XY} - n_{XY}.n_{\overline{X}Y} - n_{XY}.n_{X\overline{Y}} - n_{X\overline{Y}}.n_{\overline{X}Y}}{n.n_{XY} - n^2_{XY} - n_{XY}.n_{X\overline{Y}} - n_{XY}.n_{\overline{X}Y}}.
\end{aligned}$$

Therefore, if $\forall(k_1;k_2) \in \mathbb{N}^{*2}$, $n_{X_1Y_1} = k_1 n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}\, and$ $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, we have :

$\mathrm{OR}^f_{hn}(X_1 \to Y_1)$

$$\begin{aligned}
&= \frac{k_1.n.n_{X_2Y_2} - k_1^2.n^2_{X_2Y_2} - k_1.n_{X_2Y_2}.k_2.n_{\overline{X}_2Y_2} - k_1.n_{X_2Y_2}.k_1.n_{X_2\overline{Y}_2} - k_1.n_{X_2\overline{Y}_2}.k_2.n_{\overline{X}_2Y_2}}{k_1.n.n_{X_2Y_2} - k_1^2.n^2_{X_2Y_2} - k_1.n_{X_2Y_2}.k_1.n_{X_2\overline{Y}_2} - k_1.n_{X_2Y_2}.k_2.n_{\overline{X}_2Y_2}} \\
&= \frac{n.n_{X_2Y_2} - k_1.n^2_{X_2Y_2} - k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k_1.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k_2.n_{X_2\overline{Y}_2}.n_{\overline{X}_2Y_2}}{n.n_{X_2Y_2} - k_1.n^2_{X_2Y_2} - k_1.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}} \\
&\neq \frac{n.n_{X_2Y_2} - n^2_{X_2Y_2} - n_{X_2Y_2}.n_{\overline{X_2}Y_2} - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{X_2\overline{Y_2}}.n_{\overline{X_2}Y_2}}{n.n_{X_2Y_2} - n^2_{X_2Y_2} - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{X_2Y_2}.n_{\overline{X_2}Y_2}} = OR^f_{hn}(X_2 \to Y_2).
\end{aligned}$$

Hence OR^f_{hn} is a measure that varies with the growth of certain numbers.

In the end,

$$\begin{aligned}
OR^d_{hn}(X \to Y) &= \frac{P(Y/X) - P(Y)}{(1 - P(Y/X))(P(Y) - P(Y/X)P(X))} \\
&= \frac{\frac{n_{XY}}{n_X} - \frac{n_Y}{n}}{(1 - \frac{n_{XY}}{n_X})(\frac{n_Y}{n} - \frac{n_{XY}}{n})} \\
&= \frac{n.n_{XY} - n_X.n_Y}{n_X.n_Y - n_X.n_{XY} - n_Y.n_{XY} + n^2_{XY}} \\
&= \frac{n.n_{XY} - (n_{XY} + n_{X\overline{Y}}).(n_{XY} + n_{\overline{X}Y})}{(n_{XY} + n_{X\overline{Y}}).(n_{XY} + n_{\overline{X}Y}) - (n_{XY} + n_{X\overline{Y}}).n_{XY} - (n_{XY} + n_{\overline{X}Y}).n_{XY} + n^2_{XY}} \\
&= \frac{n.n_{XY} - n^2_{XY} - n_{XY}.n_{\overline{X}Y} - n_{XY}.n_{X\overline{Y}} - n_{\overline{X}Y}.n_{X\overline{Y}}}{n_{\overline{X}Y}.n_{X\overline{Y}}}
\end{aligned}$$

Therefore, if $\forall(k_1;k_2) \in \mathbb{N}^{*2}$, $n_{X_1Y_1} = k_1 n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k_1 n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k_2 n_{\overline{X}_2Y_2}\, and$ $n_{\overline{X}_1\overline{Y}_1} = k_2 n_{\overline{X}_2\overline{Y}_2}$, we have :

$\mathrm{OR}^d_{hn}(X_1 \to Y_1)$

$$\begin{aligned}
&= \frac{k_1.n.n_{X_2Y_2} - k_1^2.n^2_{X_2Y_2} - k_1.n_{X_2Y_2}.k_2.n_{\overline{X}_2Y_2} - k_1.n_{X_2Y_2}.k_1 n_{X_2\overline{Y}_2} - k_2.n_{\overline{X}_2Y_2}.k_1.n_{X_2\overline{Y}_2}}{k_2.n_{\overline{X}_2Y_2}.k_1.n_{X_2\overline{Y}_2}} \\
&= \frac{n.n_{X_2Y_2} - k_1.n^2_{X_2Y_2} - k_2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k_1.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k_2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\
&\neq \frac{n.n_{X_2Y_2} - n^2_{X_2Y_2} - n_{X_2Y_2}.n_{\overline{X_2}Y_2} - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{\overline{X_2}Y_2}.n_{X_2\overline{Y_2}}}{n_{\overline{X_2}Y_2}.n_{X_2\overline{Y_2}}} = OR^d_{hn}(X_2 \to Y_2)
\end{aligned}$$

Hence OR^d_{hn} is a measure that varies with the growth of certain numbers.

Hence we have : $\boxed{\mathrm{P}_{15}(M_{GK}) = P_{15}(OR) = P_{15}(OR_{nh}) = 0}$.

Property 16 (P_{16}) : ***A measure capable of differentiating between the rules $X \to Y$ and $\overline{X} \to Y$ according to a relation of opposition***

- If $\exists X \to Y, m(X \to Y) \neq -m(\overline{X} \to Y)$, then $P_{16}(m) = 0$;
- If $\forall X \to Y, m(X \to Y) = -m(\overline{X} \to Y)$, then $P_{16}(m) = 1$.

The following propositions show that the measures M_{GK}, OR and OR_{hn} are not measures capable of differentiating between $X \to Y$ and $\overline{X} \to Y$ rules according to an opposition relation..

Proposition 38. *Let X and Y be two patterns and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$M_{GK}(\overline{X} \to Y) = -M_{GK}(X \to Y) \times \frac{P(X)}{1 - P(X)}. \quad (51)$$

Proposition 39. *Let X and Y be two patterns and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$OR(\overline{X} \to Y) = \frac{1}{OR(X \to Y)}. \quad (52)$$

Proposition 40. *(a) If X favours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$OR_{hn}(\overline{X} \to Y) = -OR^{d}_{hn}(X \to Y). \quad (53)$$

(b) If X disfavours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :

$$OR_{hn}(\overline{X} \to Y) = -OR^{f}_{hn}(X \to Y). \quad (54)$$

Demonstration :

1. Indeed, $\forall X \to Y \in \mathbb{K}$,

$$\begin{aligned} M^{f}_{GK}(\overline{X} \to Y) &= \frac{\mathrm{P}_{\overline{X}}(Y) - \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} = \frac{\frac{P(\overline{X} \cap Y)}{P(\overline{X})} - P(Y)}{1 - P(Y)} = \frac{P_Y(\overline{X})P(Y) - P(Y) + P(X)P(Y)}{(1 - P(X))(1 - P(Y))} \\ &= -\frac{P_X(Y) - P(Y)}{1 - P(Y)} \times \frac{P(X)}{1 - P(X)} = -M^{f}_{GK}(X \to Y) \times \frac{P(X)}{1 - P(X)} \\ &\neq -M^{f}_{GK}(X \to Y) \end{aligned}$$

Hence M^{f}_{GK} is not a measure capable of differentiating between $X \to Y$ and $\overline{X} \to Y$ the rules according to a relation of opposition.
And for

$$\begin{aligned} M^{d}_{GK}(\overline{X} \to Y) &= \frac{\mathrm{P}_{\overline{X}}(Y) - \mathrm{P}(Y)}{\mathrm{P}(Y)} = \frac{\frac{P(\overline{X} \cap Y)}{P(\overline{X})} - P(Y)}{P(Y)} = \frac{P_Y(\overline{X})P(Y) - P(Y) + P(X)P(Y)}{(1 - P(X)).P(Y)} \\ &= -\frac{P_X(Y) - P(Y)}{P(Y)} \times \frac{P(X)}{1 - P(X)} = -M^{d}_{GK}(X \to Y) \times \frac{P(X)}{1 - P(X)} \\ &\neq -M^{d}_{GK}(X \to Y) \end{aligned}$$

Hence M^{d}_{GK} is not a measure capable of differentiating between $X \to Y$ and $\overline{X} \to Y$ rules according to a relation of opposition.

2. Then, $\forall X \to Y \in \mathbb{K}$,

$$\begin{aligned} OR(\overline{X} \to Y) &= \frac{\mathrm{P}(Y/\overline{X})(1 - \mathrm{P}(\overline{X}) - \mathrm{P}(Y) + \mathrm{P}(Y/\overline{X}).\mathrm{P}(\overline{X}))}{(\mathrm{P}(Y) - \mathrm{P}(Y/\overline{X}).\mathrm{P}(\overline{X}))(1 - \mathrm{P}(Y/\overline{X}))} \\ &= \frac{\frac{P(\overline{X} \cap Y)}{P(\overline{X})}\big(P(X) - P(Y) + P(\overline{X} \cap Y)\big)}{\big(P(Y) - P(\overline{X} \cap Y)\big)\big(1 - \frac{P(\overline{X} \cap Y)}{P(\overline{X})}\big)} \\ &= \frac{\frac{P(Y) - P(X \cap Y)}{1 - P(X)}\big(P(X) - P(Y) + P(Y) - P(X \cap Y)\big)}{\big(P(Y) - P(Y) + P(X \cap Y)\big)\big(1 - \frac{P(Y) - P(X \cap Y)}{1 - P(X)}\big)} \\ &= \frac{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))}{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))} \\ &= \frac{1}{OR(X \to Y)} \\ &\neq -OR(X \to Y) \end{aligned}$$

Hence OR is not a measure capable of differentiating between $X \to Y$ and $\overline{X} \to Y$ rules according to a relation of opposition.

3. In the end, $\forall X \to Y \in \mathbb{K}$,

$$\begin{aligned}
OR^f_{hn}(\overline{X} \to Y) &= \frac{P(Y/\overline{X}) - P(Y)}{P(Y/\overline{X})(1 - P(\overline{X}) - P(Y) + P(Y/\overline{X})P(\overline{X}))} \\
&= \frac{\frac{P(\overline{X}\cap Y)}{P(\overline{X})} - P(Y)}{\frac{P(\overline{X}\cap Y)}{P(\overline{X})}(P(X) - P(Y) + \frac{P(\overline{X}\cap Y)}{P(\overline{X})}P(\overline{X}))} \\
&= \frac{P(Y) - P(X \cap Y) - P(Y) + P(X)P(Y)}{\big(P(Y) - P(X \cap Y)\big)\big(P(X) - P(X \cap Y)\big)} \\
&= -\frac{P(Y/X) - P(Y)}{(1 - P(Y/X))(P(Y) - P(Y/X)P(X))} \\
&= -OR^d_{hn}(X \to Y) \\
&\neq -OR^f_{hn}(X \to Y)
\end{aligned}$$

Hence OR^f_{hn} is not a measure capable of differentiating between $X \to Y$ and $\overline{X} \to Y$ rules according to a relation of opposition. And for

$$\begin{aligned}
OR^d_{hn}(\overline{X} \to Y) &= \frac{P(Y/\overline{X}) - P(Y)}{(1 - P(Y/\overline{X}))(P(Y) - P(Y/\overline{X})P(\overline{X}))} \\
&= \frac{\frac{P(\overline{X}\cap Y)}{P(\overline{X})} - P(Y)}{(1 - \frac{P(\overline{X}\cap Y)}{P(\overline{X})})(P(Y) - \frac{P(\overline{X}\cap Y)}{P(\overline{X})}P(\overline{X}))} \\
&= \frac{P(Y) - P(X \cap Y) - P(Y) + P(X)P(Y)}{\big(1 - P(X) - P(Y) + P(X \cap Y)\big)\big(P(X \cap Y)\big)} \\
&= -\frac{P(Y/X) - P(Y)}{P(Y/X)(1 - P(X) - P(Y) + P(Y/X)P(X))} \\
&= -OR^f_{hn}(X \to Y) \\
&\neq -OR^d_{hn}(X \to Y)
\end{aligned}$$

Hence OR^d_{hn} is not a measure capable of differentiating between $X \to Y$ and $\overline{X} \to Y$ rules according to a relation of opposition.

Hence we have : $\boxed{P_{16}(M_{GK}) = P_{16}(OR) = P_{16}(OR_{nh}) = 0}$.

Property 17 (P_{17}) : ***A measure capable of differentiating between the rules $X \to Y$ and $X \to \overline{Y}$ according to an oppositional relationship***

- If $\exists X \to Y, m(X \to Y) \neq -m(X \to \overline{Y})$, then $P_{17}(m) = 0$;
- If $\forall X \to Y, m(X \to Y) = -m(X \to \overline{Y})$, then $P_{17}(m) = 1$.

The following propositions show that the measures M_{GK}, OR and OR_{hn} are not measures capable of differentiating between the rules $X \to Y$ and $X \to \overline{Y}$ according to an opposition relation..

Proposition 41. *(a) According to [Feno(2007)Feno], if X favours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$M_{GK}(X \to \overline{Y}) = -M^d_{GK}(X \to Y). \tag{55}$$

(b) If X disfavours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :

$$M_{GK}(X \to \overline{Y}) = -M^f_{GK}(X \to Y). \tag{56}$$

Proposition 42. *Let X and Y be two patterns and$\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$OR(X \to \overline{Y}) = \frac{1}{OR(X \to Y)}. \tag{57}$$

Proposition 43. *(a) If X favours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$OR_{hn}(X \to \overline{Y}) = -OR^d_{hn}(X \to Y). \tag{58}$$

(b) If X disfavours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :

$$OR_{hn}(X \to \overline{Y}) = -OR^f_{hn}(X \to Y). \tag{59}$$

Demonstration :

1. In fact,$\forall X \to Y \in \mathbb{K}$,

$$\begin{aligned}
M^f_{GK}(X \to \overline{Y}) &= \frac{\mathrm{P}_X(\overline{Y}) - \mathrm{P}(\overline{Y})}{1 - \mathrm{P}(\overline{Y})} = \frac{1 - \mathrm{P}_X(Y) - 1 + \mathrm{P}(Y)}{1 - 1 + \mathrm{P}(Y)} \\
&= \frac{-\mathrm{P}_X(Y) + \mathrm{P}(Y)}{\mathrm{P}(Y)} \\
&= -M^d_{GK}(X \to Y) \\
&\neq -M^f_{GK}(X \to Y)
\end{aligned}$$

Hence M^f_{GK} is not a measure capable of differentiating between the rules $XtoY$ and $Xto\overline{Y}$ according to an opposition relation.

And for

$$\begin{aligned}
M^d_{GK}(X \to \overline{Y}) &= \frac{\mathrm{P}_X(\overline{Y}) - \mathrm{P}(\overline{Y})}{\mathrm{P}(\overline{Y})} = \frac{1 - \mathrm{P}_X(Y) - 1 + \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} \\
&= \frac{-\mathrm{P}_X(Y) + \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} \\
&= -M^f_{GK}(X \to Y) \\
&\neq -M^d_{GK}(X \to Y)
\end{aligned}$$

Hence M^d_{GK} is not a measure capable of differentiating between the rules $X \to Y$ and $X \to \overline{Y}$ according to an opposition relation.

2. Then, $\forall X \to Y \in \mathbb{K}$,

$$\begin{aligned}
OR(X \to \overline{Y}) &= \frac{\mathrm{P}(\overline{Y}/X)(1 - \mathrm{P}(X) - \mathrm{P}(\overline{Y}) + \mathrm{P}(\overline{Y}/X).\mathrm{P}(X))}{(\mathrm{P}(\overline{Y}) - \mathrm{P}(\overline{Y}/X).\mathrm{P}(X))(1 - \mathrm{P}(\overline{Y}/X))} \\
&= \frac{\big(1 - \mathrm{P}(Y/X)\big)\big(1 - \mathrm{P}(X) - 1 + \mathrm{P}(Y) + \mathrm{P}(X) - \mathrm{P}(Y/X).\mathrm{P}(X)\big)}{(1 - \mathrm{P}(Y) - \mathrm{P}(X) + \mathrm{P}(Y/X).\mathrm{P}(X))(1 - 1 + \mathrm{P}(Y/X))} \\
&= \frac{\big(1 - \mathrm{P}(Y/X)\big)\big(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X)\big)}{(1 - \mathrm{P}(Y) - \mathrm{P}(X) + \mathrm{P}(Y/X).\mathrm{P}(X))(\mathrm{P}(Y/X))} \\
&= \frac{1}{OR(X \to Y)} \\
&\neq -OR(X \to Y)
\end{aligned}$$

Hence OR is not a measure capable of differentiating between the rules $X \to Y$ and $X \to \overline{Y}$ according to a relation of opposition.

3. Then, $\forall X \to Y \in \mathbb{K}$,

$$\begin{aligned}
OR^f_{hn}(X \to \overline{Y}) &= \frac{P(\overline{Y}/X) - P(\overline{Y})}{P(\overline{Y}/X)(1 - P(X) - P(\overline{Y}) + P(\overline{Y}/X)P(X))} \\
&= \frac{1 - P(Y/X) - 1 + P(Y)}{\big(1 - P(Y/X)\big)\big(1 - P(X) - 1 + P(Y) + P(X) - P(Y/X)P(X)\big)} \\
&= \frac{-P(Y/X) + P(Y)}{\big(1 - P(Y/X)\big)\big(P(Y) - P(Y/X)P(X)\big)} \\
&= -OR^d_{hn}(X \to Y) \\
&\neq -OR^f_{hn}(X \to Y)
\end{aligned}$$

Hence OR^d_{hn} is not a measure capable of differentiating between the rules $X \to Y$ and $X \to \overline{Y}$ according to an oppositional relationship.

At the end

$$\begin{aligned} OR^d_{hn}(X \to \overline{Y}) &= \frac{P(\overline{Y}/X) - P(\overline{Y})}{(1 - P(\overline{Y}/X))(P(\overline{Y}) - P(\overline{Y}/X)P(X))} \\ &= \frac{1 - P(Y/X) - 1 + P(Y)}{(1 - 1 + P(Y/X))(1 - P(Y) - P(X) + P(Y/X)P(X))} \\ &= \frac{-P(Y/X) + P(Y)}{(P(Y/X))(1 - P(Y) - P(X) + P(Y/X)P(X))} \\ &= -OR^f_{hn}(X \to Y) \\ &\neq -OR^d_{hn}(X \to Y) \end{aligned}$$

Hence OR^f_{hn} is not a measure capable of differentiating between the rules $X \to Y$ and $X \to \overline{Y}$ according to an opposition relation.

Hence we have : $\boxed{\mathrm{P}_{17}(M_{GK}) = P_{17}(OR) = P_{17}(OR_{nh}) = 0}$.

Property 18 (P_{18}) : *Measure evaluating th $X \to Y$ and $\overline{X} \to \overline{Y}$ rules in the same way*

- If $\exists X \to Y, m(X \to Y) \neq -m(\overline{X} \to \overline{Y})$, then $P_{18}(m) = 0$;
- If $\forall X \to Y, m(X \to Y) = -m(\overline{X} \to \overline{Y})$, then $P_{18}(m) = 1$.

The following propositions show that the measures M_{GK}, OR and OR_{hn} are measures evaluating in the same way the rules $X \to Y$ and $\overline{X} \to \overline{Y}$.

Proposition 44. *(a) If X favours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$M_{GK}(\overline{X} \to \overline{Y}) = M^d_{GK}(X \to Y) \times \frac{\mathrm{P}(X)}{1 - \mathrm{P}(X)}. \tag{60}$$

(b) If X disfavours Y and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :

$$M_{GK}(\overline{X} \to \overline{Y}) = M^f_{GK}(X \to Y) \times \frac{\mathrm{P}(X)}{1 - \mathrm{P}(X)}. \tag{61}$$

Proposition 45. *Let X and Y be two patterns and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$OR(\overline{X} \to \overline{Y}) = OR(X \to Y). \tag{62}$$

Proposition 46. *Let X and Y be two patterns and $\forall X \to Y \in \mathbb{K}$, we have the following relationship :*

$$OR_{hn}(\overline{X} \to \overline{Y}) = OR_{hn}(X \to Y). \tag{63}$$

Demonstration :

1. Indeed,

$$\begin{aligned} M^f_{GK}(\overline{X} \to \overline{Y}) &= \frac{\mathrm{P}_{\overline{X}}(\overline{Y}) - \mathrm{P}(\overline{Y})}{1 - \mathrm{P}(\overline{Y})} = \frac{1 - \mathrm{P}_{\overline{X}}(Y) - 1 + \mathrm{P}(Y)}{1 - 1 + \mathrm{P}(Y)} \\ &= \frac{-\mathrm{P}_{\overline{X}}(Y) + \mathrm{P}(Y)}{\mathrm{P}(Y)} = \frac{-\frac{\mathrm{P}(\overline{X} \cap Y)}{\mathrm{P}(\overline{X})} + \mathrm{P}(Y)}{\mathrm{P}(Y)} \\ &= \frac{\frac{-\mathrm{P}(Y) + \mathrm{P}_X(Y)\mathrm{P}(X)}{1 - \mathrm{P}(X)} + \mathrm{P}(Y)}{\mathrm{P}(Y)} = \frac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{\mathrm{P}(Y)} \times \frac{\mathrm{P}(X)}{1 - \mathrm{P}(X)} \\ &= M^d_{GK}(X \to Y) \times \frac{\mathrm{P}(X)}{1 - \mathrm{P}(X)} \\ &\neq M^f_{GK}(X \to Y) \end{aligned}$$

Hence M_{GK}^f is not a measure evaluating in the same way the rules $X \to Y$ and $\overline{X} \to \overline{Y}$. Then,

$$\begin{aligned}
M_{GK}^d(\overline{X} \to \overline{Y}) &= \frac{\mathrm{P}_{\overline{X}}(\overline{Y}) - \mathrm{P}(\overline{Y})}{\mathrm{P}(\overline{Y})} = \frac{1 - \mathrm{P}_{\overline{X}}(Y) - 1 + \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} \\
&= \frac{-\mathrm{P}_{\overline{X}}(Y) + \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} = \frac{-\frac{\mathrm{P}(\overline{X} \cap Y)}{\mathrm{P}(\overline{X})} + \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} \\
&= \frac{\frac{-\mathrm{P}(Y) + \mathrm{P}_X(Y)\mathrm{P}(X)}{1 - \mathrm{P}(X)} + \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} = \frac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} \times \frac{\mathrm{P}(X)}{1 - \mathrm{P}(X)} \\
&= M_{GK}^f(X \to Y) \times \frac{\mathrm{P}(X)}{1 - \mathrm{P}(X)} \\
&\neq M_{GK}^d(X \to Y)
\end{aligned}$$

Hence M_{GK}^d is not a measure evaluating in the same way the rules $X \to Y$ and $\overline{X} \to \overline{Y}$

2. Then,

$$\begin{aligned}
OR(\overline{X} \to \overline{Y}) &= \frac{\mathrm{P}(\overline{Y}/\overline{X})\big(1 - \mathrm{P}(\overline{X}) - \mathrm{P}(\overline{Y}) + \mathrm{P}(\overline{Y}/\overline{X}).\mathrm{P}(\overline{X})\big)}{\big(\mathrm{P}(\overline{Y}) - \mathrm{P}(\overline{Y}/\overline{X}).\mathrm{P}(\overline{X})\big)\big(1 - \mathrm{P}(\overline{Y}/\overline{X})\big)} \\
&= \frac{\big(1 - \mathrm{P}(Y/\overline{X})\big)\big(1 - 1 + \mathrm{P}(X) - 1 + \mathrm{P}(Y) + (1 - \mathrm{P}(Y/\overline{X})).(1 - \mathrm{P}(X))\big)}{\big(1 - \mathrm{P}(Y) - (1 - \mathrm{P}(Y/\overline{X})).(1 - \mathrm{P}(X))\big)\big(1 - 1 + \mathrm{P}(Y/\overline{X})\big)} \\
&= \frac{\big(1 - \frac{\mathrm{P}(\overline{X} \cap Y)}{\mathrm{P}(\overline{X})}\big)\big(\mathrm{P}(Y) - \frac{\mathrm{P}(\overline{X} \cap Y)}{\mathrm{P}(\overline{X})} + \frac{\mathrm{P}(\overline{X} \cap Y).\mathrm{P}(X)}{\mathrm{P}(\overline{X})}\big)}{\big(-\mathrm{P}(Y) + \mathrm{P}(X) - \frac{\mathrm{P}(\overline{X} \cap Y)}{\mathrm{P}(\overline{X})} - \frac{\mathrm{P}(\overline{X} \cap Y).\mathrm{P}(X)}{\mathrm{P}(\overline{X})}\big)\big(\frac{\mathrm{P}(\overline{X} \cap Y)}{\mathrm{P}(\overline{X})}\big)} \\
&= \frac{\big(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}_X(Y)\mathrm{P}(X)\big)\big(1 - \mathrm{P}(X)\big).\mathrm{P}_X(Y)\mathrm{P}(X)}{\big(\mathrm{P}(X) - \mathrm{P}^2(X) - \mathrm{P}_X(Y)\mathrm{P}(X) + \mathrm{P}_X(Y)\mathrm{P}^2(X)\big)\big(\mathrm{P}(Y) - \mathrm{P}_X(Y)\mathrm{P}(X)\big)} \\
&= \frac{\big(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}_X(Y)\mathrm{P}(X)\big)\big(1 - \mathrm{P}(X)\big).\mathrm{P}_X(Y)\mathrm{P}(X)}{\mathrm{P}(X)\big(1 - \mathrm{P}(X)\big)\big(1 - \mathrm{P}_X(Y)\big)\big(\mathrm{P}(Y) - \mathrm{P}_X(Y)\mathrm{P}(X)\big)} \\
&= \frac{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))} \\
&= OR(X \to Y) \\
&\neq -OR(X \to Y).
\end{aligned}$$

Hence OR is not a measure evaluating in the same way the rules $X \to Y$ and $\overline{X} \to \overline{Y}$

3. And after

$$\begin{aligned}
OR_{hn}^f(\overline{X} \to \overline{Y}) &= \frac{P(\overline{Y}/\overline{X}) - P(\overline{Y})}{P(\overline{Y}/\overline{X})(1 - P(\overline{X}) - P(\overline{Y}) + P(\overline{Y}/\overline{X})P(\overline{X}))} \\
&= \frac{1 - P(Y/\overline{X}) - 1 + P(Y)}{\big(1 - P(Y/\overline{X})\big)\big(1 - 1 + P(X) - 1 + P(Y) + (1 - P(Y/\overline{X}))(1 - P(X))\big)} \\
&= \frac{-P(Y/\overline{X}) + P(Y)}{\big(1 - P(Y/\overline{X})\big)\big(P(Y) - P(Y/\overline{X}) + P(Y/\overline{X})P(X)\big)} \\
&= \frac{-\frac{P(\overline{X} \cap Y)}{P(\overline{X})} + P(Y)}{\big(1 - \frac{P(\overline{X} \cap Y)}{P(\overline{X})}\big)\big(P(Y) - \frac{P(\overline{X} \cap Y)}{P(\overline{X})} + \frac{P(\overline{X} \cap Y)}{P(\overline{X})P(X)}\big)} \\
&= \frac{\frac{P_X(Y)P(X) - P(X)P(Y)}{1 - P(X)}}{\big(\frac{1 - P(X) - P(Y) + P_X(Y)P(X)}{1 - P(X)}\big)\frac{P_X(Y)P(X)(1 - P(X))}{1 - P(X)}} \\
&= \frac{P(Y/X) - P(Y)}{P(Y/X)(1 - P(X) - P(Y) + P(Y/X)P(X))} \\
&= OR_{hn}^f(X \to Y) \\
&\neq -OR_{hn}^f(X \to Y).
\end{aligned}$$

Hence OR^f_{hn} is not a measure evaluating in the same way the rules $X \to Y$ and $\overline{X} \to \overline{Y}$

In the end,

$$\begin{aligned}
OR^d_{hn}(\overline{X} \to \overline{Y}) &= \frac{P(\overline{Y}/\overline{X}) - P(\overline{Y})}{(1 - P(\overline{Y}/\overline{X}))(P(\overline{Y}) - P(\overline{Y}/\overline{X})P(\overline{X}))} \\
&= \frac{1 - P(Y/\overline{X}) - 1 + P(Y)}{\big(1 - 1 + P(Y/\overline{X})\big)\big(1 - P(Y) - (1 - P(Y/\overline{X}))(1 - P(X))\big)} \\
&= \frac{-\frac{P(\overline{X} \cap Y)}{P(\overline{X})} + P(Y)}{\big(\frac{P(\overline{X} \cap Y)}{P(\overline{X})}\big)\big(- P(Y) + P(X) + \frac{P(\overline{X} \cap Y)}{P(\overline{X})} - \frac{P(\overline{X} \cap Y)}{P(\overline{X})}P(X)\big)} \\
&= \frac{\big(P_X(Y) - P(Y)\big)P(X)}{\big(P(Y) - P_X(Y)\big)\frac{(1-P(X))P(X)(1-P_X(Y))}{1-P(X)}} \\
&= \frac{P(Y/X) - P(Y)}{(1 - P(Y/X))(P(Y) - P(Y/X)P(X))} \\
&= OR^d_{hn}(X \to Y) \\
&\neq -OR^d_{hn}(X \to Y)
\end{aligned}$$

Hence OR^d_{hn} is not a measure evaluating in the same way the rules $X \to Y$ and $\overline{X} \to \overline{Y}$

Hence we have : $\boxed{\mathrm{P}_{18}(M_{GK}) = P_{18}(OR) = P_{18}(OR_{nh}) = 0}$.

Property 19 (P_{19}) : *Measurement having a size the random premise*

A measure m necessarily has a random premise size if it is based on one of these probabilistic models: normal, binomial, poisson or hypergeometric distribution.

- If m is not based on a probabilistic model, then $P_{19}(m) = 0$ (taille fixe);
- If m is based on a probabilistic model, then $P_{19}(m) = 1$.

In effect

The measures M_{GK}, OR and OR_{hn} are not based on one of the models: normal, binomial, poisson or hypergeometric distribution.

Hence M_{GK}, OR and OR_{hn} are not based on any probabilistic models

Hence we have : $\boxed{\mathrm{P}_{19}(M_{GK}) = P_{19}(OR) = P_{19}(OR_{nh}) = 0}$.

Propriété 20 (P_{20}) : *Mesure statistique*

Une mesure m est dite statistique si elle est sensible (croissante) à l'augmentation des données, par contre, elle est dite descriptive si elle est invariante à la croissance du nombre d'enregistrements.

- Si $\forall k \in \mathbb{N}^*, \forall X_1 \to Y_1, \forall X_2 \to Y_2$, tel que ($n_{X_1Y_1} = k.n_{X_2Y_2}$ et $n_{X_1\overline{Y}_1} = k.n_{X_2\overline{Y}_2}$ et $n_{\overline{X}_1Y_1} = k.n_{\overline{X}_2Y_2}$ et $n_{\overline{X}_1\overline{Y}_1} = k.n_{\overline{X}_2\overline{Y}_2}$) $\Rightarrow m(X_1 \to Y_1) = m(X_2 \to Y_2)$, alors $P_{20}(m) = 0$ (descriptive);
- Si $\exists k \in \mathbb{N}^*, \exists X_1 \to Y_1, \exists X_2 \to Y_2$, tel que ($n_{X_1Y_1} = k.n_{X_2Y_2}$ et $n_{X_1\overline{Y}_1} = k.n_{X_2\overline{Y}_2}$ et $n_{\overline{X}_1Y_1} = k.n_{\overline{X}_2Y_2}$ et $n_{\overline{X}_1\overline{Y}_1} = k.n_{\overline{X}_2\overline{Y}_2}$) et $m(X_1 \to Y_1) \neq m(X_2 \to Y_2)$ alors $P_{20}(m) = 1$ (statistique).

The following propositions show that the measures M_{GK}, OR and OR_{hn} are statistical measures.

Proposition 47. *Let X and Y be two patterns, $\forall k \in \mathbb{N}^*$ and $\forall X \to Y \in \mathbb{K}$, we have the following relation :*

$$M_{GK}(X_1 \to Y_1) \neq M_{GK}(X_2 \to Y_2). \tag{64}$$

Proposition 48. *Let X and Y be two patterns, $\forall k \in \mathbb{N}^*$ and $\forall X \to Y \in \mathbb{K}$, we have the following relation :*

$$OR(X_1 \to Y_1) \neq OR(X_2 \to Y_2). \tag{65}$$

Proposition 49. *Let X and Y be two patterns, $\forall k \in \mathbb{N}^*$ and $\forall X \to Y \in \mathbb{K}$, we have the following relation :*

$$OR_{hn}(X_1 \to Y_1) \neq OR_{hn}(X_2 \to Y_2). \tag{66}$$

Demonstration :

1. Indeed

$$\begin{aligned} M_{GK}^f(X \to Y) &= \frac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{1 - \mathrm{P}(Y)} \\ &= \frac{n.n_{XY} - n_{XY}^2 - n_{XY}.n_{\overline{X}Y} - n_{X\overline{Y}}.n_{\overline{X}Y}}{n.n_{XY} + n.n_{X\overline{Y}} - n_{XY}^2 - n_{XY}.n_{\overline{X}Y} - n_{X\overline{Y}}.n_{\overline{X}Y}}. \end{aligned}$$

So, if we have : $\forall k \in \mathbb{N}^*, n_{X_1Y_1} = k.n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k.n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k.n_{\overline{X}_2Y_2}\, and\, n_{\overline{X}_1\overline{Y}_1} = k.n_{\overline{X}_2\overline{Y}_2}$, donc on a:

$$\begin{aligned} M_{GK}^f(X_1 \to Y_1) &= \frac{k.n.n_{X_2Y_2} - k^2.n_{X_2Y_2}^2 - k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k^2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k.n.n_{X_2Y_2+k.n.n_{X_2\overline{Y}_2}} - k^2n_{X_2Y_2}^2 - k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k^2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &= \frac{n.n_{X_2Y_2} - k.n_{X_2Y_2}^2 - k.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{n.n_{X_2Y_2+n.n_{X_2\overline{Y}_2}} - k.n_{X_2Y_2}^2 - k.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &\neq \frac{n.n_{X_2Y_2} - n_{X_2Y_2}^2 - n_{X_2Y_2}.n_{\overline{X}_2Y_2} - n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{n.n_{X_2Y_2+n.n_{X_2\overline{Y}_2}} - n_{X_2Y_2}^2 - n_{X_2Y_2}.n_{\overline{X}_2Y_2} - n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} = M_{GK}^f(X_2 \to Y_2) \end{aligned}$$

So $\exists k \in \mathbb{N}^*, \exists X_1 \to Y_1, \exists X_2 \to Y_2, M_{GK}^f(X_1 \to Y_1) \neq M_{GK}^f(X_2 \to Y_2)$

Hence M_{GK}^f is a statistical measure.
Then,

$$\begin{aligned} M_{GK}^d(X_1 \to Y_1) &= \frac{\mathrm{P}_X(Y) - \mathrm{P}(Y)}{\mathrm{P}(Y)} \\ &= \frac{n.n_{XY} - n_{XY}^2 - n_{XY}.n_{\overline{X}Y} - n_{X\overline{Y}}.n_{\overline{X}Y}}{n_{XY}^2 + n_{XY}.n_{\overline{X}Y} + n_{X\overline{Y}}.n_{\overline{X}Y}}. \end{aligned}$$

So, if we have : $\forall k \in \mathbb{N}^*, n_{X_1Y_1} = k.n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k.n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k.n_{\overline{X}_2Y_2}\, and\, n_{\overline{X}_1\overline{Y}_1} = k.n_{\overline{X}_2\overline{Y}_2}$, donc on a:

$$\begin{aligned} M_{GK}^d(X_1 \to Y_1) &= \frac{k.n.n_{X_2Y_2} - k^2.n_{X_2Y_2} - k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k^2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k^2.n_{X_2Y_2} + k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} + k^2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &= \frac{n.n_{X_2Y_2} - k.n_{X_2Y_2} - k.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k.n_{X_2Y_2} + k.n_{X_2Y_2}.n_{\overline{X}_2Y_2} + k.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &\neq \frac{n.n_{X_2Y_2} - n_{X_2Y_2} - n_{X_2Y_2}.n_{\overline{X}_2Y_2} - n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{n_{X_2Y_2} + n_{X_2Y_2}.n_{\overline{X}_2Y_2} + n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} = M_{GK}^d(X_2 \to Y_2) \end{aligned}$$

So $\exists k \in \mathbb{N}^*, \exists X_1 \to Y_1, \exists X_2 \to Y_2, M_{GK}^f(X_1 \to Y_1) \neq M_{GK}^f(X_2 \to Y_2)$
Hence M_{GK}^d is a statistical measure.

2. After

$$\begin{aligned} OR(X \to Y) &= \frac{\mathrm{P}(Y/X)(1 - \mathrm{P}(X) - \mathrm{P}(Y) + \mathrm{P}(Y/X).\mathrm{P}(X))}{(\mathrm{P}(Y) - \mathrm{P}(Y/X).\mathrm{P}(X))(1 - \mathrm{P}(Y/X))} \\ &= \frac{n.n_{XY} - n_{XY}^2 - n_{XY}.n_{X\overline{Y}} - n_{XY}.n_{\overline{X}Y}}{n_{X\overline{Y}}.n_{\overline{X}Y}}. \end{aligned}$$

Therefore, if we have : $\forall k \in \mathbb{N}^*, n_{X_1Y_1} = k.n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k.n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k.n_{\overline{X}_2Y_2}\, and\, n_{\overline{X}_1\overline{Y}_1} = k.n_{\overline{X}_2\overline{Y}_2}$, therefore, we have:

$$\begin{aligned} OR(X_1 \to Y_1) &= \frac{k.n.n_{X_2Y_2} - k^2.n^2_{X_2Y_2} - k^2.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}}{k^2.n_{X_2\overline{Y}_2}.n_{\overline{X}_2Y_2}} \\ &= \frac{n.n_{X_2Y_2} - k.n^2_{X_2Y_2} - k.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k.n_{X_2Y_2}.n_{\overline{X}_2Y_2}}{k.n_{X_2\overline{Y}_2}.n_{\overline{X}_2Y_2}} \\ &\neq = \frac{n.n_{X_2Y_2} - n^2_{X_2Y_2} - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{X_2Y_2}.n_{\overline{X_2}Y_2}}{n_{X_2\overline{Y_2}}.n_{\overline{X_2}Y_2}} = OR(X_2 \to Y_2). \end{aligned}$$

So $\exists k \in \mathbb{N}^*, \exists X_1 \to Y_1, \exists X_2 \to Y_2, OR(X_1 \to Y_1) \neq OR(X_2 \to Y_2)$
Hence OR is a statistical measure.

3. Then,

$$\begin{aligned} OR^f_{hn}(X \to Y) &= \frac{P(Y/X) - P(Y)}{P(Y/X)(1 - P(X) - P(Y) + P(Y/X)P(X))} \\ &= \frac{n.n_{XY} - n^2_{XY} - n_{XY}.n_{\overline{X}Y} - n_{XY}.n_{X\overline{Y}} - n_{X\overline{Y}}.n_{\overline{X}Y}}{n.n_{XY} - n^2_{XY} - n_{XY}.n_{X\overline{Y}} - n_{XY}.n_{\overline{X}Y}}. \end{aligned}$$

So, if we have : $\forall k \in \mathbb{N}^*, n_{X_1Y_1} = k.n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k.n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k.n_{\overline{X}_2Y_2}\, and\, n_{\overline{X}_1\overline{Y}_1} = k.n_{\overline{X}_2\overline{Y}_2}$, therefore, we have:

$$\begin{aligned} &= \frac{k.n.n_{X_2Y_2} - k^2.n^2_{X_2Y_2} - k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k^2.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k^2.n_{X_2\overline{Y}_2}.n_{\overline{X}_2Y_2}}{k.n.n_{X_2Y_2} - k^2.n^2_{X_2Y_2} - k^2.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2}} \\ \mathrm{OR}^f_{hn}(X_1 \to Y_1) &= \frac{n.n_{X_2Y_2} - k.n^2_{X_2Y_2} - k.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k.n_{X_2\overline{Y}_2}.n_{\overline{X}_2Y_2}}{n.n_{X_2Y_2} - k.n^2_{X_2Y_2} - k.n_{X_2Y_2}.n_{X_2\overline{Y}_2} - k.n_{X_2Y_2}.n_{\overline{X}_2Y_2}} \\ &\neq \frac{n.n_{X_2Y_2} - n^2_{X_2Y_2} - n_{X_2Y_2}.n_{\overline{X_2}Y_2} - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{X_2\overline{Y_2}}.n_{\overline{X_2}Y_2}}{n.n_{X_2Y_2} - n^2_{X_2Y_2} - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{X_2Y_2}.n_{\overline{X_2}Y_2}} = OR^f_{hn}(X_2 \to Y_2). \end{aligned}$$

So $\exists k \in \mathbb{N}^*, \exists X_1 \to Y_1, \exists X_2 \to Y_2, OR^f_{hn}(X_1 \to Y_1) \neq OR^f_{hn}(X_2 \to Y_2)$
Hence OR^f_{hn} is a statistical measure.

At the end

$$\begin{aligned} OR^d_{hn}(X \to Y) &= \frac{P(Y/X) - P(Y)}{(1 - P(Y/X))(P(Y) - P(Y/X)P(X))} \\ &= \frac{n.n_{XY} - n^2_{XY} - n_{XY}.n_{\overline{X}Y} - n_{XY}.n_{X\overline{Y}} - n_{\overline{X}Y}.n_{X\overline{Y}}}{n_{\overline{X}Y}.n_{X\overline{Y}}} \end{aligned}$$

So, if we have : $\forall k \in \mathbb{N}^*, n_{X_1Y_1} = k.n_{X_2Y_2}\, and\, n_{X_1\overline{Y}_1} = k.n_{X_2\overline{Y}_2}\, and\, n_{\overline{X}_1Y_1} = k.n_{\overline{X}_2Y_2}\, and\, n_{\overline{X}_1\overline{Y}_1} = k.n_{\overline{X}_2\overline{Y}_2}$, therefore, we have:

$$\begin{aligned} &= \frac{k.n.n_{X_2Y_2} - k^2.n^2_{X_2Y_2} - k^2.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k^2.n_{X_2Y_2}n_{X_2\overline{Y}_2} - k^2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k^2.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ \mathrm{OR}^d_{hn}(X_1 \to Y_1) &= \frac{n.n_{X_2Y_2} - k.n^2_{X_2Y_2} - k.n_{X_2Y_2}.n_{\overline{X}_2Y_2} - k.n_{X_2Y_2}n_{X_2\overline{Y}_2} - k.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}}{k.n_{\overline{X}_2Y_2}.n_{X_2\overline{Y}_2}} \\ &\neq \frac{n.n_{X_2Y_2} - n^2_{X_2Y_2} - n_{X_2Y_2}.n_{\overline{X_2}Y_2} - n_{X_2Y_2}.n_{X_2\overline{Y_2}} - n_{\overline{X_2}Y_2}.n_{X_2\overline{Y_2}}}{n_{\overline{X_2}Y_2}.n_{X_2\overline{Y_2}}} = OR^d_{hn}(X_2 \to Y_2) \end{aligned}$$

So $\exists k \in \mathbb{N}^*, \exists X_1 \to Y_1, \exists X_2 \to Y_2, OR^d_{hn}(X_1 \to Y_1) \neq OR^d_{hn}(X_2 \to Y_2)$
Hence OR^d_{hn} is a statistical measure.
Hence we have : $\boxed{\mathrm{P}_{20}(M_{GK}) = P_{20}(OR) = P_{20}(OR_{nh}) = 1}$.

Propriété 21 (P_{21}) : *Discriminating measure*

A measure m is said to be discriminative if it is able to distinguish rules of interest as the size of the training set n increases. In other words, a measure is able to return distinct values to the rules for different levels of involvement.

- If $\exists\eta \in \mathbb{N}^*, \forall n > \eta, \forall X_1 \to Y_1, \forall X_2 \to Y_2$ such that $\big(P_{X_1}(Y_1) > P(Y_1)\, and\, P_{X_2}(Y_2) > P(Y_2)\big) \Rightarrow m(X_1 \to Y_1) \approx m(X_2 \to Y_2)$, then $P_{21}(m) = 0$ (non-discriminant);
- If $\forall\eta \in \mathbb{N}^*, \exists n > \eta, \exists X_1 \to Y_1, \exists X_2 \to Y_2$ such that $\big(P_{X_1}(Y_1) > P(Y_1)\, and\, P_{X_2}(Y_2) > P(Y_2)\big) \Rightarrow m(X_1 \to Y_1) \neq m(X_2 \to Y_2)$, , then $P_{21}(m) = 1$(discriminante).

Théorème .4. *(a) A probabilistic quality measure μ is a discriminating measure if, and only if, for any association rule $X \to Y$, the following condition is verified at the reference situation : $\mu_{imp} \neq \mu_{ind} \neq \mu inc$.*

(b) All normalisable and normalised measures (affine or homograph) are discriminating measures.

Proof :

Following the process of normalization of the measurement values between [-1 ;1] and following the definition of normalized measurements, it is obvious that normalizable and normalized measurements (affine or homographies) are discriminating measurements.

Demonstration :

It is obvious that the measures $M_{GK}, OR, and\, OR_{hn}$ are discriminating, because they are admitted different values to the independence and to the logical implication $(m_{ind}(X \to Y) \neq m_{imp}(X \to Y))$. This means to attraction or $P_{X_1}(Y_1) > P(Y_1)\, and\, P_{X_2}(Y_2) > P(Y_2)$, $\exists X_1 \to Y_1, \exists X_2 \to Y_2$ such that $m(X_1 \to Y_1) \neq m(X_2 \to Y_2)$

Hence we have : $\boxed{\mathrm{P}_{21}(M_{GK}) = P_{21}(OR) = P_{21}(OR_{nh}) = 1}$.

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